

## SIMILARITY METHOD FOR BOUNDARY-LAYER FLOW OF A NON-NEWTONIAN VISCOUS FLUID AT A CONVECTIVELY HEATED SURFACE

by

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*The similarity method is presented for the determination of the velocity and the temperature distribution in the boundary-layer next to a horizontal moving surface heated convectively from below. The basic partial differential equations are transformed to a system of ordinary differential equations subjected to boundary conditions.*

*Keywords: boundary-layer flow, non-Newtonian fluid, heated surface, similarity solution, convective boundary condition*

### Introduction

The study of flow generated by a moving surface in an otherwise quiescent fluid plays a significant role in many material processing applications such as hot rolling, metal forming, and continuous casting (see *e. g.*, [1, 2]). The boundary-layer fluid flow over a continuous solid surface moving with constant velocity in a Newtonian fluid has been analytically studied by Sakiadis [3] and experimentally by Tsou *et al.* [4]. The thermal behavior of the problem was studied by Erickson *et al.* [5]. A polymer sheet extruded continuously from a die traveling between a feed roll and a wind-up roll was investigated by Sakiadis [3, 6], who pointed out that the known solutions for the boundary-layer on surfaces of finite length were not applicable to the boundary-layer on continuous surfaces. In the case of a moving sheet of finite length, the boundary-layer grows in a direction opposite to the direction of motion of the sheet. Tsou *et al.* [4] showed in their analytical and experimental study that the obtained analytical result for the laminar velocity field was in excellent agreement with the measured data. In tribology it is important and useful to study behavior of lubricants on solid surfaces and their role in friction. In tribological systems, lubricant reduces adhesion, friction and wear. Among the lubricant properties, viscosity and its dependence on shear rate are investigated in the literature [7-10]. It is known that the relative velocity between the moving surface and each layer of the lubricant is affected by the lubricant viscosity. In a thin boundary-layer, the wall shear stress and from this the friction drag caused by the shear next to the wall can be estimated. This drag depends on the fluid properties, and on the shape, size and speed of the solid object submerged in the fluid. Heat and mass transfer aspects with Newtonian/non-Newtonian fluids are studied by several authors [1, 11-18] under different physical situations.

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The problem of laminar hydrodynamic and thermal boundary-layers over a flat plate with convective boundary condition was examined for Newtonian fluid when the bottom surface of the plate is heated by convection from the hot fluid [13, 19-21]. The similarity solutions to the convective heat transfer problems have been studied by Aziz [19] and Magyari [22] for impermeable plate and Ishak [23] for permeable plate. The work of Aziz [19] was extended by investigating the hydrodynamic and thermal boundary-layers for power-law non-Newtonian fluid on an impermeable plate in [20].

All the previous studies deal with fluid flows and heat transfer in the absence of a buoyancy force. In many practical situations the material moves in a quiescent fluid due to the fluid flow induced by the motion of a solid material and the thermal buoyancy. Therefore, the resulting flow and the thermal field are determined by these two mechanisms, *i. e.*, surface motion and thermal buoyancy, see the paper by Uddin *et al.* [17] for Newtonian flow. It is well known that the buoyancy force arising from the heating or cooling of the continuous stretching sheet alters the flow and the thermal fields and thereby the heat transfer characteristics of the manufacturing processes.

In our investigations, we generalize the results of [17] to non-Newtonian cases and show how to transform the system of four PDE to a system of two ODE via a similarity transformation. A mathematical method for determination of the velocity and temperature distribution is presented in a steady 2-D boundary-layer flow from a heated horizontal surface in a non-Newtonian power-law fluid.

### Basic equations

The problem considered here is the steady boundary-layer flow due to a moving horizontal flat surface in an otherwise quiescent non-Newtonian fluid medium moving at a speed of  $U(\bar{x})$ . We consider a steady laminar 2-D flow of a viscous, incompressible fluid of density,  $\rho$ , and temperature,  $T_\infty$ , over the top surface of a semi-infinite horizontal impermeable stationary flat plate. It is assumed that the bottom surface of the plate is heated by convection from the hot fluid of temperature,  $T_f$ . The problem is a model for the first approximation to the laminar incompressible flow of a non-Newtonian power-law fluid past a plate surface. Here,  $\bar{x} \geq 0$  and  $\bar{y} \geq 0$  are the Cartesian co-ordinates along and normal to the plate. The plate is located at  $\bar{y} = 0$  and its origin at  $\bar{x} = \bar{y} = 0$ .

Within the framework of the previously noted assumptions and boundary-layer approximations, the governing equations of continuity, motion, and heat transfer for non-Newtonian flow can be described by the following equations:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \quad (1)$$

$$\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\rho} \frac{\partial p}{\partial \bar{x}} + \frac{1}{\rho} \frac{\partial \tau_{xy}}{\partial \bar{y}} \quad (2)$$

$$-\frac{1}{\rho} \frac{\partial p}{\partial \bar{y}} + g\beta(T - T_\infty) = 0 \quad (3)$$

$$\bar{u} \frac{\partial T}{\partial \bar{x}} + \bar{v} \frac{\partial T}{\partial \bar{y}} = \frac{\partial}{\partial \bar{y}} \left( \alpha_t \frac{\partial T}{\partial \bar{y}} \right) \quad (4)$$

where  $\bar{u}$ ,  $\bar{v}$  are the velocity components along  $\bar{x}$  and  $\bar{y}$  co-ordinates, respectively,  $T$  – the temperature of the fluid in the boundary-layer. Here, it is assumed that the flow behavior of

the non-Newtonian fluid is described by the Ostwald-de Waele power law model, where the shear stress is related to the strain rate  $\partial \bar{u} / \partial \bar{y}$  by the expression:

$$\tau_{xy} = K \left| \frac{\partial \bar{u}}{\partial \bar{y}} \right|^{n-1} \frac{\partial \bar{u}}{\partial \bar{y}} \quad (5)$$

where  $n > 0$  is called the power-law index,  $K$  – the consistency coefficient for non-Newtonian viscosity, and  $\alpha_t$  – the thermal diffusivity. The case  $0 < n < 1$  corresponds to pseudoplastic fluids (or shear-thinning fluids), the case  $n > 1$  is known as dilatant or shear-thickening fluids and for the Newtonian fluid  $n = 1$ .

We consider the boundary-layer flow induced by a continuous surface stretching with velocity  $U(\bar{x})$ . The surface is assumed to be impermeable. Accordingly, the boundary conditions at the surface  $\bar{y} = 0$  are:

$$\bar{u}(\bar{x}, 0) = U(\bar{x}), \quad \bar{v}(\bar{x}, 0) = 0, \quad \text{and} \quad -k \frac{\partial T}{\partial \bar{y}} = h_f(T_f - T_w) \quad (6)$$

moreover, far from the surface matching with the free stream as  $\bar{y} \rightarrow \infty$ :

$$\lim_{\bar{y} \rightarrow \infty} \bar{u}(\bar{x}, \bar{y}) = 0 \quad \text{and} \quad T(\bar{x}, \infty) = T_\infty \quad (7)$$

The third condition in eq. (6) expresses the convective heat conduction at the surface with thermal conductivity,  $k$ , and heat transfer coefficient,  $h_f = h_f(\bar{x})$ . If  $T_w$  denotes the uniform temperature over the top surface of the plate we have the following relations:

$$T_f > T_w > T_\infty$$

Taking the derivative of eq. (2) with respect to  $\bar{y}$  and the derivative of eq. (3) with respect to  $\bar{x}$ , the pressure terms can be eliminated:

$$\frac{\partial \bar{u}}{\partial \bar{y}} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{u} \frac{\partial^2 \bar{u}}{\partial \bar{x} \partial \bar{y}} + \bar{v} \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\partial \bar{v}}{\partial \bar{y}} \frac{\partial \bar{u}}{\partial \bar{y}} = \frac{K}{\rho} \frac{\partial^2}{\partial \bar{y}^2} \left( \left| \frac{\partial \bar{u}}{\partial \bar{y}} \right|^{n-1} \frac{\partial \bar{u}}{\partial \bar{y}} \right) - g\beta \frac{\partial(T - T_\infty)}{\partial \bar{x}} \quad (8)$$

### Similarity transformation

We first look at non-dimensionalization of the differential eqs. (1), (4), and (8). As usual, the co-ordinates  $x$  and  $y$  will be given by:

$$x = \frac{\bar{x}}{L}, \quad y = \frac{\bar{y}}{L} \text{Gr}^\alpha$$

with  $\bar{x}$ ,  $\bar{y}$  being the physical co-ordinates,  $L$  denotes the reference length, and the Grashof number will be defined for power-law viscosity:

$$\text{Gr} = \left( \frac{\rho}{K} L^n U_F^{2-n} \right)^{\frac{1}{\alpha(n+1)}}$$

Let us define  $T_f - T_\infty = \Delta T_0 \text{Gr}^\alpha g(x)$  and introduce new variables with:

$$u = \frac{\bar{u}}{U_F}, \quad v = \frac{\bar{v}}{U_F} \text{Gr}^\beta, \quad \theta = \frac{T - T_\infty}{T_f - T_\infty}, \quad U = \frac{\bar{U}}{U_F}$$

Introducing the stream function  $\psi$  with  $u = \partial \psi / \partial y$ ,  $v = -\partial \psi / \partial x$ , the equation of continuity (1) is satisfied identically. From the similarity analysis it follows that  $\alpha = \beta$ , the reference velocity is:

$$U_F = \sqrt{g\beta\Delta T_0 L}$$

and the Prandtl number is  $\text{Pr}_\alpha = LU_F / \alpha_t \sqrt[2\alpha]{\text{Gr}}$ . For  $\alpha = 1/4$  we get that  $\text{Pr} = LU_F / \alpha_t \sqrt{\text{Gr}}$  as for the Newtonian case.

On the other hand, we have from eqs. (8) and (4) that:

$$\frac{\partial^2}{\partial y^2} \left( \left| \frac{\partial^2 \psi}{\partial y^2} \right|^{n-1} \frac{\partial^2 \psi}{\partial y^2} \right) - \lambda \frac{\partial \theta g(x)}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial^3 \psi}{\partial y^3} - \frac{\partial \psi}{\partial y} \frac{\partial^3 \psi}{\partial x \partial y^2} = 0 \quad (9)$$

$$\frac{1}{\text{Pr}} g(x) \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial \psi}{\partial x} g(x) \frac{\partial \theta}{\partial y} - \frac{\partial \psi}{\partial y} g(x) \frac{\partial \theta}{\partial x} - \theta \frac{\partial \psi}{\partial y} \frac{\partial g(x)}{\partial x} = 0 \quad (10)$$

where for the buoyancy parameter  $\lambda = g\beta\Delta T_0 L / U_F^2$ .

Conditions (6) and (7) can be written:

$$\frac{\partial \psi}{\partial y}(x, 0) = \frac{\bar{U}(x)}{U_F}, \quad \psi_x(x, 0) = 0, \quad \frac{\partial \theta}{\partial y}(x, 0) = -\frac{Lh_f(xL)}{k\sqrt[4]{\text{Gr}}}(1-\theta) \quad (11)$$

$$\lim_{y \rightarrow \infty} \frac{\partial \psi(x, y)}{\partial y} = 0, \quad \lim_{y \rightarrow \infty} \frac{\partial^2 \psi(x, y)}{\partial y^2} = 0, \quad \lim_{y \rightarrow \infty} \theta(x, y) = 0 \quad (12)$$

We introduce the following dimensionless variables:

$$\psi = x^\delta f(\eta)$$

$$\eta = \frac{y}{x^\gamma}$$

$$\theta = \theta(\eta)$$

$$g(x) = x^\sigma$$

with some exponents  $\gamma$ ,  $\delta$ , and  $\sigma$ .

In order to have similarity solutions to eqs. (9) and (10) subject to boundary conditions (11) and (12) the powers of  $x$  in each term must be equal. Therefore, we obtain for parameters  $\gamma$ ,  $\delta$ , and  $\sigma$  that:

$$\delta = \frac{(2n-1)\gamma-1}{n-2}, \quad \sigma = \frac{(n+4)\gamma-2}{n-2}, \quad \gamma = 1-\delta$$

i. e.,  $\gamma = \sigma = 1/3$ ,  $\delta = 2/3$ . For the similarity functions  $f$  and  $\theta$  we have:

$$3(|f|^{n-1} f)'' + \lambda \eta \theta' - \lambda \theta + 2f f''' = 0 \quad (13)$$

$$\frac{3}{\text{Pr}} \theta'' + 2f \theta' - f' \theta = 0 \quad (14)$$

where the prime denotes differentiation with respect to  $\eta$ . Boundary conditions (11) and (12) are transformed to the following forms:

at  $\eta = 0$ :  $f'(0) = 1$ , if the stretching velocity is  $\bar{U}(x) = U_F \sqrt[3]{x}$

$$f(0) = 0$$

$$\theta'(0) = -b[1-\theta(0)] \text{ for } b = Lh_{f0} / k\sqrt[4]{\text{Gr}} \text{ and } h_f(x) = h_{f0} x^{-1/3}$$

$$\text{for } \eta \rightarrow \infty: \quad \begin{aligned} f'(\eta) &\rightarrow 0 \\ f''(\eta) &\rightarrow 0 \\ \theta(\eta) &\rightarrow 0 \end{aligned}$$

In order to evaluate the velocity and thermal distribution in the boundary-layer next to the surface, the boundary value problem of the connected system of ODE has to be solved instead of the system of four PDE. From the numerical solutions, the effect of the parameters can be examined. Applying the similarity transformation method, we arrived to system (13) and (14) subjected to the previous boundary conditions.

### Results

In paper [22] the numerical calculations were done by finite difference method on a truncated interval  $\eta_{\max} = 10$ . In this paper, numerical results to the boundary value problem of

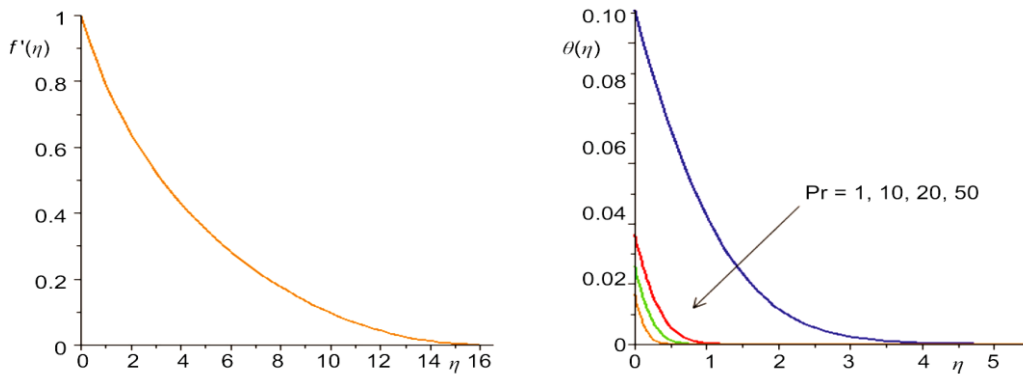


Figure 1. Velocity profile  $f'$  and temperature profiles  $\theta$  for  $n = 1$ ,  $b = 0.1$ ,  $\lambda = 0$ , and  $Pr = 1, 10, 20, 50$

(13), (14) have been generated by MAPLE bvp midpoint submethod. The effects of various physical parameters such as Prandtl number, convective heat transfer parameter,  $b$ , and buoyancy parameter,  $\lambda$ , on the similarity velocity and temperature profiles are exhibited in figs.1-3. The computations are carried on the assumption that the liquid is Newtonian fluid. Figure 1 shows the velocity profiles  $f'$  and temperature profiles  $\theta$  for  $b = 0.1$ ,  $\lambda = 0$ , and  $Pr = 1, 10, 20, 50$ . It can be seen from the figures that increase in the Prandtl number results in decrease of temperature profiles while the velocity profile is the same for each case. The effect of the heat transfer parameter,  $b$ , is shown by fig. 2, in which the temperature profiles increase with increase of  $b$ . It shows a good agreement with those results obtained in [22]. Figure 3 displays the velocity and temperature profiles for  $b = 0.1$ ,  $Pr = 1$ , and  $\lambda = 0, 0.1, 0.5, 1$ . It is observed that increasing  $\lambda$  leads to an increase of the velocity in the boundary-layer and a decrease in the temperature. It can be observed that the wall temperature is smaller for larger Prandtl number or  $\lambda$ . But the higher parameter value  $b$  will provide higher wall temperature.

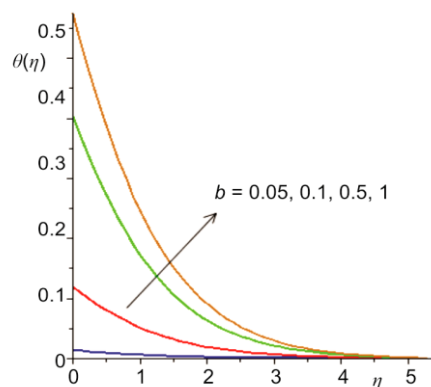


Figure 2. Temperature profiles for  $n = 1$ ,  $\lambda = 0$ ,  $Pr = 1$ , and  $b = 0.05, 0.1, 0.5, 1$

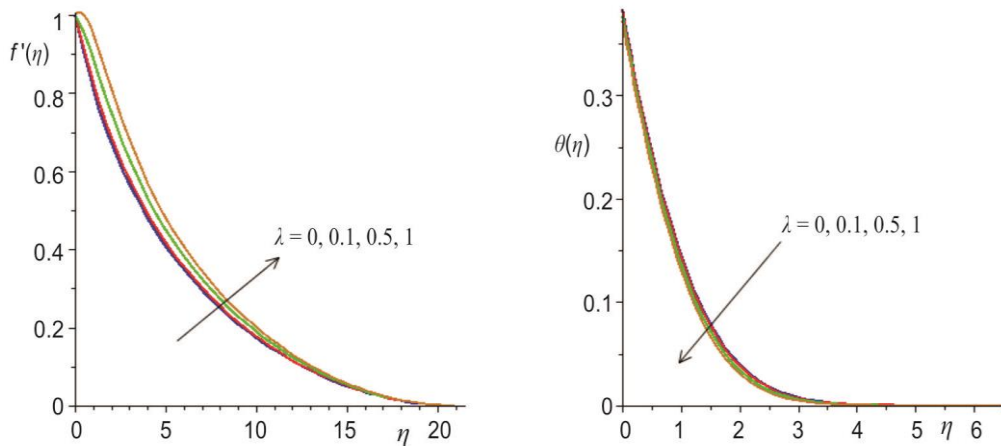


Figure 3. Velocity profile  $f'$  and temperature profiles  $\theta$  for  $n = 1, b = 0.1, Pr = 1,$  and  $\lambda = 0, 0.1, 0.5, 1$

The dependence on the buoyancy parameter  $\lambda$  of the dimensionless friction factor  $f'(0)$  and wall temperature is presented in tab. 1. The absolute value of the friction factor and the wall temperature decrease with an increase in  $\lambda$ .

Influence of the power-law exponent  $n$  on the friction factor  $-|f''(0)|^n$  and on the temperature  $\theta(0)$  is shown in tab. 2. Both the absolute value of the skin friction factor and the wall temperature decrease with an increase in the power-law exponent.

Table 1. Numerical values of  $f'(0)$  and  $\theta(0)$  for  $n = 1, b = 0.1, Pr = 1,$  and  $\lambda = 0, 0.1, 0.5, 1$

	$\lambda = 0$	$\lambda = 0.1$	$\lambda = 0.5$	$\lambda = 1$
$-f'(0)$	0.2257	0.1933	0.0732	0.0618
$\theta(0)$	0.3831	0.3816	0.3766	0.3717

Table 2. Numerical values of  $f'(0)$  and  $\theta(0)$  for  $\lambda = 0, b = 1, Pr = 1,$  and  $n = 1.0, 1.2, 1.4, 1.8$

	$n = 1.0$	$n = 1.2$	$n = 1.4$	$n = 1.8$
$- f''(0) ^n$	0.4257	0.3291	0.2565	0.1629
$\theta(0)$	0.5700	0.5674	0.5656	0.5636

The velocity profiles are shown in fig. 4 for  $n = 1.0, 1.2, 1.4, 1.8$ .

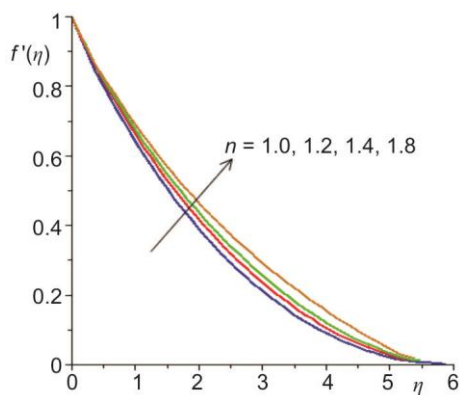


Figure 4. Effects of power-law exponent on the velocity profiles

### Conclusions

Similarity method has been used for the transformation of the system of PDE to a system of ODE when the surface velocity is proportional to  $(x)^{1/3}$  and the heat transfer coefficient is proportional to  $1/(x)^{1/3}$ . The obtained boundary value problem has been solved numerically. Profiles of dimensionless velocity and temperature for different values of the parameters have been presented. The effect of the parameters has been observed:

- An increase in the Prandtl number leads to a decrease in the wall temperature, in the temperature in the thermal boundary-layer and in the thermal boundary-layer thickness.



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