Dual spaces of multi-parameter martingale Hardy spaces

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Abstract

In this paper we introduce the generalized multi-parameter martingale BMO spaces. The atomic decomposition of the multi-parameter martingale Hardy-Lorentz space $H^s_{p,q}$ is given. With the help of this, the dual space of $H^s_{p,q}$ is characterized as the generalized BMO space. Finally, as an application, John-Nirenberg inequality is generalized for multi-parameters.

Key words and phrases: Martingale Hardy-Lorentz spaces, multi-parameter martingales, generalized BMO spaces, atomic decomposition, John-Nirenberg inequality.

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1 Introduction

Martingale Hardy-Lorentz spaces $H^S_{p,q}$ and $H^s_{p,q}$ defined by the quadratic and conditional quadratic variation are considered. The atomic decomposition is a useful characterization of martingale Hardy spaces by the help of which some duality theorems, interpolation results and martingale inequalities can be proved. The atomic decompositions of five different martingale Hardy spaces (amongst others the one of H^s_p) were given in [10]. More recently, using an idea of Abu-Shammala and Torchinsky [1], Jiao, Xie and Zhou [7] have extended the atomic decomposition to the martingale Hardy-Lorentz spaces $H^s_{p,q}$ (see also Jiao, Peng and Liu [6] and Ho [5]).

Multi-parameter martingales were investigated in several papers, see [10] and the references therein and they can be well applied in Fourier analysis (see [11]). The proofs for multi-parameter martingales are usually not simple adaptations of that of the one-parameter proofs. They need new ideas. The atomic decomposition for multi-parameter martingale Hardy space H^s_p is more complicated and is due to the author

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[10]. In this paper we generalize the preceding result of Jiao, Xie and Zhou [7] and characterize the atomic decomposition of the multi-parameter $H^s_{p,q}$.

A very classical result in harmonic analysis is that the dual of H_1 is BMO (Fefferman and Stein [2]). For martingale Hardy spaces see Garsia [3], Long [8] and Weisz [10]. In [10] we proved that the dual of H^s_p is $BMO_2(\alpha)$ if $0 < p \leq 1, \alpha = 1/p - 1$ and the dual of H^s_p is $H^s_{p'}$ if $1 < p < \infty, 1/p + 1/p' = 1$. The same holds also for multi-parameter Hardy spaces (see [10]). Note that the situation is very different from the duals of the L_p spaces if $0 < p < 1$, because the dual of L_p is trivial. More recently, in the one-parameter case, Ho [5] characterized the dual of $H^s_{p,q}$ as the space $BMO_2(\alpha)$ when $0 < p \leq 1, 0 < q \leq p, \alpha = 1/p - 1$. Later Jiao, Xie and Zhou [7] and Ren [9] have generalized this result for all $0 < p \leq 2, 0 < q \leq 1$. They [7] have introduced generalized $BMO_{2,q}(\alpha)$ spaces and have proved that the dual of $H^s_{p,q}$ is $BMO_{2,q}(\alpha)$ if $0 < p \leq 2, 1 < q < \infty, \alpha = 1/p-1$. As a consequence, they [7] obtained a generalization of John-Nirenberg inequality, more exactly, $BMO_2(\alpha)$ is equivalent to $BMO_r(\alpha)$ and $BMO_{2,q}(\alpha)$ is equivalent to $BMO_{r,q}(\alpha)$ $(1 \leq r < \infty)$. In this paper we generalize these results to multi-parameter Hardy-Lorentz spaces and generalized BMO spaces.

2 Martingales and dyadic Hardy spaces

For a set $\mathbb{X} \neq \emptyset$ let $\mathbb{X}^d$ be its Cartesian product $\mathbb{X} \times \ldots \times \mathbb{X}$ taken with itself d-times. Let $d \geq 1$ be a fixed integer and let us introduce the following partial ordering on $\mathbb{N}^d$. For $n = (n_1, \ldots, n_d)$, $m = (m_1, \ldots, m_d) \in \mathbb{N}^d$ set $n \leq m$ if $n_j \leq m_j$ for all $j = 1, \ldots, d$. We say that $n < m$ if $n \leq m$ and $n \neq m$. Moreover, $n \ll m$ means that the inequalities $n_j < m_j$ hold for all $j = 1, \ldots, d$. For $n = (n_1, \ldots, n_d)$ let $n - 1 = (n_1 - 1, \ldots, n_d - 1)$.

For two arbitrary sets $H, G \subset \mathbb{N}^d$ consisting of *incomparable number pairs* (i.e. if $n, m \in H$ (or G) then neither of the inequalities $n \leq m$ and $m \leq n$ hold) we write $H \ll G$ (resp. $H \leq G$) if for all $n \in G$ there exists $m \in H$, such that $m \ll n$ (resp. $m \leq n$). Denote by $\inf H$ the set of the number pairs $m \in H$ for which there does not exist any $n \in H$, $n \neq m$ such that $n \leq m$. We shall use the convention $\inf \emptyset = \infty$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\mathcal{F} = (\mathcal{F}_n, n \in \mathbb{N}^d)$ be a non-decreasing sequence of σ-algebras with respect to the partial ordering on $\mathbb{N}^d$. The expectation operator and the conditional expectation operator relative to $\mathcal{F}_n$ are denoted by E and E_n. We suppose that

$$E_n(fg) = E_n f E_n g \qquad (n \in \mathbb{N}^d)$$

for all bounded $\cup_{n_i \in \mathbb{N}, i \neq k} \mathcal{F}_n$-measurable functions f and all $\cup_{n_i \in \mathbb{N}, i \neq l} \mathcal{F}_n$-measurable functions g $(k \neq l)$.

An integrable sequence $f = (f_n, n \in \mathbb{N}^d)$ is said to be a *martingale* if f_n is $\mathcal{F}_n$ measurable $(n \in \mathbb{N}^d)$ and $E_n f_m = f_n$ for all $n \leq m$. For simplicity, we always suppose that for a martingale f we have $f_n = 0$ if $n_1 \cdots n_d = 0$. The stochastic basis $\mathcal{F}$ is said to be *regular* if there exists a number $R > 0$ such that

$$f_n \leq R f_{n_1 - \epsilon_1, \ldots, n_d - \epsilon_d} \qquad (n \in \mathbb{N}^d)$$

holds for all non-negative martingales $(f_n, n \in \mathbb{N}^d)$ and all numbers $\epsilon_j \in \{0, 1\}$ with $\epsilon_1 + \cdots + \epsilon_d = 1$.

We briefly write L_p instead of the $L_p(\Omega, \mathcal{A}, P)$ space while the norm (or quasinorm) of this space is defined by $\|f\|_p := (E|f|^p)^{1/p}$ $(0 < p \leq \infty)$. For a measurable function f, the *non-increasing rearrangement* is defined by

$$\tilde{f}(t) := \inf\{\rho : P(|f| > \rho) \leq t\}.$$

A measurable function is in the *Lorentz space* $L_{p,q} = L_{p,q}(\Omega, \mathcal{A}, P)$ $(0 < p < \infty, 0 < q \leq \infty)$ if

$$\begin{aligned} \|f\|_{p,q} &:= \left(\frac{q}{p}\int_0^\infty t^{q/p}\tilde{f}(t)^q \frac{dt}{t}\right)^{1/q} \qquad (0 < q \leq \infty), \\ \|f\|_{p,\infty} &:= \sup_{t>0} t^{1/p}\tilde{f}(t) \qquad (q = \infty). \end{aligned}$$

It is known (see Grafakos [4]) that

$$\begin{aligned} \|f\|_{p,q} &= \left(q\int_0^\infty t^q P(|f| > t)^{q/p} \frac{dt}{t}\right)^{1/q} \qquad (0 < q \leq \infty), \\ \|f\|_{p,\infty} &= \sup_{t>0} tP(|f| > t)^{1/p} \qquad (q = \infty). \end{aligned}$$

We recall that $L_{p,p} = L_p$ and $L_{p,q}$ increase as the second exponent q increases, and decrease as the first exponent p increases.

The *quadratic variation* and the *conditional quadratic variation* of a martingale $f = (f_n, n \in \mathbb{N}^d)$ are defined by

$$S(f) := \left(\sum_{n\in\mathbb{N}^d} |d_n f|^2\right)^{1/2}, \qquad s(f) := \left(\sum_{n\in\mathbb{N}^d} E_{n-1}|d_n f|^2\right)^{1/2},$$

respectively, where the martingale differences are given with

$$d_n f = \sum_{\epsilon_i\in\{0,1\}} (-1)^{\epsilon_1+\cdots+\epsilon_d} f_{n_1-\epsilon_1,\ldots,n_d-\epsilon_d}.$$

For $0 < p, q \leq \infty$ the *martingale Hardy-Lorentz spaces* $H^S_{p,q}$ and $H^s_{p,q}$ consist of all d-parameter martingales for which

$$\|f\|_{H^S_{p,q}} := \|S(f)\|_{p,q} < \infty, \qquad \|f\|_{H^s_{p,q}} := \|s(f)\|_{p,q} < \infty,$$

respectively. It is known that $H^S_{p,q} \sim L_{p,q}$ for $1 < p < \infty$ and $0 < q \leq \infty$.

In this paper the constants C and C_p may vary from line to line and the constants C_p are depending only on p.

3 Atomic decomposition of multi-parameter Hardy spaces

Lemma 1 *Let $0 < p < \infty$, $0 < q \leq \infty$, $(2^k\eta_k) \in \ell_q$, $\eta_k \geq 0$ and $f \geq 0$. Suppose that there exists $0 < \delta < 1 < \epsilon < \infty$ such that for all $N \in \mathbb{Z}$, $f \leq g_N + h_N$,*

$$P(g_N > 2^N) \leq C2^{-N\epsilon p} \sum_{k=-\infty}^{N} 2^{k\epsilon p}\eta_k^p \tag{1}$$

and

$$P(h_N > 2^N) \le C 2^{-N\delta p} \sum_{k=N}^{\infty} 2^{k\delta p} \eta_k^p. \tag{2}$$

Then $f \in L_{p,q}$ *and*

$$\|f\|_{p,q} \le C \left\| (2^k \eta_k) \right\|_{\ell_q}.$$

Proof. It is enough to prove that

$$\left\| \left(2^k P(f > 2 \cdot 2^k)^{1/p} \right) \right\|_{\ell_q} \le C \left\| (2^k \eta_k) \right\|_{\ell_q}.$$

It is easy to see that

$$\left\| \left(2^N P(f > 2 \cdot 2^N)^{1/p} \right) \right\|_{\ell_q} \le C \left\| \left(2^N P(g_N > 2^N)^{1/p} \right) \right\|_{\ell_q} + C \left\| \left(2^N P(h_N > 2^N)^{1/p} \right) \right\|_{\ell_q}.$$

The inequality

$$\left\| \left(2^N P(h_N > 2^N)^{1/p} \right) \right\|_{\ell_q} \le C \left\| (2^k \eta_k) \right\|_{\ell_q}$$

was proved in [1] and [7]. If $q = \infty$, then

$$2^N P(g_N > 2^N)^{1/p} \le C_p 2^{N - N\epsilon} \left(\sum_{k=-\infty}^{N} 2^{k(\epsilon-1)p} 2^{kp} \eta_k^p \right)^{1/p} \le C_p \left\| (2^k \eta_k) \right\|_{\ell_\infty},$$

which proves the result. If $0 < q < \infty$ and $p < q$, then apply Hölder's inequality with the exponent $r = q/p$ and its conjugate r' to obtain

$$\begin{aligned} 2^N P(g_N > 2^N)^{1/p} &\le C_p 2^{N - N\epsilon} \left(\sum_{k=-\infty}^{N} 2^{k\gamma p} 2^{k(\epsilon-\gamma)p} \eta_k^p \right)^{1/p} \\ &\le C_p 2^{N(1-\epsilon)} \left(\sum_{k=-\infty}^{N} 2^{k\gamma p r'} \right)^{1/pr'} \left(\sum_{k=-\infty}^{N} 2^{k(\epsilon-\gamma)q} \eta_k^q \right)^{1/q} \\ &\le C_p 2^{N(1-\epsilon+\gamma)} \left(\sum_{k=-\infty}^{N} 2^{k(\epsilon-\gamma)q} \eta_k^q \right)^{1/q}, \end{aligned}$$

where $\gamma > 0$ is arbitrary. If $p \ge q$, then we obtain the same inequality as follows:

$$\begin{aligned} 2^N P(g_N > 2^N)^{1/p} &\le C_p 2^{N(1-\epsilon+\gamma)} \left(\sum_{k=-\infty}^{N} 2^{k(\epsilon-\gamma)p} \eta_k^p \right)^{1/p} \\ &\le C_p 2^{N(1-\epsilon+\gamma)} \left(\sum_{k=-\infty}^{N} 2^{k(\epsilon-\gamma)q} \eta_k^q \right)^{1/q}. \end{aligned}$$

Hence

$$\begin{aligned}
\left\| \left(2^N P(g_N > 2^N)^{1/p}\right) \right\|_{\ell_q}^q &\leq C_p \sum_{N\in\mathbb{Z}} 2^{N(1-\epsilon+\gamma)q} \sum_{k=-\infty}^{N} 2^{k(\epsilon-\gamma)q}\eta_k^q \\
&\leq C_p \sum_{k\in\mathbb{Z}} 2^{k(\epsilon-\gamma)q}\eta_k^q \sum_{N=k}^{\infty} 2^{N(1-\epsilon+\gamma)q} \\
&\leq C_p \sum_{k\in\mathbb{Z}} 2^k \eta_k^q,
\end{aligned}$$

whenever we choose γ such that $0 < \gamma < \epsilon - 1$. The lemma is proved. ∎

The atomic decomposition for $H_{p,q}^s$ is much more complicated in the multi-parameter setting. In this case, instead of the ∞-norm of the atoms, we have to use the 2-norm. The next theorem generalizes the atomic decomposition of H_p^s (see Weisz [10]). We [10] generalized the stopping times for the multi-parameter setting as follows. A function τ which maps Ω into the set of subspaces of $\mathbb{N}^d \cup \{\infty\}$ is said to be a *stopping time* relative to $(\mathcal{F}_n, n \in \mathbb{N}^d)$ if the elements of $\tau(\omega)$ are incomparable for all $\omega \in \Omega$ and

$$\{n \in \tau\} \in \mathcal{F}_n \qquad (n \in \mathbb{N}^d).$$

The set of stopping times will be denoted by $\mathcal{T}$. A function $a \in L_2$ is called a *p-atom* if there exists a stopping time τ such that

(i) $a_n := E_n a = 0$ if $\tau \not\ll n$

(ii) $\|a^*\|_2 \leq P(\tau \neq \infty)^{1/2-1/p}$ $(0 < p < 2)$.

Theorem 1 *A d-parameter martingale f is in $H_{p,q}^s$ $(0 < p < 2, 0 < q \leq \infty)$ if and only if there exists a sequence $(a^k, k \in \mathbb{Z})$ of p-atoms with associated stopping times $(\tau_k, k \in \mathbb{Z})$ such that*

$$\left(\sum_{k\in\mathbb{Z}} |\mu_k|^q\right)^{1/q} < \infty \qquad \text{and} \qquad \sum_{k\in\mathbb{Z}} \mu_k a_n^k = f_n \quad \text{a.e. } (n \in \mathbb{N}^d), \tag{3}$$

where $\mu_k = \sqrt{2} \cdot 2^{k+1} P(\tau_k \neq \infty)^{1/p}$. Moreover,

$$\|f\|_{H_{p,q}^s} \sim \inf \left(\sum_{k\in\mathbb{Z}} |\mu_k|^q\right)^{1/q}$$

where the infimum is taken over all decompositions of f of the form (3).

Proof. Assume that $f \in H_{p,q}^s$. Here we have to use finer stopping times than in the one-parameter case. Let

$$\tau_k := \inf\left\{n \in \mathbb{N}^d : E_n 1_{\{s(f) > 2^k\}} > 1/2\right\}.$$

It is easy to see that

$$f_n = \sum_{k\in\mathbb{Z}} (f_n^{\tau_{k+1}} - f_n^{\tau_k}) = \sum_{k\in\mathbb{Z}} \mu_k a_n^k \quad \text{a.e.} \qquad (n \in \mathbb{N}^d),$$

where

$$a_n^k := \frac{f_n^{\tau_{k+1}} - f_n^{\tau_k}}{\mu_k}.$$

For a fixed k, (a_n^k) is a martingale. In [10] we have shown that a^k is a p-atom. For the sake of completeness, we give a short proof here. If $\tau_k \not\ll n$, then obviously $f_n^{\tau_{k+1}} = f_n^{\tau_k}$, thus (i) holds. Since L_2 is isometric to H_2^s, for (ii) we get that

$$E(f_n^{\tau_{k+1}} - f_n^{\tau_k})^2 \le E\left(\sum_{n\in\mathbb{N}^d} E_{n-1}|d_n f|^2 1_{\{\tau_k \ll n \not\gg \tau_{k+1}\}}\right) = (A) + (B),$$

where

$$(A) = \sum_{n\in\mathbb{N}^d} E\Big(E_{n-1}|d_n f|^2 1_{\{\tau_k \ll n \not\gg \tau_{k+1}\}} 1_{\{s(f)\le 2^{k+1}\}}\Big)$$

and

$$(B) = \sum_{n\in\mathbb{N}^d} E\Big(E_{n-1}|d_n f|^2 1_{\{\tau_k \ll n \not\gg \tau_{k+1}\}} 1_{\{s(f)> 2^{k+1}\}}\Big).$$

Clearly,

$$(A) \le 4^{k+1} P(\tau_k \ne \infty).$$

It follows from the definition of τ_{k+1} that if $\tau_{k+1} \not\ll n$, then

$$E_{n-1} 1_{\{s(f)>2^{k+1}\}} \le 1/2.$$

Hence

$$\begin{aligned}(B) &= \sum_{n\in\mathbb{N}^d} E\Big(E_{n-1}|d_n f|^2 1_{\{\tau_k \ll n \not\gg \tau_{k+1}\}} E_{n-1} 1_{\{s(f)> 2^{k+1}\}}\Big) \\ &\le \frac{1}{2} E\left(\sum_{n\in\mathbb{N}^d} E_{n-1}|d_n f|^2 1_{\{\tau_k \ll n \not\gg \tau_{k+1}\}}\right),\end{aligned}$$

which implies

$$E(f_n^{\tau_{k+1}} - f_n^{\tau_k})^2 \le 2\cdot 4^{k+1} P(\tau_k \ne \infty).$$

Thus

$$E\left((a_n^k)^2\right) \le P(\tau_k \ne \infty)^{1-2/p} \qquad (n \in \mathbb{N}^d).$$

Hence there exists a function $a^k \in L_2$ such that $E_n a^k = a_n^k$ $(n \in \mathbb{N}^d)$ and (ii) holds.

Next we obtain

$$\begin{aligned}P(\tau_k \ne \infty) &= P\left(\sup_{n\in\mathbb{N}^d} E_n 1_{\{s(f)>2^k\}} > 1/2\right) \\ &\le 4E\left(\sup_{n\in\mathbb{N}^d} (E_n 1_{\{s(f)>2^k\}})^2\right) \le CP(s(f) > 2^k) \le CP(s(f) > u),\end{aligned}$$

where $2^{k-1} \le u < 2^k$. If $0 < q < \infty$, then

$$\sum_{k\in\mathbb{Z}} |\mu_k|^q \le C \sum_{k\in\mathbb{Z}} \int_{2^{k-1}}^{2^k} \left(uP(s(f) > u)^{1/p}\right)^q \frac{du}{u} \le C \|f\|_{H^s_{p,q}}^q .$$

If $q = \infty$, then

$$\sup_{k\in\mathbb{Z}} |\mu_k| \le C \sup_{k\in\mathbb{Z}} 2^k P(s(f) > 2^k)^{1/p} \le C \|f\|_{H_{p,\infty}} .$$

Conversely, if the martingale f has the above decomposition, then for an arbitrary integer N let

$$f_n = \sum_{k=-\infty}^{\infty} \mu_k a_n^k = g_n + h_n \qquad (n \in \mathbb{N}^d),$$

where

$$g_n := \sum_{k=-\infty}^{N-1} \mu_k a_n^k \qquad \text{and} \qquad h_n := \sum_{k=N}^{\infty} \mu_k a_n^k.$$

Obviously, $s(f) \le s(g) + s(h)$. Since for a fixed m the sets $\{\nu_k \ll m \not\gg \nu_{k+1}\}$ are disjoint and

$$\mu_k a_n^k = \sum_{m\le n} (d_m f) 1_{\{\nu_k \ll m \not\gg \nu_{k+1}\}},$$

we obtain

$$\begin{aligned} P(s(g) > 2^N) &\le 2^{-2N} \|s(g)\|_2^2 = 2^{-2N} \int_\Omega \left| \sum_{k=-\infty}^{N-1} \mu_k a^k \right|^2 dP \\ &= 2^{-2N} \sum_{k=-\infty}^{N-1} \int_\Omega \left|\mu_k a^k\right|^2 dP = C 2^{-2N} \sum_{k=-\infty}^{N} 2^{2k} P(\tau_k \ne \infty). \end{aligned}$$

Choosing $\epsilon = 2/p > 1$, we obtain (1). Moreover,

$$P(s(h) > 2^N) \le P(s(h) > 0) \le \sum_{k=N}^{\infty} P(s(a^k) > 0) \le \sum_{k=N}^{\infty} P(\tau_k \ne \infty),$$

which proves (2). By Lemma 1 we conclude that

$$\|s(f)\|_{p,q} \le C \left\| \left(2^k P(\tau_k \ne \infty)^{1/p}\right) \right\|_{\ell_q} = C \|(\mu_k)\|_{\ell_q} .$$

The proof of the theorem is complete. ■

If $\mathcal{F}$ is regular, then the previous theorem can be shown for the $H^S_{p,q}$ spaces as well.

Theorem 2 *Suppose that the stochastic basis $\mathcal{F}$ is regular. A d-parameter martingale f is in $H_{p,q}^S$ $(0 < p < 2, 0 < q \le \infty)$ if and only if there exists a sequence $(a^k, k \in \mathbb{Z})$ of p-atoms with associated stopping times $(\tau_k, k \in \mathbb{Z})$ such that*

$$\left(\sum_{k\in\mathbb{Z}} |\mu_k|^q \right)^{1/q} < \infty \qquad \text{and} \qquad \sum_{k\in\mathbb{Z}} \mu_k a_n^k = f_n \quad a.e.\ (n \in \mathbb{N}^d), \tag{4}$$

where $\mu_k = \sqrt{2} \cdot 2^{k+1} P(\tau_k \ne \infty)^{1/p}$. Moreover,

$$\|f\|_{H_{p,q}^S} \sim \inf \left(\sum_{k\in\mathbb{Z}} |\mu_k|^q \right)^{1/q}$$

where the infimum is taken over all decompositions of f of the form (4).

Corollary 1 *If $\mathcal{F}$ is regular, then $H_{p,q}^s \sim H_{p,q}^S$ for all $0 < p < 2, 0 < q \le \infty$.*

Note that this corollary was already proved in [10] with another method.

4 Duality theorems

In [10] we have introduced the $BMO_r(\alpha)$ space (there it was denoted by $\Lambda_r(\alpha)$) and proved that the dual of H_p^s is $BMO_2(\alpha)$ with $0 < p \le 1$, $\alpha = 1/p - 1$. $BMO_r(\alpha)$ $(1 \le r < \infty, \alpha > -1/r)$ denotes the space of those functions $f \in L_r$ for which

$$\|f\|_{BMO_r(\alpha)} = \sup_{\tau\in\mathcal{T}} P(\tau \ne \infty)^{-1/r-\alpha} \|f - f^\tau\|_r < \infty.$$

We generalize these spaces as follows. A functions $f \in L_r$ is in $BMO_{r,q}(\alpha)$ $(1 \le r < \infty, \alpha > -1/r, 0 < q < \infty)$ if

$$\|f\|_{BMO_{r,q}(\alpha)} = \sup_{\tau_k\in\mathcal{T}} \frac{\sum_{k\in\mathbb{Z}} 2^k P(\tau_k \ne \infty)^{1-1/r} \|f - f^{\tau_k}\|_r}{\left(\sum_{k\in\mathbb{Z}} (2^k P(\tau_k \ne \infty)^{1+\alpha})^q\right)^{1/q}} < \infty,$$

where the supremum is taken over all stopping times τ_k, for which $\left(2^k P(\tau_k \ne \infty)^{1+\alpha}\right) \in \ell_q$. If we take only one stopping time in the supremum of the $BMO_{r,q}(\alpha)$-norm, then we get back the $BMO_r(\alpha)$-norm, i.e., $\|f\|_{BMO_r(\alpha)} \le \|f\|_{BMO_{r,q}(\alpha)}$. On the other hand, if $0 < q \le 1$, then

$$\|f\|_{BMO_{r,q}(\alpha)} \le \sup_{\tau_k\in\mathcal{T}} \frac{\sum_{k\in\mathbb{Z}} 2^k P(\tau_k \ne \infty)^{1+\alpha} \|f\|_{BMO_r(\alpha)}}{\left(\sum_{k\in\mathbb{Z}} (2^k P(\tau_k \ne \infty)^{1+\alpha})^q\right)^{1/q}} \le \|f\|_{BMO_r(\alpha)},$$

so in this case $BMO_r(\alpha) \sim BMO_{r,q}(\alpha)$. In case $\alpha = 0$, we denote the spaces by BMO_r and $BMO_{r,q}$.

These spaces were first introduced and investigated in the one-parameter case by Jiao, Xie and Zhou [7] (see also Ho [5]). They proved the one-parameter version of Theorems 3–6. Since the following theorems can be shown similarly as in the one-parameter case (see [7]), we omit the proofs.

Theorem 3 *The dual space of $H^s_{p,q}$ is $BMO_2(\alpha)$, $(0 < p < 2, 0 < q \leq 1, \alpha = 1/p - 1)$.*

Theorem 4 *The dual space of $H^s_{p,q}$ is $BMO_{2,q}(\alpha)$, $(0 < p < 2, 1 < q < \infty, \alpha = 1/p - 1)$.*

Theorem 5 *If $\mathcal{F}$ is regular, then the dual space of $H^s_{p,q}$ is $BMO_r(\alpha)$, $(0 < p < r' \leq 2, 0 < q \leq 1, \alpha = 1/p - 1)$.*

Theorem 6 *If $\mathcal{F}$ is regular, then the dual space of $H^s_{p,q}$ is $BMO_{r,q}(\alpha)$, $(0 < p < r' \leq 2, 1 < q < \infty, \alpha = 1/p - 1)$.*

Corollary 2 *Suppose that $\mathcal{F}$ is regular, $2 \leq r < \infty, 1 < q < \infty$ and $\alpha > -1/r$. Then $BMO_2(\alpha)$ is equivalent to $BMO_r(\alpha)$ and $BMO_{2,q}(\alpha)$ is equivalent to $BMO_{r,q}(\alpha)$.*

Note that the first half of Corollary 2 was proved in [10] for $\alpha = 0$, i.e., $BMO_2 \sim BMO_r$ $(2 \leq r < \infty)$.

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