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# Solving the Least Squares Method problem in the AHP for $3 \times 3$ and $4 \times 4$ matrices*

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**Abstract** The Analytic Hierarchy Process (AHP) is one of the most popular methods used in Multi-Attribute Decision Making. The Eigenvector Method ($EM$) and some distance minimizing methods such as the Least Squares Method ($LSM$) are of the possible tools for computing the priorities of the alternatives. A method for generating all the solutions of the $LSM$ problem for $3 \times 3$ and $4 \times 4$ matrices is discussed in the paper. Our algorithms are based on the theory of resultants.

Keywords: decision theory, pairwise comparison matrix, least squares method, polynomial system.

## 1 Introduction

The Analytic Hierarchy Process was developed by Thomas L. Saaty [26]. It is a procedure for representing the elements of any problem, hierarchically. It breaks a problem into smaller parts and then guides decision makers through a series of pairwise comparison judgments to express the relative strength or intensity of the impact of the elements in the hierarchy. These judgments are converted into numbers.

We will study only one part of the decision problem, i.e. when one matrix is obtained from pairwise comparisons. Suppose that we have an $n \times n$ positive reciprocal matrix in the form

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$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & 1 & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & 1 \end{pmatrix},$$

where for any $i, j = 1, \dots, n$,

$$a_{ij} > 0,$$
$$a_{ij} = \frac{1}{a_{ji}}.$$

We want to find a weight vector $w = (w_1, w_2, \dots, w_n)^T \in \mathbb{R}^n_+$ representing the priorities where $\mathbb{R}^n_+$ is the positive orthant. The Eigenvector Method [26] and some distance minimizing methods such as the Least Squares Method [6,17], Logarithmic Least Squares Method [10,13,9,8,1,14], Weighted Least Squares Method [6,2], Chi Squares Method [17] and Logarithmic Least Absolute Values Method [7,16], Singular Value Decomposition [24,25] are of the tools for computing the priorities of the alternatives.

After some comparative analyses [4,27,8,31,29] Golany and Kress [15] have compared most of the scaling methods above by seven criteria and concluded that every method has advantages and weaknesses, none of them is prime.
Since $LSM$ problem has not been solved fully, comparisons to other methods are restricted to a few specific examples. The aim of the paper is to present a method for solving $LSM$ for $3 \times 3$ and $4 \times 4$ matrices in order to ground for further research of comparisons to other methods and examining its real life application possibilities.

Before studying $LSM$ we show a few examples to interpret the variety of decision problems based on pairwise comparisons. Let $A$ be a $3 \times 3$ matrix from pairwise comparisons:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 5 \\ 1/3 & 1/5 & 1 \end{pmatrix}.$$

Saaty's original Eigenvector Method gives the result

$$w^{EM} = \begin{pmatrix} 0.508 \\ 0.379 \\ 0.113 \end{pmatrix},$$

with 0.155 inconsistency ratio as Saaty [26] defined. Since the first alternative is (2 and 3 times) better than the others, it seems to be correct that it is the winner. One may ask about the second alternative: Is not the value

5 enough to compensate $\frac{1}{2}$? It depends on a decision principle which alternative should be desired for the first place. If we look for a relatively high result and we are clement with small weak results, we will choose the second alternative. Which scaling method handles this problem?

Second matrix is very similar to Jensen's [17] but here we have fours instead of nines. Let $A$ be a $3 \times 3$ matrix as follows:

$$A = \begin{pmatrix} 1 & 4 & 1/4 \\ 1/4 & 1 & 4 \\ 4 & 1/4 & 1 \end{pmatrix}.$$

$EM$-solution is $w^{EM} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, while $LSM$ generates triple solutions with a symmetry of the weights:

$$\begin{aligned} w^{LSM_1} &= (0.215, 0.317, 0.468), \\ w^{LSM_2} &= (0.468, 0.215, 0.317), \\ w^{LSM_3} &= (0.317, 0.468, 0.215). \end{aligned}$$

Note that inconsistency ratio is high (2.14) which is unexpected in practice, this phenomenon rather has a theoretical content. We have observed that $EM$-solution is closer and closer to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ as the inconsistency increases. $LSM$-solution is often not unique in case of higher inconsistency.

Third question is about measure of inconsistency. Given an $n \times n$ pairwise comparison matrix, $\lambda_{max}$ denotes the maximal eigenvalue. $\bar{\lambda}_{max}$ is the expected value of $\lambda_{max}$ computed from matrices with elements taken at random from the scale $\frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \ldots, \frac{1}{2}, 1, 2, \ldots, 9$. Consistency Ratio is, by definition,

$$CR = \frac{CI}{MRCI_n},$$

where

$$CI = \frac{\lambda_{max} - n}{n - 1},$$

$$MRCI_n = \frac{\bar{\lambda}_{max} - n}{n - 1}.$$

Saaty suggested that a consistency ratio of about 10% or less should be usually considered acceptable. This 10% limit is often holds for small matrices. Computations by first author show that the number of random matrices of consistency ratio less than 10% decreases dramatically as $n$ increases. $10^7$ random matrices have been generated for every $n = 3, 4, \ldots, 10$.

| n | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| Number of matrices under 10% | $2.08 \cdot 10^6$ | $3.16 \cdot 10^5$ | $2.41 \cdot 10^4$ | 787 | 14 | 0 | 0 | 0 |

Similar results are given by Standard [28]. She has examined the consistency of pairwise comparison matrices from real life, too, and concluded that it is rather hard to stay under 10%.

Each of distance minimizing methods has an objective function. In the consistent case each of them is zero. It may be a task of further research to choose functions which can be used for measuring the inconsistency. More numerical examples are shown in last section.

In the paper we study the Least Squares Method ($LSM$) which is a minimization problem of the Frobenius norm of $(A - w\frac{1}{w}^T)$, where $\frac{1}{w}^T$ denotes the row vector $(\frac{1}{w_1}, \frac{1}{w_2}, \ldots, \frac{1}{w_n})$.

### *1.1 Least Squares Method (LSM)*

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left( a_{ij} - \frac{w_i}{w_j} \right)^2$$

$$\sum_{i=1}^{n} w_i = 1,$$

$$w_i > 0, \qquad i = 1, 2, \ldots, n.$$

$LSM$ is rather difficult to solve because the objective function is nonlinear and usually nonconvex, moreover, no unique solution exists [17,18] and the solutions are not easily computable. Farkas [12] applied Newton's method of successive approximation. His method requires a good initial point to find the solution.

## 2 Solving the *LSM* problem for 3×3 matrices

Bozóki [3] developed an algorithm for generating all the $LSM$ solutions of any $3 \times 3$ matrix. We summarize the method in short. Suppose that $A$ is a $3 \times 3$ matrix obtained from pairwise comparisons in the form

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} \\ 1/a_{12} & 1 & a_{23} \\ 1/a_{13} & 1/a_{23} & 1 \end{pmatrix}.$$

The aim is to find a positive reciprocal consistent matrix X in the form

$$X = \begin{pmatrix} 1 & w_1/w_2 & w_1/w_3 \\ w_2/w_1 & 1 & w_2/w_3 \\ w_3/w_1 & w_3/w_2 & 1 \end{pmatrix},$$

which minimizes the Frobenius norm

$$\| A - X \|_F^2 = \left(a_{12} - \frac{w_1}{w_2}\right)^2 + \left(a_{13} - \frac{w_1}{w_3}\right)^2 + \left(\frac{1}{a_{12}} - \frac{w_2}{w_1}\right)^2 + \left(a_{23} - \frac{w_2}{w_3}\right)^2 + \left(\frac{1}{a_{13}} - \frac{w_3}{w_1}\right)^2 + \left(\frac{1}{a_{23}} - \frac{w_3}{w_2}\right)^2,$$

where

$$w_1 + w_2 + w_3 = 1, \tag{1}$$

$$w_1, w_2, w_3 > 0. \tag{2}$$

Introducing new variables $x, y$

$$x = \frac{w_1}{w_2}, \qquad y = \frac{w_2}{w_3}, \tag{3}$$

the optimization problem is reduced to

$$\min \quad f(x, y)$$
$$x, y > 0,$$

where

$$f(x, y) = \| A - X \|_F^2 = (a_{12} - x)^2 + (a_{13} - xy)^2 + \left(\frac{1}{a_{12}} - \frac{1}{x}\right)^2 + (a_{23} - y)^2 + \left(\frac{1}{a_{13}} - \frac{1}{xy}\right)^2 + \left(\frac{1}{a_{23}} - \frac{1}{y}\right)^2.$$

A necessary condition of optimality is that $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$. The partial derivatives of $f$ are rational functions of $x, y$ and can be directly transformed to polynomials $p(x, y)$ and $q(x, y)$ by multiplication by common denominators. We seek for $(x, y) \in \mathbb{R}^2_+$ for which both $p(x, y)$ and $q(x, y)$ become zero.

$$p(x, y) = x^4y^4 + x^4y^2 - a_{13}x^3y^3 - a_{12}x^3y^2 + \frac{xy^2}{a_{12}} + \frac{xy}{a_{13}} - y^2 - 1 = 0, \tag{4}$$

$$q(x, y) = x^4y^4 + x^2y^4 - a_{13}x^3y^3 - a_{23}x^2y^3 + \frac{x^2y}{a_{23}} + \frac{xy}{a_{13}} - x^2 - 1 = 0. \tag{5}$$

Resultant method [20] is a possible way to solve systems like (4)-(5). The number of variables can be reduced to 1 from 2 by taking only $x$ as a variable and considering $y$ as a parameter. Computing the Sylvester-determinant from the coefficients of polynomials $p$ and $q$, we get a polynomial $P$ in $y$ of degree 28. Using a polynomial-solver algorithm (e.g. in Maple) to find all the positive real roots of $P$, we have the solutions $y_1, y_2, \ldots, y_t$, where $1 \leq t \leq 28$. Substituting these solutions $y_i$, $i = 1, \ldots, t$, back in $p(x, y)$ and $q(x, y)$, we get polynomials in $x$ of degree 4. Solving these polynomials in $x$, we have to check whether $p(x, y)$ and $q(x, y)$ have common positive real roots. If $(x, y)$ is a common root of $p(x, y)$ and $q(x, y)$, we need to check the Hessian matrix of $f$ to be sure that it is a local minimum point. If the Hessian matrix is positive definite at $(x, y)$, we have a strict local minimum point. Then, from (1)-(3) the $LSM$-optimal weight vector is given by

$$w_1 = \frac{xy}{xy+y+1}, \qquad w_2 = \frac{y}{xy+y+1}, \qquad w_3 = \frac{1}{xy+y+1}.$$

We note again that $LSM$ solution is not unique in general.

## 3 The case of 4 × 4 matrices

We have a matrix from pairwise comparisons in the form

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 1/a_{12} & 1 & a_{23} & a_{24} \\ 1/a_{13} & 1/a_{23} & 1 & a_{34} \\ 1/a_{14} & 1/a_{24} & 1/a_{34} & 1 \end{pmatrix}.$$

We seek for a positive reciprocal consistent matrix X in the form

$$X = \begin{pmatrix} 1 & w_1/w_2 & w_1/w_3 & w_1/w_4 \\ w_2/w_1 & 1 & w_2/w_3 & w_2/w_4 \\ w_3/w_1 & w_3/w_2 & 1 & w_3/w_4 \\ w_4/w_1 & w_4/w_2 & w_4/w_3 & 1 \end{pmatrix},$$

which minimizes the Frobenius norm $\| A - X \|_F^2$.

$$\begin{aligned} \| A - X \|_F^2 = & \left(a_{12} - \frac{w_1}{w_2}\right)^2 + \left(a_{13} - \frac{w_1}{w_3}\right)^2 + \left(a_{14} - \frac{w_1}{w_4}\right)^2 \\ & + \left(\frac{1}{a_{12}} - \frac{w_2}{w_1}\right)^2 + \left(a_{23} - \frac{w_2}{w_3}\right)^2 + \left(a_{24} - \frac{w_2}{w_4}\right)^2 \\ & + \left(\frac{1}{a_{13}} - \frac{w_3}{w_1}\right)^2 + \left(\frac{1}{a_{23}} - \frac{w_3}{w_2}\right)^2 + \left(a_{34} - \frac{w_3}{w_4}\right)^2 \\ & + \left(\frac{1}{a_{14}} - \frac{w_4}{w_1}\right)^2 + \left(\frac{1}{a_{24}} - \frac{w_4}{w_2}\right)^2 + \left(\frac{1}{a_{34}} - \frac{w_4}{w_3}\right)^2, \end{aligned}$$

where

$$w_1 + w_2 + w_3 + w_4 = 1, \tag{6}$$

$$w_1, w_2, w_3, w_4 > 0. \tag{7}$$

With new variables $x, y, z$,

$$x = \frac{w_1}{w_2}, \qquad y = \frac{w_1}{w_3}, \qquad z = \frac{w_1}{w_4}, \tag{8}$$

we get the matrix

$$X = \begin{pmatrix} 1 & x & y & z \\ 1/x & 1 & y/x & z/x \\ 1/y & x/y & 1 & z/y \\ 1/z & x/z & y/z & 1 \end{pmatrix},$$

where $x, y, z > 0$. This matrix is composed of three variables instead of four. If $f : \mathbb{R}^3_+ \to \mathbb{R}$ is given by

$$\begin{aligned} \| A - X \|_F^2 = (a_{12} - x)^2 + (a_{13} - y)^2 + (a_{14} - z)^2 + \left(\frac{1}{a_{12}} - \frac{1}{x}\right)^2 \\ + \left(a_{23} - \frac{y}{x}\right)^2 + \left(a_{24} - \frac{z}{x}\right)^2 + \left(\frac{1}{a_{13}} - \frac{1}{y}\right)^2 + \left(\frac{1}{a_{23}} - \frac{x}{y}\right)^2 \\ + \left(a_{34} - \frac{z}{y}\right)^2 + \left(\frac{1}{a_{14}} - \frac{1}{z}\right)^2 + \left(\frac{1}{a_{24}} - \frac{x}{z}\right)^2 + \left(\frac{1}{a_{34}} - \frac{y}{z}\right)^2, \end{aligned}$$

then the optimization problem is as follows:

$$\begin{aligned} \min \quad & f(x, y, z) \\ & x, y, z > 0. \end{aligned} \tag{9}$$

We need to know the $x, y, z$ values for which the first partial derivatives of $f$ become zero, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$. After computing partial derivatives of $f$, dividing by 2, and multiplying $\frac{\partial f}{\partial x}$ by $x^3y^2z^2$, $\frac{\partial f}{\partial y}$ by $x^2y^3z^2$, and $\frac{\partial f}{\partial z}$ by $x^2y^2z^3$, we get the $p, q, r$ polynomials in variables $x, y, z$ :

$$\begin{aligned} p(x, y, z) = & -a_{12}x^3y^2z^2 + x^4y^2z^2 + \frac{xy^2z^2}{a_{12}} - y^2z^2 + a_{23}xy^3z^2 \\ & - y^4z^2 + a_{24}xy^2z^3 - y^2z^4 - \frac{x^3yz^2}{a_{23}} + x^4z^2 - \frac{x^3y^2z}{a_{24}} + x^4y^2, \\ q(x, y, z) = & -a_{13}x^2y^3z^2 + x^2y^4z^2 - a_{23}xy^3z^2 + +y^4z^2 + \frac{x^2yz^2}{a_{13}} \\ & - x^2z^2 + \frac{x^3yz^2}{a_{23}} - x^4z^2 + a_{34}x^2yz^3 - x^2z^4 - \frac{x^2y^3z}{a_{34}} + x^2y^4, \\ r(x, y, z) = & -a_{14}x^2y^2z^3 + x^2y^2z^4 - a_{24}xy^2z^3 + y^2z^4 - a_{34}x^2yz^3 \\ & + x^2z^4 + \frac{x^2y^2z}{a_{14}} - x^2y^2 + \frac{x^3y^2z}{a_{24}} - x^4y^2 + \frac{x^2y^3z}{a_{34}} - x^2y^4. \end{aligned}$$

We seek for $(x, y, z) \in \mathbb{R}^3_+$ solution(s) of the system

$$\begin{aligned} p(x, y, z) &= 0, \\ q(x, y, z) &= 0, \\ r(x, y, z) &= 0, \\ x, y, z &> 0. \end{aligned} \tag{10}$$

A method for solving polynomial systems is described in the next section. The algorithm below finds all the common roots of multivariate polynomials.

## 4 Generalized resultants

Here, we present a more general solving method for polynomial systems. Given a system of three equations with three unknowns such as (10); we want common solutions. First, we introduce a general theory of resultants.

### *4.1 Bezout-Dixon-Kapur-Saxena-Yang Method*

Consider a system of $n + 1$ polynomial equations in $n$ variables $x, y, z, \ldots$ and $m$ parameters $a, b, c, \ldots$.

$$f_1(x, y, z, \ldots, a, b, c, \ldots) = 0$$
$$f_2(x, y, z, \ldots, a, b, c, \ldots) = 0$$
$$\ldots\ldots$$

We want to eliminate the variables and derive a *resultant* polynomial in the parameters; the system has a common solution only when the resultant is 0.

Let us consider the one-variable, two polynomials case. Bezout [30], and later Cayley, presented the following method: given $f(x), g(x) \in \mathcal{R}[x]$, where $\mathcal{R}$ is an integral domain. Let $t$ be a new variable and consider

$$\delta(x, t) = \frac{1}{x - t} \begin{vmatrix} f(x) & g(x) \\ f(t) & g(t) \end{vmatrix}.$$

This polynomial is symmetric in two variables $x, t$. Note that if $x_0$ is a common zero of $f$ and $g$, then $\delta(x_0, t)$ vanishes identically in $t$. Let $d = max\{degree(f), degree(g)\} - 1$. Then, the degree of $\delta(x, t)$ in $x$ and $t$ is at most $d$, and is equal to $d$ unless $f$ and $g$ are linearly dependent. Write $\delta(x, t)$ as a polynomial in $t$ with coefficients in $\mathcal{R}[x]$:

$$\delta(t, x) = (Ax^d + \cdots + F)t^d + (Bx^d + \cdots + G)t^{d-1} + \cdots + (Sx^d + \cdots + W)t^0,$$

where $A, B$, etc. are elements of $\mathcal{R}$. For a common root $x_0$, $\delta(x_0, t)$ becomes zero for all $t$. Therefore, every coefficient polynomial above in $x$ vanishes. This produces a sequence of equations that we can write as a matrix product:

$$M \equiv \begin{bmatrix} A \cdots\cdots F \\ B \cdots\cdots G \\ \cdots\cdots \\ S \cdots\cdots W \end{bmatrix} \begin{bmatrix} x^d \\ \cdots \\ x \\ 1 \end{bmatrix} = 0,$$

where $M$ denotes the square matrix on the left. We can interpret this as a system of linear equations, by replacing the column vector with one of indeterminates $\{v_d, v_{d-1}, \dots, v_0\}$:

$$M \equiv \begin{bmatrix} A \cdots\cdots F \\ B \cdots\cdots G \\ \cdots\cdots \\ S \cdots\cdots W \end{bmatrix} \begin{bmatrix} v_d \\ \cdots \\ v_1 \\ v_0 \end{bmatrix} = 0.$$

Since we have $v_0 = 1$, the linear system has a non-trivial solution, $\{v_k = {x_0}^k\}$. Therefore, the determinant of $M$ must be 0. We have proven:

**Theorem 1** *The Bezoutian or Dixon Resultant, denoted by $DR$, of $f$ and $g$ is the determinant of $M$. If there exists a common zero of $f$ and $g$, then $DR = 0$.*

**Example.** *Suppose*

$$f(x) = (x + a - 1)(a + 3)(x - a),$$
$$g(x) = (x + 3a)(x + a).$$

*We have $DR = 8a^2(2a + 1)(a + 3)^2$. Setting $DR = 0$ gives a necessary condition, and yields $a = -\frac{1}{2}, -3, -3, 0, 0$. The solutions are*

$$(a = -3, x = 9), \quad (a = -3, x = 3), \quad (a = 0, x = 0), \quad (a = -\frac{1}{2}, x = \frac{3}{2}).$$

The values of $a$ may lie in an extension field of $\mathcal{R}$.

Dixon [11] generalized the above idea to $n + 1$ equations in $n$ variables. To illustrate this, suppose we have three equations in two variables:

$$f(x, y) = 0, \quad g(x, y) = 0, \quad h(x, y) = 0.$$

Add two new variables $s, t$ and define

$$\delta(x,y,s,t) = \frac{1}{(x-s)(y-t)} \begin{vmatrix} f(x,y) & g(x,y) & h(x,y) \\ f(s,y) & g(s,y) & h(s,y) \\ f(s,t) & g(s,t) & h(s,t) \end{vmatrix}.$$

As before, $\delta$ is a polynomial, but it is not symmetrical in $x$ and $s$ nor in $y$ and $t$. Generalizing the one-variable case, we write $\delta$ in terms of monomials in $s$ and $t$ with coefficients in $\mathcal{R}[x,y]$.

$$\delta = (Ax^{d_1}y^{d_2} + \cdots + F)s^{e_1}t^{e_2} + \cdots + (Bx^{d_1}y^{d_2} + \cdots + G)s^i t^j + \cdots$$

It is not easy to predict what $d_1, d_2, e_1$, and $e_2$ will exactly be. $d_1$ is the largest power of $x$ that occurs in $\delta$, $e_1$ the largest power of $s$, etc. We again get a matrix equation

$$\begin{bmatrix} A\cdots & \cdots F \\ \cdots & \cdots \\ B\cdots & \cdots G \\ \cdots & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \end{bmatrix} \begin{bmatrix} x^{d_1}y^{d_2} \\ \cdots \\ y \\ x^{d_1} \\ \cdots \\ x \\ 1 \end{bmatrix} = 0.$$

However, the coefficient matrix $M$ may not be square. When it is square, we may again define the Dixon Resultant $DR$ as the determinant of $M$, and so at any common zero, $DR = 0$. The procedure generalizes to $n+1$ equations $\{f_1, f_2, \ldots, f_{n+1}\}$ in $n$ variables (and any number of parameters).

Dixon proved that for *generic polynomials* $DR = 0$ is necessary and sufficient for the existence of a common root. *Generic* means that each polynomial $\{f_1, f_2, \ldots, f_{n+1}\}$ has every possible coefficient and all the coefficients are independent parameters, so that each equation may be written

$$f_j = \sum_{i_1=0}^{k_{j1}} \cdots \sum_{i_n=0}^{k_{jn}} a_{ji_1\cdots i_n} x_1^{i_1} \cdots x_n^{i_n},$$

where the $a_{ji_1\cdots i_n}$ are distinct parameters, and $k_{jm}$ is the degree of $f_j$ in the $m^{th}$ variable ($j = 1, \ldots, n+1$, $m = 1, \ldots, n$).

However, problems arising in applications do not have so many parameters. Therefore, Dixon's sufficient criterion is of little value in practice.

Moreover, often the large matrix $M$ has rows or columns entirely zero, consequently the determinant vanishes identically – when the determinant can be defined at all. Thus, the Dixon method for multivariate problems seemed to be of little value, until the 1994 paper of Kapur, Saxena, and Yang [19]:

**Theorem 2** *(Kapur-Saxena-Yang) Let $DR$ be the determinant of any maximal rank submatrix of $M$. Then, if a certain condition holds, $DR = 0$ is necessary for the existence of a common zero.*

The condition they used is rather technical and often does not hold in applications [22]. Nonetheless, even in cases when the equation $DR = 0$, arising from any maximal rank submatrix of $M$, was found to be correct, in the sense that correct solution values of the parameters are among the roots of $DR = 0$. This was explained in [5].

An important variation (used in the paper) occurs when we have $n$ variables but only $n$ equations. Then, one of the variables, say $x_1$, is treated as a parameter, and the resultant provides an equation for $x_1$ in terms of the parameters.

## 5 Implementation in Fermat

The computer algebra system *Fermat* [21] is very good at polynomial and matrix problems [23], [22]. Co-author Lewis used it for implementing the Kapur-Saxena-Yang method. Starting with polynomials $\{f_1, f_2, \ldots, f_{n+1}\}$ in variables $x_1, x_2, \ldots, x_n$ and parameters $a_1, \ldots, a_m$ over the ring $\mathbb{Z}$ of integers, the determinant polynomial $\delta(x_1, x_2, \ldots, a_1, \ldots)$ is computed as above. Matrix $M$ is then created, as indicated above. The entries of this matrix are polynomials in the parameters $a_1, \ldots, a_m$. We find a maximal rank submatrix by replacing some or all of the parameters with prime integers, running a standard column normalizing algorithm, and keeping track of which rows and columns of $M$ are being used. This is easy to do in *Fermat* with the builtin command *Pseudet*. We then extract these rows and columns from $M$ to form $M_2$. The equation $DR = \text{determinant}(M_2) = 0$ contains all the desired solutions.

In practice, however, two problems arise. The polynomial $DR$ may contain millions of terms and be too large to compute, or even store, in the RAM of a desktop computer system. Secondly, $DR$ is usually larger than necessary; it contains *spurious factors*. For theoretical reasons [5], we expect the true resultant to be an irreducible factor of $DR$. In practical problems it is often a very small factor of $DR$; indeed $DR$ may have millions of terms but the resultant only hundreds. Several techniques can be used to overcome these problems (see also [22]):

1. Compute several maximal rank determinants and take their greatest common divisor.
2. Work modulo $\mathbb{Z}_p$ for primes $p$. Sometimes this is good enough.

3. Plug in constants for some or all of the parameters.
4. Rather than compute a large $DR$ and face the daunting task of factoring it, Lewis has developed a technique that is often useful. Column normalize the matrix $M_2$, but at each step remove any common factors in the entries of each row and column, and pull out any denominators that arise. Keep track of all of these polynomials, canceling common factors as they arise. In the end, $M_2$ contains only 0 and units, and we have a list of polynomials the product of which is $DR$. Since $DR$ tends to have many factors, the list is nontrivial. We have observed that the last item in the list is usually the desired irreducible resultant.

### *5.1 Applying Dixon Resultants for solving the LSM problem*

*Fermat* provides a language in which one can write programs to invoke the *Fermat* primitives. The collection of *Fermat* programs that implements the strategies described above is available from the second author by E-mail. Using them on the polynomial system (10) calculated from a $4 \times 4$ matrix, we first substituted constants for $a_{12}, a_{13}, a_{14}, a_{23}$, and $a_{24}$, leaving $a_{34}$ as symbolic. The method in Sections 4 and 5 computes the answer in 45 minutes. When a constant is plugged in for $a_{34}$ as well, it finishes computing in 49 seconds. In either case, the spurious factor is much smaller than the resultant. The algorithm results in a polynomial of one variable (e.g. $x$). The degree is between 26 and 137 depending on the $4\times 4$ matrix, so we could find its positive real roots with Maple. The next step is to find the corresponding $y$ and $z$ solutions, which can be solved by using the algorithm for 2 variables. It works like in the case of $3\times 3$ matrices. Suppose that $(x, y, z)$ is a solution of system (10). If the Hessian matrix of $f$ is positive definite at $(x, y, z)$, then we have a strict local minimum point. Thus $(x, y, z)$ is a solution of (9) and the $LSM$-optimal weight vector can be computed from (6)-(8):

$$w_1 = \frac{xyz}{xyz + xy + xz + yz}, \qquad w_2 = \frac{yz}{xyz + xy + xz + yz},$$

$$w_3 = \frac{xz}{xyz + xy + xz + yz}, \qquad w_4 = \frac{xy}{xyz + xy + xz + yz}.$$

## 6 Numerical results

Here we present two examples of Eigenvector and Least Squares approximation. We calculated the weight vectors in two ways:
$w^{EM}$ denotes the solution by Eigenvector Method suggested by Saaty [26],
$w^{LSM}$ denotes the approximation vector by Least Squares Method.

*6.1 A 3 × 3 matrix*

We tested all the $3 \times 3$ matrices with elements $\frac{1}{9}, \frac{1}{8}, \ldots, \frac{1}{2}, 1, 2, \ldots, 9$. Thus we have $17^3 = 4913$ matrices and found that $LSM$-solution is always unique while Saaty's inconsistency ratio is less than 0.292 (29.2%). The $3\times 3$ matrix having non-unique $LSM$-solution with the smallest $EM$-inconsistency is as follows:

$$A = \begin{pmatrix} 1 & 6 & 7 \\ 1/6 & 1 & 6 \\ 1/7 & 1/6 & 1 \end{pmatrix}.$$

Two $LSM$-solutions exist in this case. We present the $LSM$-solutions, the approximating matrices by definition $[\frac{w_i}{w_j}]$ $(i, j = 1, 2, 3)$. $A^{LSM_1}$ is computed from $w^{LSM_1}$ and $A^{LSM_2}$ is from $w^{LSM_2}$. The errors of the approximation are calculated as the Frobenius-norm of $A - A^{LSM_1}$ and $A - A^{LSM_2}$.

$$w^{LSM_1} = \begin{pmatrix} 0.722 \\ 0.188 \\ 0.090 \end{pmatrix}, \qquad A^{LSM_1} = \begin{pmatrix} 1 & 3.833 & 8.039 \\ 0.261 & 1 & 2.098 \\ 0.124 & 0.477 & 1 \end{pmatrix},$$

$$\| A - A^{LSM_1} \|_F^2 = 21.11,$$

while the second solution gives

$$w^{LSM_2} = \begin{pmatrix} 0.624 \\ 0.298 \\ 0.078 \end{pmatrix}, \qquad A^{LSM_2} = \begin{pmatrix} 1 & 2.098 & 8.037 \\ 0.477 & 1 & 3.831 \\ 0.124 & 0.261 & 1 \end{pmatrix},$$

$$\| A - A^{LSM_2} \|_F^2 = 21.11.$$

Saaty's original Eigenvector Method gives the result

$$w^{EM} = \begin{pmatrix} 0.730 \\ 0.210 \\ 0.060 \end{pmatrix}, \qquad A^{EM} = \begin{pmatrix} 1 & 3.480 & 12.09 \\ 0.287 & 1 & 3.475 \\ 0.083 & 0.289 & 1 \end{pmatrix},$$

where $A^{EM}$ is computed from $w^{EM}$. Inconsistency ratio as Saaty [26] defined is 0.293 in this case.

Both $LSM$-ranks are the same as $EM$'s, $w^{LSM_1}$ is quite close to $w^{EM}$, $w^{LSM_2}$ is a little bit different. However, the $LSM$-approximation errors are equal. Considering the approximating matrices, the most spectacular difference is that a 7 is approximated by 12.09 ($EM$) and 8.037 ($LSM$).

*6.2 A 4 × 4 matrix*

Let $B$ be a $4 \times 4$ pairwise comparison matrix:

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1/2 & 1 & 7 & 2 \\ 1/3 & 1/7 & 1 & 1 \\ 1/4 & 1/2 & 1 & 1 \end{pmatrix}.$$

Now, $LSM$-solution is unique:

$$w^{LSM} = \begin{pmatrix} 0.339 \\ 0.452 \\ 0.078 \\ 0.131 \end{pmatrix}, \qquad B^{LSM} = \begin{pmatrix} 1 & 0.750 & 4.326 & 2.588 \\ 1.333 & 1 & 5.766 & 3.450 \\ 0.231 & 0.173 & 1 & 0.598 \\ 0.386 & 0.290 & 1.672 & 1 \end{pmatrix},$$

$$\| B - B^{LSM} \|_F^2 = 7.17.$$

$EM$-solution is

$$w^{EM} = \begin{pmatrix} 0.443 \\ 0.345 \\ 0.096 \\ 0.116 \end{pmatrix}, \qquad B^{EM} = \begin{pmatrix} 1 & 3.121 & 6.303 & 0.448 \\ 0.320 & 1 & 2.020 & 0.144 \\ 0.159 & 0.495 & 1 & 0.071 \\ 2.230 & 6.959 & 14.06 & 1 \end{pmatrix},$$

$$\text{Inconsistency ratio} = 0.1.$$

The $EM$-winner is the first alternative, while the $LSM$-winner is the second one. Although the first alternative is better than the others in pairwise comparisons as the elements of the first row show but the second alternative has a topping result 7 compared to the third alternative.

Matrix $B$ is better approximated by $B^{EM}$ at some relatively small values but $LSM$ is much better at the biggest element (7). This is the same situation as earlier: $LSM$ concentrates on big values.

## 7 Conclusion

In the paper we showed a method for solving the Least Squares Problem for $3 \times 3$ matrices and a more difficult method for solving $LSM$ for $4 \times 4$ matrices. The algorithms find all the solutions of the least squares optimization problem. One may be interested in case of larger matrices. In these cases

Dixon Resultant can be used but the size of matrices increases very quickly. At the moment, we can give results in a few seconds in the case of $3 \times 3$ and $4 \times 4$ matrices.

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