Multivariable (φ, Γ)-modules and products of Galois groups

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14th March 2016

Abstract

We show that the category of continuous representations of the dth direct power of the absolute Galois group of $\mathbb{Q}_p$ on finite dimensional $\mathbb{F}_p$-vector spaces (resp. finitely generated $\mathbb{Z}_p$-modules, resp. finite dimensional $\mathbb{Q}_p$-vector spaces) is equivalent to the category of étale (φ, Γ)-modules over a d-variable Laurent-series ring over $\mathbb{F}_p$ (resp. over $\mathbb{Z}_p$, resp. over $\mathbb{Q}_p$).

1 Introduction

This note serves as a complement to the work [11] where we relate multivariable (φ, Γ)-modules to smooth modulo p^n representations of a split reductive group G over $\mathbb{Q}_p$. The goal here is to show that the category of d-variable (φ, Γ)-modules is equivalent to the category of representations of the dth direct power of the absolute Galois group of $\mathbb{Q}_p$.

Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$, prime element ϖ, and residue field κ. For a finite set Δ let $G_{\mathbb{Q}_p,\Delta} := \prod_{\alpha\in\Delta} \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ denote the direct power of the absolute Galois group of $\mathbb{Q}_p$ indexed by Δ. We denote by $\mathrm{Rep}_\kappa(G_{\mathbb{Q}_p,\Delta})$ (resp. by $\mathrm{Rep}_{\mathcal{O}_K}(G_{\mathbb{Q}_p,\Delta})$, resp. by $\mathrm{Rep}_K(G_{\mathbb{Q}_p,\Delta})$) the category of continuous representations of the profinite group $G_{\mathbb{Q}_p,\Delta}$ on finite dimensional κ-vector spaces (resp. finitely generated $\mathcal{O}_K$-modules, resp. finite dimensional K-vector spaces). On the other hand, for independent commuting variables X_α ($\alpha \in \Delta$) we put

$$\begin{aligned} E_{\Delta,\kappa} &:= \kappa[[X_\alpha \mid \alpha\in\Delta]][X_\alpha^{-1} \mid \alpha\in\Delta] \ , \\ \mathcal{O}_{\mathcal{E}_{\Delta,K}} &:= \varprojlim_h \left(\mathcal{O}_K/\varpi^h[[X_\alpha \mid \alpha\in\Delta]][X_\alpha^{-1} \mid \alpha\in\Delta]\right) \ , \\ \mathcal{E}_{\Delta,K} &:= \mathcal{O}_{\mathcal{E}_{\Delta,K}}[p^{-1}] \ . \end{aligned}$$

Moreover, for each element $\alpha \in \Delta$ we have the partial Frobenius φ_α, and group $\Gamma_\alpha \cong \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ acting on the variable X_α in the usual way and commuting with the other

*This research was supported by a Hungarian OTKA Research grant K-100291 and by the János Bolyai Scholarship of the Hungarian Academy of Sciences. I would like to thank the Arithmetic Geometry and Number Theory group of the University of Duisburg–Essen, campus Essen, for its hospitality and for financial support from SFB TR45 where parts of this paper was written.

variables X_β ($\beta \in \Delta\setminus\{\alpha\}$) in the above rings. A $(\varphi_\Delta, \Gamma_\Delta)$-module over $E_{\Delta,\kappa}$ (resp. over $\mathcal{O}_{\mathcal{E}_{\Delta,K}}$, resp. over $\mathcal{E}_{\Delta,K}$) is a finitely generated $E_{\Delta,\kappa}$-module (resp. $\mathcal{O}_{\mathcal{E}_{\Delta,K}}$-module, resp. $\mathcal{E}_{\Delta,K}$-module) D together with commuting semilinear actions of the operators φ_α and groups Γ_α ($\alpha \in \Delta$). In case the coefficient ring is $E_{\Delta,\kappa}$ or $\mathcal{O}_{\mathcal{E}_{\Delta,K}}$, we say that D is étale if the map $\mathrm{id}\otimes\varphi_\alpha \colon \varphi_\alpha^* D \to D$ is an isomorphism for all $\alpha \in \Delta$. For the coefficient ring $\mathcal{E}_{\Delta,K}$ we require the stronger assumption for the étale property that D comes from an étale $(\varphi_\Delta, \Gamma_\Delta)$-module over $\mathcal{O}_{\mathcal{E}_{\Delta,K}}$ by inverting p. The main result of the paper is that $\mathrm{Rep}_\kappa(G_{\mathbb{Q}_p,\Delta})$ (resp. $\mathrm{Rep}_{\mathcal{O}_K}(G_{\mathbb{Q}_p,\Delta})$, resp. $\mathrm{Rep}_K(G_{\mathbb{Q}_p,\Delta})$) is equivalent to the category of étale $(\varphi_\Delta, \Gamma_\Delta)$-modules over $E_{\Delta,\kappa}$ (resp. over $\mathcal{O}_{\mathcal{E}_{\Delta,K}}$, resp. over $\mathcal{E}_{\Delta,K}$).

Passing from the Galois side to $(\varphi_\Delta, \Gamma_\Delta)$-modules is rather straightforward. One constructs a big ring E_Δ^{sep} as an inductive limit of completed tensor products of finite separable extensions E'_α of $E_\alpha = \mathbb{F}_p((X_\alpha))$ ($\alpha \in \Delta$) over which the action of $H_{\mathbb{Q}_p,\Delta} = \mathrm{Ker}(G_{\mathbb{Q}_p,\Delta} \twoheadrightarrow \prod_{\alpha\in\Delta}\Gamma_\Delta)$ trivializes. The other direction is more involved. In order to trivialize the action of the partial Frobenii φ_α ($\alpha \in \Delta$) using induction, the main step is to find a lattice $D_{\overline{\alpha}}^{+*}$ integral in the variable X_α for some fixed $\alpha \in \Delta$ which is an étale $(\varphi_{\Delta\setminus\{\alpha\}}, \Gamma_{\Delta\setminus\{\alpha\}})$-module over the ring $\mathbb{F}_p[\![X_\beta \mid \beta \in \Delta]\!][X_\beta^{-1} \mid \beta \in \Delta \setminus \{\alpha\}]$. This uses the ideas of Colmez [3] constructing lattices D^+ and D^{++} in usual (φ, Γ)-modules.

We remark here that Scholze [7] recently realized $G_{\mathbb{Q}_p,\Delta}$ (using Drinfeld's Lemma for diamonds) as a geometric fundamental group $\pi_1((\mathrm{Spd}\ \mathbb{Q}_p)^{|\Delta|}/\mathrm{p.Fr.})$ of the diamond $(\mathrm{Spd}\ \mathbb{Q}_p)^{|\Delta|}$ modulo the partial Frobenii φ_β ($\beta \in \Delta \setminus \{\alpha\}$) for some fixed $\alpha \in \Delta$: one can endow $E_\Delta^+ = \mathbb{F}_p[\![X_\alpha \mid \alpha \in \Delta]\!]$ with its natural compact topology, and look at the subset of its adic spectrum $\mathrm{Spa}\, E_\Delta^+$ where all X_α ($\alpha \in \Delta$) are invertible. This defines an analytic adic space over $\mathbb{F}_p$, whose perfection modulo the action of all Γ_α's is a model for $(\mathrm{Spd}\,\mathbb{Q}_p)^d$. Thus, after taking the action modulo partial Frobenii φ_β ($\beta \in \Delta \setminus \{\alpha\}$ for some fixed $\alpha \in \Delta$), the fundamental group will be $G_{\mathbb{Q}_p,\Delta}$. Now, quite generally étale local systems on diamonds are equivalent to φ-modules. This introduces the last missing Frobenius, and one ends up with an equivalence between representations of $G_{\mathbb{Q}_p,\Delta}$, and some sheaf of modules with Γ_Δ-action and commuting actions of φ_α for all $\alpha \in \Delta$. However, this will not produce an actual module over a ring, but a sheaf of modules over a sheaf of rings. One can perhaps deduce the result of this paper along these lines, but that would require some further nontrivial input (replacing the above method of finding a lattice $D_{\overline{\alpha}}^{+*}$).

1.1 Acknowledgements

I would like to thank Christophe Breuil, Elmar Große-Klönne, Kiran Kedlaya, and Vytas Paškūnas for useful discussions on the topic. I would like to thank Peter Scholze for clarifying the relation of this work to his theory of realizing $G_{\mathbb{Q}_p,\Delta}$ as the étale fundamental group of a diamond.

2 Algebraic properties of multivariable (φ, Γ)-modules

2.1 Definition and projectivity

For a finite set Δ (which is the set of simple roots of G in [11]) consider the Laurent series ring $E_\Delta := E_\Delta^+[X_\Delta^{-1}]$ where $E_\Delta^+ := \mathbb{F}_p[\![X_\alpha \mid \alpha \in \Delta]\!]$ and $X_\Delta := \prod_{\alpha\in\Delta} X_\alpha \in E_\Delta^+$. E_Δ^+ is a regular noehterian local ring of global dimension $|\Delta|$, therefore E_Δ is a regular noetherian ring of global dimension $|\Delta|-1$. For each index α we define the action of the partial Frobenius φ_α and of the group Γ_α with $\chi_\alpha\colon \Gamma_\alpha \xrightarrow{\sim} \mathbb{Z}_p^\times$ on E_Δ as

$$\begin{aligned}
\varphi_\alpha(X_\beta) &:= \begin{cases} X_\beta & \text{if } \beta\in\Delta\setminus\{\alpha\} \\ (X_\alpha+1)^p-1 = X_\alpha^p & \text{if } \beta=\alpha \end{cases} \\
\gamma_\alpha(X_\beta) &:= \begin{cases} X_\beta & \text{if } \beta\in\Delta\setminus\{\alpha\} \\ (X_\alpha+1)^{\chi_\alpha(\gamma_\alpha)}-1 & \text{if } \beta=\alpha \end{cases}
\end{aligned} \tag{1}$$

for all $\gamma_\alpha\in\Gamma_\alpha$ extending the above formulas to continuous ring endomorphisms of E_Δ in the obvious way. By an étale $(\varphi_\Delta,\Gamma_\Delta)$-module over E_Δ we mean a (unless otherwise mentioned) finitely generated module D over E_Δ together with a semilinear action of the (commutative) monoid $T_{+,\Delta} := \prod_{\alpha\in\Delta}\varphi_\alpha^{\mathbb{N}}\Gamma_\alpha$ (also denote by φ_t the action of $\varphi_t\in T_{+,\Delta}$) such that the maps

$$\mathrm{id}\otimes\varphi_t\colon \varphi_t^*D := E_\Delta\otimes_{E_\Delta,\varphi_t} D \to D$$

are isomorphisms for all elements $\varphi_t\in T_{+,\Delta}$. Here we put $\Gamma_\Delta := \prod_{\alpha\in\Delta}\Gamma_\alpha$. We denote by $\mathcal{D}^{et}(\varphi_\Delta,\Gamma_\Delta,E_\Delta)$ the category of étale $(\varphi_\Delta,\Gamma_\Delta)$-modules over E_Δ.

The category $\mathcal{D}^{et}(\varphi_\Delta,\Gamma_\Delta,E_\Delta)$ has the structure of a neutral Tannakian category: For two objects D_1 and D_2 the tensor product $D_1\otimes_{E_\Delta}D_2$ is an étale $T_{+,\Delta}$-module with the action $\varphi_t(d_1\otimes d_2) := \varphi_t(d_1)\otimes\varphi_t(d_2)$ for $\varphi_t\in T_{+,\Delta,}$, $d_i\in D_i$ $(i=1,2)$. Moreover, since E_Δ is a free module over itself via φ_t, putting $(\cdot)^* := \mathrm{Hom}_{E_\Delta}(\cdot,E_\Delta)$ we have an identification $(\varphi_t^*D)^*\cong\varphi_t^*(D^*)$. So the isomorphism $\mathrm{id}\otimes\varphi_t\colon\varphi_t^*D\to D$ dualizes to an isomorphism $D^*\to\varphi_t^*(D^*)$. The inverse of this isomorphism (for all $\varphi_t\in T_{+,\Delta}$) equips D^* with the structure of an étale $T_{+,\Delta}$-module.

Lemma 2.1. *There exists a Γ_Δ-equivariant injective resolution of E_Δ^+ as a module over itself.*

Proof. Consider the Cousin complex (see IV.2 in [6])

$$0\to E_\Delta\to E_{\Delta,(0)}\to\cdots\to\bigoplus_{\mathfrak{p}\in\mathrm{Spec}(E_\Delta),\mathrm{codim}\,\mathfrak{p}=r} J(\mathfrak{p})\to\dots$$

where $J(\mathfrak{p})$ is the injective envelope of the residue field $\kappa(\mathfrak{p})$ as a module over the local ring $E_{\Delta,\mathfrak{p}}$. This is a Γ_Δ-equivariant injective resolution since the action of Γ_Δ on $\mathrm{Spec}(E_\Delta)$ respects the codimension. $\square$

Proposition 2.2. *Any object D in $\mathcal{D}^{et}(\varphi_\Delta,\Gamma_\Delta,E_\Delta)$ is a projective module over E_Δ.*

Proof. Since E_Δ has finite global dimension, let n be the projective dimension of D. Then by Lemma 4.1.6 in [9] we have $\mathrm{Ext}^i(D,M)=0$ for all $i>n$ and E_Δ-module M and there exists an R-module M_0 with $\mathrm{Ext}^n(D,M_0)\neq 0$. By the long exact sequence of Ext and choosing an onto module homomorphism $F\twoheadrightarrow M_0$ from a free module F we find that $\mathrm{Ext}^n(D,F)\neq 0$ whence $\mathrm{Ext}^n(D,E_\Delta)\neq 0$. However, $\mathrm{Ext}^n(D,E_\Delta)$ is a finitely generated torsion E_Δ-module for $n>0$ admitting a semilinear action of Γ_Δ. Therefore the global annihilator of $\mathrm{Ext}^n(D,E_\Delta)$ in E_Δ is a nonzero Γ_Δ-invariant ideal in E_Δ hence equals E_Δ by Lemma 2.1 in [11]. So $n=0$ and D is projective. $\square$

Lemma 2.3. *We have $K_0(E_\Delta) \cong \mathbb{Z}$, ie. any finitely generated projective module over E_Δ is stably free.*

Proof. $E_\Delta^+ \cong \mathbb{F}_p[[X_\alpha \mid \alpha \in \Delta]]$ is a regular local ring, so it has finite global dimension and $K_0(E_\Delta^+) \cong G_0(E_\Delta^+) \cong \mathbb{Z}$ (Thm. II.7.8 in [10]). Therefore the localization $E_\Delta = E_\Delta^+[X_\Delta^{-1}]$ also has finite global dimension whence we have $K_0(E_\Delta) \cong G_0(E_\Delta)$. The statement follows noting that the map $G_0(E_\Delta^+) \to G_0(E_\Delta)$ is onto by the localization exact sequence of algebraic K-theory (Thm. II.6.4 in [10]). $\square$

Remark. I am not aware of the analogue of the Theorem of Quillen and Suslin on the freeness of projective modules over E_Δ. However, using the equivalence of categories of $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, E_\Delta)$ with $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta})$ we shall see later on (Cor. 3.16) that any object D in $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, E_\Delta)$ is in fact free over E_Δ.

We equip E_Δ^+ with the X_Δ-adic topology. Then (E_Δ, E_Δ^+) is a Huber pair (in the sense of [7]) if we equip E_Δ with the inductive limit topology $E_\Delta = \bigcup_n X_\Delta^{-n} E_\Delta^+$. In fact, E_Δ is a complete noetherian Tate ring (op. cit.). Note that this is *not* the natural compact topology on E_Δ^+ as in the compact topology E_Δ^+ would not be open in E_Δ since the index of E_Δ^+ in $X_\Delta^{-n} E_\Delta^+$ is not finite. On the other hand, the inclusion $\mathbb{F}_p((X_\alpha)) \hookrightarrow E_\Delta$ is *not* continuous in the X_Δ-adic topology therefore we cannot apply Drinfeld's Lemma (Thm. 17.2.4 in [7]) directly in this situation.

Let D be an object in $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, E_\Delta)$. By Banach's Theorem for Tate rings (Prop. 6.18 in [8]), there is a unique E_Δ-module topology on D that we call the X_Δ-adic topology. Moreover, any E_Δ-module homomorphism is continuous in the X_Δ-adic topology.

2.2 Integrality properties

Put $\varphi_s := \prod_{\alpha\in\Delta} \varphi_\alpha \in T_{+,\Delta}$ and define $D^{++} := \{x \in D \mid \lim_{k\to\infty} \varphi_s^k(x) = 0\}$ where the limit is considered in the X_Δ-adic topology (cf. II.2.1 in [3] in case $|\Delta| = 1$). Note that φ_s is the absolute Frobenius on E_Δ, it takes any element to its pth power.

Lemma 2.4. *Let M be a finitely generated E_Δ^+-submodule in D. Then $E_\Delta^+\varphi_s(M)$ is also finitely generated.*

Proof. If M is generated by $m_1, \dots, m_n$ then $\varphi_s(m_1), \dots, \varphi_s(m_n)$ generate $E_\Delta^+\varphi_s(M)$. $\square$

Proposition 2.5. *D^{++} is a finitely generated E_Δ^+-submodule in D that is stable under the action of $T_{+,\Delta}$ and we have $D = D^{++}[X_\Delta^{-1}]$.*

Proof. Choose an arbitrary finitely generated E_Δ^+-submodule M of D with $M[X_\Delta^{-1}] = D$ (e.g. take $M = E_\Delta^+ e_1 + \cdots + E_\Delta^+ e_n$ for some E_Δ-generating system $e_1, \dots, e_n$ of D). By Lemma 2.4 we have an integer $r \geq 0$ such that $\varphi(M) \subseteq X_\Delta^{-r} M$, since E_Δ^+ is noetherian and we have $D = \bigcup_r X_\Delta^{-r} M$. Then we have

$$\varphi_s(X_\Delta^k M) = X_\Delta^{pk} \varphi_s(M) \subseteq X_\Delta^{pk-r} M \subseteq X_\Delta^{k+1} M$$

for any integer $k \geq \frac{r+1}{p-1}$. Therefore we have $X_\Delta^{\left[\frac{r+1}{p-1}\right]+1} M \subseteq D^{++}$ whence $D^{++}[X_\Delta^{-1}] = M[X_\Delta^{-1}] = D$.

Since $T_{+\Delta}$ is commutative and the action of each φ_t ($t \in T_{+,\Delta}$) is continuous, D^{++} is stable under the action of $T_{+,\Delta}$. There is a system of neighbourhoods of 0 in D consisting of E_Δ^+-submodules therefore D^{++} is an E_Δ^+-submodule.

To prove that D^{++} is finitely generated over E_Δ^+ suppose first that D is a free module over E_Δ generated by $e_1, \dots, e_n$ and put $M := E_\Delta^+ e_1 + \dots + E_\Delta^+ e_n$. We may assume $M \subseteq D^{++}$ by replacing M with $X_\Delta^{[\frac{r+1}{p-1}]+1} M$. Moreover, further multiplying $M = E_\Delta^+ e_1 + \dots + E_\Delta^+ e_n$ by a power of X_Δ, we may assume that the matrix $A := [\varphi_s]_{e_1,\dots,e_n}$ of φ_s in the basis $e_1, \dots, e_n$ lies in $E_\Delta^{+n\times n}$ as we have $[\varphi_s]_{X_\Delta^r e_1,\dots,X_\Delta^r e_n} = X_\Delta^{(p-1)r}[\varphi_s]_{e_1,\dots,e_n}$. Now we choose the integer $r > 0$ so that it is bigger than $\mathrm{val}_{X_\alpha}(\det A)$ for all $\alpha \in \Delta$ and claim that $D^{++} \subseteq X_\Delta^{-r} M$ whence D^{++} is finitely generated over E_Δ^+ as E_Δ^+ is noetherian. Assume for contradiction that $d = \sum_{i=1}^n d_i e_i$ lies in D^{++} for some $d_i \in E_\Delta$ ($i = 1, \dots, n$) such that at least one d_i, say d_1, does not lie in $X_\Delta^{-r} E_\Delta^+$. In particular, there exists an α in Δ such that $\mathrm{val}_{X_\alpha}(d_1) < -r$. Since M is open in D and $d \in D^{++}$, there exists an integer $k > 0$ such that $\varphi_s^k(d)$ is in M which is equivalent to saying that the column vector

$$A\varphi_s(A)\dots\varphi_s^{k-1}(A)\begin{pmatrix}\varphi_s^k(d_1)\\ \vdots \\ \varphi_s^k(d_n)\end{pmatrix}$$

lies in E_Δ^{+n}. Multiplying this by the matrix built from the $(n-1)\times(n-1)$ minors of $A\varphi_s(A)\dots\varphi_s^{k-1}(A)$ we deduce that $\det(A\varphi_s(A)\dots\varphi_s^{k-1}(A))\varphi_s^k(d_1) = \det(A)^{\frac{p^k-1}{p-1}} d_1^{p^k}$ lies in E_Δ^+. We compute

$$0 \le \mathrm{val}_{X_\alpha}(\det(A)^{\frac{p^k-1}{p-1}} d_1^{p^k}) = \frac{p^k-1}{p-1}\mathrm{val}_{X_\alpha}(\det(A)) + p^k \mathrm{val}_{X_\alpha}(d_1) <$$
$$< \frac{p^k-1}{p-1}\mathrm{val}_{X_\alpha}(\det(A)) - p^k r < 0$$

by our assumption that $r > \mathrm{val}_{X_\alpha}(\det(A))$, yielding a contradiction.

In the general case note that D is always stably free by Prop. 2.2 and Lemma 2.3. So $D_1 := D \oplus E_\Delta^k$ is a free module over E_Δ for k large enough. We make D_1 into an étale $T_{+,\Delta}$-module by the trivial action of $T_{+,\Delta}$ on E_Δ^k to deduce that D_1^{++} is finitely generated over E_Δ^+. The result follows noting that $D^{++} \subseteq D_1^{++}$ and E_Δ^+ is noetherian. $\square$

For an object D in $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, E_\Delta)$ we define

$$D^+ := \{x \in D \mid \{\varphi_s^k(x)\colon k \ge 0\} \subset D \text{ is bounded}\}\ .$$

Since $\varphi_s^k(X_\Delta)$ tends to 0 in the X_Δ-adic topology, we have $X_\Delta D^+ \subseteq D^{++}$, ie. $D^+ \subseteq X_\Delta^{-1} D^{++}$. In particular, D^+ is finitely generated over E_Δ^+. On the other hand, we also have $D^{++} \subseteq D^+$ by construction whence we deduce $D = D^+[X_\Delta^{-1}]$.

Lemma 2.6. *We have $\varphi_t(D^+) \subset D^+$ (resp. $\varphi_t(D^{++}) \subset D^{++}$) for all $\varphi_t \in T_{+,\Delta}$.*

Proof. For any generating system $e_1, \dots, e_n$ of D and any $\varphi_t \in T_{+,\Delta}$ there exists an integer $k = k(\varphi_t, M) > 0$ such that we have $\varphi_t(X_\Delta^k M) \subseteq X_\Delta^k E_\Delta^+ \varphi_t(M) \subseteq M$ where we put $M :=$

$E_\Delta^+ e_1 + \cdots + E_\Delta^+ e_n$ by Lemma 2.4. Indeed, X_Δ divides $\varphi_t(X_\Delta)$ in E_Δ^+, and we have $D = M[1/X_\Delta]$ by construction. The statement on D^{++} follows from the commutativity of the monoid $T_{+,\Delta}$ noting that there exists a basis of neighbouhoods of 0 in D consisting of E_Δ^+-submodules of the form M. To see that $\varphi_t(D^+) \subseteq D^+$ note that $\varphi_t(D^+)$ is bounded and we have $\varphi_s^k(\varphi_t(D^+)) = \varphi_t(\varphi_s^k(D^+)) \subset \varphi_t(D^+)$. $\square$

Now fix an $\alpha \in \Delta$ and define $D_{\overline{\alpha}}^+ := D^+[X_{\Delta\setminus\{\alpha\}}^{-1}]$ where for any subset $S \subseteq \Delta$ we put $X_S := \prod_{\beta \in S} X_\beta$. Then $D_{\overline{\alpha}}^+$ is a finitely generated module over $E_{\overline{\alpha}}^+ := E_\Delta^+[X_{\Delta\setminus\{\alpha\}}^{-1}]$. We denote by $T_{+,\overline{\alpha}} \subset T_{+,\Delta}$ the monoid generated by φ_β ($\beta \in \Delta \setminus \{\alpha\}$) and Γ_Δ.

Lemma 2.7. *$D_{\overline{\alpha}}^+/D^+$ is X_α-torsion free: If both $X_\alpha^{n_1} d$ and $X_{\Delta\setminus\{\alpha\}}^{n_2} d$ lie in D^+ for some element $d \in D^+$, $\alpha \in \Delta$, and integers $n_1, n_2 \geq 0$ then we have $d \in D^+$. The same statement holds if we replace D^+ by D^{++}.*

Proof. At first assume that D is free as a module over E_Δ with basis $e_1, \ldots, e_n$. Then the denominators of $\varphi_s^k(X_\alpha^{n_1} d) = X_\alpha^{n_1 p^k} \varphi_s^k(d)$ in the basis $e_1, \ldots, e_n$ are bounded for $k \geq 0$ by assumption. Therefore the X_β-valuations of the denominators of $\varphi_s^k(d)$ are bounded for all $\beta \in \Delta \setminus \{\alpha\}$ since E_Δ^+ is a unique factorization domain. On the other hand, the X_α-valuations of these denominators are also bounded since the denominators of $\varphi_s^k(X_{\Delta\setminus\{\alpha\}}^{n_2} d) = X_{\Delta\setminus\{\alpha\}}^{n_2 p^k} \varphi_s^k(d)$ are bounded. To prove the statement we have the same argument but 'being bounded' replaced by 'tends to 0'.

Finally, by Prop. 2.2 and Lemma 2.3 $D \oplus E_\Delta^k$ is free over E_Δ and we equip it with the structure of an étale (φ, Γ)-module (trivially on E_Δ^k). The statement follows from the additivity of the constructions $D \mapsto D^+$ and $D \mapsto D_{\overline{\alpha}}^+$ in direct sums. $\square$

Lemma 2.8. *Assume that D is generated by a single element $e_1 \in D$ over E_Δ. Then for any φ_t in $T_{+,\overline{\alpha}}$ we have $\varphi_t(e_1) = a_t e_1$ for some unit a_t in $(E_{\overline{\alpha}}^+)^\times$.*

Proof. Define $a_t \in E_\Delta$ and $a_\alpha \in E_\Delta$ so that $\varphi_t(e_1) = a_t e_1$ and $\varphi_\alpha(e_1) = a_\alpha e_1$. By the étale property both a_t and a_α are units in E_Δ, so it remains to show that $\mathrm{val}_{X_\alpha}(a_t) = 0$. We compute

$$\begin{aligned} \varphi_\alpha(a_t) a_\alpha e_1 &= \varphi_\alpha(a_t)\varphi_\alpha(e_1) = \varphi_\alpha(a_t e_1) = \varphi_\alpha(\varphi_t(e_1)) = \\ &= \varphi_t(\varphi_\alpha(e_1)) = \varphi_t(a_\alpha e_1) = \varphi_t(a_\alpha)\varphi_t(e_1) = \varphi_t(a_\alpha) a_t e_1 \end{aligned}$$

whence we deduce

$$p\,\mathrm{val}_{X_\alpha}(a_t) + \mathrm{val}_{X_\alpha}(a_\alpha) = \mathrm{val}_{X_\alpha}(\varphi_\alpha(a_t) a_\alpha) = \mathrm{val}_{X_\alpha}(\varphi_t(a_\alpha) a_t) = \mathrm{val}_{X_\alpha}(a_\alpha) + \mathrm{val}_{X_\alpha}(a_t)\ .$$

This yields $\mathrm{val}_{X_\alpha}(a_t) = 0$ as required. $\square$

Lemma 2.9. *There exists an integer $k = k(D) > 0$ such that for any $\varphi_t \in T_{+,\overline{\alpha}}$ we have $X_\alpha^k D_{\overline{\alpha}}^+ \subseteq E_\Delta^+ \varphi_t(D_{\overline{\alpha}}^+)$.*

Proof. At first assume that D is free, choose a basis $e_1, \ldots, e_n$ contained in D^+, and put $M := E_\Delta^+ e_1 + \ldots E_\Delta^+ e_n$, $M_\alpha := E_{\overline{\alpha}}^+ e_1 + \cdots + E_{\overline{\alpha}}^+ e_n$. There exists an integer $k_0 > 0$ such that $D^+ \subseteq X_\Delta^{-k_0} M$. In particular, we have $D_{\overline{\alpha}}^+ \subseteq X_\alpha^{-k_0} M_{\overline{\alpha}}$. Now for a fixed $\varphi_t \in T_{+,\overline{\alpha}}$ let $A_t \in E_\Delta^{n\times n}$ be the matrix of φ_t in the basis $e_1, \ldots, e_n$. Since $\varphi_t(e_i)$ lies in $D^+ \subseteq X_\alpha^{-k_0} M_{\overline{\alpha}}$, all the entries of the matrix A_t are in $X_\alpha^{-k_0}E_{\overline{\alpha}}^+$. Applying Lemma 2.8 to the single generator $e_1\wedge\cdots\wedge e_n$ of $\bigwedge^n D$ we obtain $\mathrm{val}_{X_\alpha}(\det A_t)=0$. In particular, all the entries of A_t^{-1} lie in $X_\alpha^{-(n-1)k_0}E_{\overline{\alpha}}^+$ by the formula for the inverse matrix using the $(n-1)\times(n-1)$ minors in A_t. Now note that the elements $e_1,\dots,e_n$ can be written as a linear combination of $\varphi_t(e_1),\dots,\varphi_t(e_n)$ with coefficients from A_t^{-1}. Using Lemma 2.6 this shows

$$X_\alpha^{k_0}D_{\overline{\alpha}}^+\subseteq M_{\overline{\alpha}}\subseteq X_\alpha^{-(n-1)k_0}\varphi_t(M_{\overline{\alpha}})\subseteq X_\alpha^{-(n-1)k_0}D_{\overline{\alpha}}^+\ .$$

So we may choose $k:=nk_0$ independent of φ_t.

The general case follows from Prop. 2.2 and Lemma 2.3 noting that the functor $D\mapsto D_{\overline{\alpha}}^+$ commutes with direct sums. □

In view of the above Lemma we define

$$D_{\overline{\alpha}}^{+*}:=\bigcap_{\varphi_t\in T_{+,\overline{\alpha}}}E_{\overline{\alpha}}^+\varphi_t(D_{\overline{\alpha}}^+)\ .$$

$D_{\overline{\alpha}}^{+*}$ is finitely generated over $E_{\overline{\alpha}}^+$ as it is contained in $D_{\overline{\alpha}}^+$ and $E_{\overline{\alpha}}^+$ is noetherian. On the other hand, by Lemma 2.9 we have $X_\alpha^kD_{\overline{\alpha}}^+\subseteq D_{\overline{\alpha}}^{+*}$ for some integer $k=k(D)>0$ whence, in particular, $D=D_{\overline{\alpha}}^{+*}[X_\alpha^{-1}]$.

Proposition 2.10. *$D_{\overline{\alpha}}^{+*}$ is an étale $T_{+,\overline{\alpha}}$-module over $E_{\overline{\alpha}}^+$, ie. the maps*

$$\mathrm{id}\otimes\varphi_t\colon \varphi_t^*D_{\overline{\alpha}}^{+*}=E_{\overline{\alpha}}^+\otimes_{E_{\overline{\alpha}}^+,\varphi_t}D_{\overline{\alpha}}^{+*}\to D_{\overline{\alpha}}^{+*}\qquad(2)$$

are bijective for all $\varphi_t\in T_{+,\alpha}$.

Proof. At first note that we have $\varphi_t(D_{\overline{\alpha}}^{+*})\subseteq D_{\overline{\alpha}}^{+*}$ for all $\varphi_t\in T_{+,\overline{\alpha}}$ by Lemma 2.6 and the commutativity of $T_{+,\overline{\alpha}}$, so the map (2) exists. Now let $\varphi_{t_0}\in T_{+,\overline{\alpha}}$ be arbitrary. Since $E_{\overline{\alpha}}^+$ (resp. E_Δ) is a finite free module over $\varphi_{t_0}(E_{\overline{\alpha}}^+)$ (resp. over $\varphi_{t_0}(E_\Delta)$) with generators contained in E_Δ^+, we have a natural identification $\varphi_{t_0}^*D_{\overline{\alpha}}^{+*}\cong E_\Delta^+\otimes_{E_\Delta^+,\varphi_{t_0}}D_\Delta^{+*}$ (resp. $\varphi_{t_0}^*D\cong E_\Delta^+\otimes_{E_\Delta^+,\varphi_{t_0}}D$). Since E_Δ^+ is finite free (hence flat) over $\varphi_{t_0}(E_\Delta^+)$, the inclusion $D_{\overline{\alpha}}^+\subset D$ induces an inclusion $\varphi_{t_0}^*D_{\overline{\alpha}}^+\subset\varphi_{t_0}^*D$. It follows that (2) is injective since D is étale. Similarly, for each $\varphi_t\in T_{+,\overline{\alpha}}$, the map

$$\mathrm{id}\otimes\varphi_{t_0}\colon \varphi_{t_0}^*(E_{\overline{\alpha}}^+\varphi_t(D_{\overline{\alpha}}^+))\to E_{\overline{\alpha}}^+\varphi_t(D_{\overline{\alpha}}^+)$$

is injective with image $E_{\overline{\alpha}}^+\varphi_{t_0}\varphi_t(D_{\overline{\alpha}}^+)$. On the other hand, since E_Δ^+ is finite free over $\varphi_{t_0}(E_\Delta^+)$, we have $\varphi_{t_0}^*D_{\overline{\alpha}}^{+*}=\bigcap_{t\in T_{+,\overline{\alpha}}}\varphi_{t_0}^*(E_{\overline{\alpha}}^+\varphi_t(D_{\overline{\alpha}}^+))$ where the intersection is taken inside $\varphi_{t_0}^*D$. Therefore (2) is bijective as we have $D_{\overline{\alpha}}^{+*}=\bigcap_{\varphi_t\in T_{+,\overline{\alpha}}}E_{\overline{\alpha}}^+\varphi_{t_0}\varphi_t(D_{\overline{\alpha}}^+)$. □

Lemma 2.11. *There exists a finitely generated E_Δ^+-submodule $D_0\subset D_{\overline{\alpha}}^{+*}$ such that $D_0\subseteq E_\Delta^+\varphi_{\overline{\alpha}}(D_0)$ and $D_{\overline{\alpha}}^{+*}=D_0[X_{\Delta\setminus\{\alpha\}}^{-1}]$ where $\varphi_{\overline{\alpha}}:=\prod_{\beta\in\Delta\setminus\{\alpha\}}\varphi_\beta$. Moreover, we have $D_{\overline{\alpha}}^{+*}=\bigcup_{r\geq0}E_\Delta^+\varphi_{\overline{\alpha}}^r(X_{\Delta\setminus\{\alpha\}}^{-1}D_0)$.*

Proof. Put $D_1:=D^+\cap D_{\overline{\alpha}}^{+*}$. By Prop. 2.10 and the fact that $D_{\overline{\alpha}}^{+*}=D_1[X_{\Delta\setminus\{\alpha\}}^{-1}]$ we find an integer $k_0>0$ such that $X_{\Delta\setminus\{\alpha\}}^{k_0}D_1\subseteq E_\Delta^+\varphi_{\overline{\alpha}}(D_1)$. So for $k>\frac{k_0}{p-1}$ we have

$$X_{\Delta\setminus\{\alpha\}}^{-k}D_1\subseteq X_{\Delta\setminus\{\alpha\}}^{-k-k_0}E_\Delta^+\varphi_{\overline{\alpha}}(D_1)\subseteq X_{\Delta\setminus\{\alpha\}}^{-pk}E_\Delta^+\varphi_{\overline{\alpha}}(D_1)=E_\Delta^+\varphi_{\overline{\alpha}}(X_{\Delta\setminus\{\alpha\}}^{-k}D_1)\ .$$

So we put $D_0 := X_{\Delta\setminus\{\alpha\}}^{-k} D_1$ so that the first part of the statement is satisfied. Iterating the inclusion $D_0 \subseteq E_\Delta^+ \varphi_{\overline{\alpha}}(D_0)$ we obtain $D_0 \subseteq E_\Delta^+ \varphi_{\overline{\alpha}}^r(D_0)$ for all $r \geq 1$. Finally, we compute

$$X_{\Delta\setminus\{\alpha\}}^{-p^r} D_0 \subseteq X_{\Delta\setminus\{\alpha\}}^{-p^r} E_\Delta^+ \varphi_{\overline{\alpha}}^r(D_0) = E_\Delta^+ \varphi_{\overline{\alpha}}^r(X_{\Delta\setminus\{\alpha\}}^{-1} D_0) \ .$$

The statement follows noting that we have $D_{\overline{\alpha}}^{+*} = D_0[X_{\Delta\setminus\{\alpha\}}^{-1}] = \bigcup_r X_{\Delta\setminus\{\alpha\}}^{-p^r} D_0$. $\square$

3 The equivalence of categories for $\mathbb{F}_p$-representations

3.1 The functor $\mathbb{D}$

Take a copy $G_{\mathbb{Q}_p,\alpha} \cong \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ of the absolute Galois group of $\mathbb{Q}_p$ for each element $\alpha \in \Delta$ and let $G_{\mathbb{Q}_p,\Delta} := \prod_{\alpha\in\Delta} G_{\mathbb{Q}_p,\alpha}$. Let $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta})$ be the category of continuous representations of the group $G_{\mathbb{Q}_p,\Delta}$ on finite dimensional $\mathbb{F}_p$ vectorspaces. We identify Γ_α with the Galois group $\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ as a quotient of $G_{\mathbb{Q}_p,\alpha}$ via the cyclotomic character $\chi_\alpha \colon \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \to \mathbb{Z}_p^\times$. Further, we denote by $H_{\mathbb{Q}_p,\alpha}$ the kernel of the natural quotient map $G_{\mathbb{Q}_p,\alpha} \to \Gamma_\alpha$ and put $H_{\mathbb{Q}_p,\Delta} := \prod_{\alpha\in\Delta} H_{\mathbb{Q}_p,\alpha} \lhd G_{\mathbb{Q}_p,\Delta}$. Putting $E_\alpha := \mathbb{F}_p(\!(X_\alpha)\!)$ we have the following fundamental result of Fontaine and Wintenberger (Thm. 4.16 [5]).

Theorem 3.1. *The absolute Galois group* $\mathrm{Gal}(E_\alpha^{sep}/E_\alpha)$ *is isomorphic to* $H_{\mathbb{Q}_p,\alpha}$*. Moreover,* $G_{\mathbb{Q}_p,\alpha}$ *acts on the separable closure* E_α^{sep} *via automorphisms such that the action of* $\Gamma_\alpha \cong G_{\mathbb{Q}_p,\alpha}/H_{\mathbb{Q}_p,\alpha}$ *on* $E_\alpha = (E_\alpha^{sep})^{H_{\mathbb{Q}_p,\alpha}}$ *coincides with the one given in* (1)*.*

For each $\alpha \in \Delta$ consider a finite separable extension E'_α of E_α together with the Frobenius $\varphi_\alpha \colon E'_\alpha \to E'_\alpha$ acting by raising to the power p. We denote by E'^+_α the integral closure of $E_\alpha^+ = \mathbb{F}_p[\![X_\alpha]\!]$ in E'_α. Note that E'_α is isomorphic to $\mathbb{F}_{q_\alpha}(\!(X'_\alpha)\!)$ for some power q_α of p and uniformizer X'_α such that we have $E'^+_\alpha \cong \mathbb{F}_{q_\alpha}[\![X'_\alpha]\!]$. We normalize the X_α-adic (multiplicative) valuation on E_α so that we have $|X_\alpha|_{X_\alpha} = p^{-1}$. This extends uniquely to the finite extension E'_α. Moreover, we equip the tensor product $E'_{\Delta,\circ} := \bigotimes_{\alpha\in\Delta,\mathbb{F}_p} E'_\alpha$ with a norm $|\cdot|_{prod}$ by the formula

$$|c|_{prod} := \inf\left(\max_i(\prod_{\alpha\in\Delta} |c_{\alpha,i}|_\alpha) \ \mid \ c = \sum_{i=1}^n \bigotimes_{\alpha\in\Delta} c_{\alpha,i}\right) \ . \tag{3}$$

Note that the restriction of $|\cdot|_{prod}$ to the subring $E'^+_{\Delta,\circ} := \bigotimes_{\alpha\in\Delta,\mathbb{F}_p} E'^+_\alpha$ induces the valuation with respect to the augmentation ideal $\mathrm{Ker}(E'^+_{\Delta,\circ} \twoheadrightarrow \bigotimes_{\alpha\in\Delta,\mathbb{F}_p} \mathbb{F}_{q_\alpha})$. The norm $|\cdot|_{prod}$ is not multiplicative in general, as the ring $\bigotimes_{\alpha\in\Delta,\mathbb{F}_p} \mathbb{F}_{q_\alpha})$ is not a domain. However, it is submultiplicative. We define E'^+_Δ as the completion of $E'^+_{\Delta,\circ}$ with respect to $|\cdot|_{prod}$ and put $E'_\Delta := E'^+_\Delta[1/X_\Delta]$. Note that E'_Δ is *not* complete with respect to $|\cdot|_{prod}$ (unless $|\Delta| = 1$) even though $E'_{\Delta,\circ} = E'^+_{\Delta,\circ}[1/X_\Delta]$ is a dense subring in E'_Δ. Since we have a containment

$$(\bigotimes_{\alpha\in\Delta,\mathbb{F}_p} \mathbb{F}_{q_\alpha})[X'_\alpha, \alpha\in\Delta] = \bigotimes_{\alpha\in\Delta,\mathbb{F}_p} \mathbb{F}_{q_\alpha}[X_\alpha] \leq_{dense} E'^+_{\Delta,\circ}$$

we may identify E'^+_Δ with the power series ring $(\bigotimes_{\alpha\in\Delta,\mathbb{F}_p} \mathbb{F}_{q_\alpha})[\![X'_\alpha, \alpha \in \Delta]\!]$ which is the completion of the polynomial ring above. In particular, the special case $E'_\alpha = E_\alpha$ for all $\alpha \in \Delta$ yields a ring E'_Δ isomorphic to E_Δ. Therefore E_Δ is a subring of E'_Δ for all collection of finite separable extensions E'_α of E_α ($\alpha \in \Delta$). Further, φ_α acts on $E'^{+}_{\Delta,\circ}$ (and on $E'_{\Delta,\circ}$) by the Frobenius on the component in E'_α and by the identity on all the other components in E'_β, $\beta \in \Delta \setminus \{\alpha\}$. This action is continuous in the norm $|\cdot|_{prod}$ therefore extends to the completion E'^{+}_{Δ} and the localization E'_Δ. We have the following alternative characterization of the ring E'_Δ.

Lemma 3.2. *Put $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. We have*

$$E'_\Delta \cong E'_{\alpha_1} \otimes_{E_{\alpha_1}} \left(E'_{\alpha_2} \otimes_{E_{\alpha_2}} \left(\cdots (E'_{\alpha_n} \otimes_{E_{\alpha_n}} E_\Delta)\right)\right) \ .$$

Proof. By rearranging the order of tensor products we have an identification

$$E'^{+}_{\Delta,\circ} = \bigotimes_{\alpha\in\Delta,\mathbb{F}_p} (E'^{+}_\alpha \otimes_{E^+_\alpha} E^+_\alpha) \cong E'^{+}_{\alpha_1} \otimes_{E^+_{\alpha_1}} \left(E'^{+}_{\alpha_2} \otimes_{E^+_{\alpha_2}} \left(\ldots (E'^{+}_{\alpha_n} \otimes_{E^+_{\alpha_n}} E^+_{\Delta,\circ})\right)\right) \ .$$

The statement follows by completing this with respect to the maximal ideal of E^+_Δ and inverting X_Δ. $\square$

We define the multivariable analogue of E^{sep} as

$$E^{sep}_\Delta := \varinjlim_{E_\alpha \leq E'_\alpha \leq E^{sep}_\alpha, \forall \alpha\in\Delta} E'_\Delta \ .$$

For any subset $S \subseteq \Delta$ we define the similar notions E'^{+}_S, E'_S, and E^{sep}_S with Δ replaced by S. We equip E^{sep}_Δ with the relative Frobenii φ_α for each $\alpha \in \Delta$ defined above on each E'_Δ. Further, E^{sep}_Δ admits an action of $G_{\mathbb{Q}_p,\Delta}$ satisfying

Proposition 3.3. *Assume that the extensions E'_α/E_α are Galois for all $\alpha \in \Delta$ and let $H' := \prod_{\alpha\in\Delta} H'_\alpha$ where $H'_\alpha := \mathrm{Gal}(E^{sep}_\alpha/E'_\alpha)$. Then we have $(E^{sep}_\Delta)^{H'_\Delta} = E'_\Delta$. In particular, the subring $(E^{sep}_\Delta)^{H_{\mathbb{Q}_p,\Delta}}$ of $H_{\mathbb{Q}_p,\Delta}$-invariants in E^{sep}_Δ equals E_Δ with the previously defined action of $\Gamma_\Delta \cong G_{\mathbb{Q}_p,\Delta}/H_{\mathbb{Q}_p,\Delta}$.*

Proof. Since X_Δ is H'_Δ-invariant and $\varinjlim$ can be interchanged with taking H'_Δ-invariants, it suffices to show that whenever

$$E_\alpha = \mathbb{F}_p(\!(X_\alpha)\!) \leq E'_\alpha = \mathbb{F}_{q'_\alpha}(\!(X'_\alpha)\!) \leq E''_\alpha = \mathbb{F}_{q''_\alpha}(\!(X''_\alpha)\!)$$

is a a sequence of finite Galois extensions for each $\alpha \in \Delta$ then we have $(E''^{+}_\Delta)^{H'_\Delta} = E'^{+}_\Delta$. The containment $(E''^{+}_\Delta)^{H'_\Delta} \supseteq E'^{+}_\Delta$ is clear. We prove the converse by induction on $|\Delta|$. Note that the ideal $\mathcal{M}_\alpha \lhd E''^{+}_\Delta$ generated by X''_α is invariant under the action of H'_Δ for any fixed α in Δ. Moreover, for any integer $k \geq 1$ the ring $E''^{+}_\alpha/\mathcal{M}^k_\alpha$ is finite dimensional over $\mathbb{F}_p$. Therefore the image of $(E''^{+}_\Delta)^{H'_\Delta}$ under the quotient map $E''^{+}_\Delta \twoheadrightarrow E''^{+}_\Delta/\mathcal{M}^k_\alpha$ is contained in

$$\left(E''^{+}_\Delta/\mathcal{M}^k_\alpha\right)^{H'_\Delta} \subseteq \left(E''^{+}_\Delta/\mathcal{M}^k_\alpha\right)^{H'_{\Delta\setminus\{\alpha\}}} = \left(E''^{+}_{\Delta\setminus\{\alpha\}} \otimes_{\mathbb{F}_p} \left(E''^{+}_\alpha/\mathcal{M}^k_\alpha\right)\right)^{H'_{\Delta\setminus\{\alpha\}}} =$$

$$= \left(E''^{+}_{\Delta\setminus\{\alpha\}}\right)^{H'_{\Delta\setminus\{\alpha\}}} \otimes_{\mathbb{F}_p} \left(E''^{+}_\alpha/\mathcal{M}^k_\alpha\right) = E'^{+}_{\Delta\setminus\{\alpha\}} \otimes_{\mathbb{F}_p} \left(E''^{+}_\alpha/\mathcal{M}^k_\alpha\right)$$

by induction. Taking the projective limit with respect to $k \geq 1$ we deduce that $(E''^{+}_{\Delta})^{H'_\Delta}$ is contained in the power series ring

$$\left(\mathbb{F}_{q''_\alpha} \otimes_{\mathbb{F}_p} \bigotimes_{\beta\in\Delta\setminus\{\alpha\},\mathbb{F}_p} \mathbb{F}_{q'_\beta}\right) [[X''_\alpha, X'_\beta \mid \beta \in \Delta \setminus \{\alpha\}]] \subseteq E''^{+}_{\Delta} \ .$$

Now using the action of H'_α in a similar argument as above (reducing modulo the kth power of the ideal generated by all the X'_β, $\beta \in \Delta \setminus \{\alpha\}$ for all $k \geq 1$) we deduce the statement. $\square$

The subring $E^{sep}_{\Delta,\circ} \cong \bigotimes_{\alpha\in\Delta,\mathbb{F}_p} E^{sep}_\alpha$ in E^{sep}_Δ is the inductive limit of $E'_{\Delta,\circ} \subseteq E'_\Delta$ where E'_α runs through the finite separable extensions of E_α for each $\alpha \in \Delta$.

Let V be a finite dimensional representation of the group $G_{\mathbb{Q}_p,\Delta}$ over $\mathbb{F}_p$. The basechange $E^{sep}_\Delta \otimes_{\mathbb{F}_p} V$ is equipped with the diagonal semilinear action of $G_{\mathbb{Q}_p,\Delta}$ and with the Frobenii φ_α for $\alpha \in \Delta$. These all commute with each other. We define the value of the functor $\mathbb{D}$ at V by putting

$$\mathbb{D}(V) := (E^{sep}_\Delta \otimes_{\mathbb{F}_p} V)^{H_{\mathbb{Q}_p,\Delta}} \ .$$

By Lemma 3.3 $\mathbb{D}(V)$ is a module over E_Δ inheriting the action of the monoid $T_{+,\Delta}$ from the action of φ_α ($\alpha \in \Delta$) and the Galois group $G_{\mathbb{Q}_p,\Delta}$ on $E^{sep}_\Delta \otimes_{\mathbb{F}_p} V$. Our key Lemma is the following.

Lemma 3.4. *The E^{sep}_Δ-module $E^{sep}_\Delta \otimes_{\mathbb{F}_p} V$ admits a basis consisting of elements fixed by $H_{\mathbb{Q}_p,\Delta}$.*

Proof. At first consider the $E^{sep}_{\Delta,\circ}$-module $E^{sep}_{\Delta,\circ} \otimes_{\mathbb{F}_p} V$. We show by induction on $|\Delta|$ that $E^{sep}_{\Delta,\circ} \otimes_{\mathbb{F}_p} V$ admits a basis consisting of $H_{\mathbb{Q}_p,\Delta}$-invariant vectors. The statement follows from this noting that $E^{sep}_{\Delta,\circ}$ is a subring in E^{sep}_Δ therefore the required basis exists also in $E^{sep}_\Delta \otimes_{\mathbb{F}_p} V \cong E^{sep}_\Delta \otimes_{E^{sep}_{\Delta,\circ}} (E^{sep}_{\Delta,\circ} \otimes_{\mathbb{F}_p} V)$.

By Hilbert's Thm. 90 the $H_{\mathbb{Q}_p,\alpha}$-module $E^{sep}_\alpha \otimes_{\mathbb{F}_p} V$ is trivial for each $\alpha \in \Delta$. So we have an E^{sep}_α-basis $e^{(\alpha)}_1, \dots, e^{(\alpha)}_d$ of $E^{sep}_\alpha \otimes_{\mathbb{F}_p} V$ consisting of $H_{\mathbb{Q}_p,\alpha}$-invariant elements. Since we have an action of the direct product $H_{\mathbb{Q}_p,\Delta}$ on V, the E_α-vector space

$$V_\alpha := E_\alpha e^{(\alpha)}_1 + \dots + E_\alpha e^{(\alpha)}_d = (E^{sep}_\alpha \otimes_{\mathbb{F}_p} V)^{H_{\mathbb{Q}_p,\alpha}}$$

admits a linear action of the group $H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$. Now note that the representations V and V_α of the group $H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$ become isomorphic over the field E^{sep}_α by construction. Since $H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$ acts through a finite quotient on V, there is a finite extension E'_α of E_α contained in E^{sep}_α such that we have an isomorphism $E'_\alpha \otimes_{\mathbb{F}_p} V \cong E'_\alpha \otimes_{E_\alpha} V_\alpha$ of $H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$-representations. Making this identification and writing $e_i := 1 \otimes e_i \in E'_\alpha \otimes_{\mathbb{F}_p} V$ (resp. $e^{(\alpha)}_i := 1 \otimes e^{(\alpha)}_i$), $i = 1, \dots, d$, for a basis $e_1, \dots, e_d$ in V (resp. for the basis $e^{(\alpha)}_1, \dots e^{(\alpha)}_d$ in V_α) by an abuse of notation, we find a matrix $B \in \mathrm{GL}_d(E'_\alpha)$ with $B\rho(h) = \rho_\alpha(h)B$ for all $h \in H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$ where $\rho(h) \in \mathrm{GL}_d(\mathbb{F}_p)$ (resp. $\rho_\alpha(h) \in \mathrm{GL}_d(E_\alpha)$) is the matrix of the action of h on V (resp. on V_α) in the basis $e_1, \dots, e_d$ (resp. $e^{(\alpha)}_1, \dots e^{(\alpha)}_d$). Now E'_α/E_α is a finite separable extension, so there exists a primitive element $u \in E'_\alpha$ with $E'_\alpha = E_\alpha(u)$. Hence we may write B is a sum $B = B(u) = B_0 + B_1 u + \dots + B_{n-1}u^{n-1}$ for some matrices $B_0, B_1, \dots, B_{n-1} \in E^{d\times d}_\alpha$ with $n := |E'_\alpha : E_\alpha|$. Since $\det B \neq 0$, the polynomial $\det(B(x)) := \det(B_0 + B_1 x + \dots + B_{n-1}x^{n-1}) \in E_\alpha[x]$ is not identically 0. As E_α is an infinite field, there exists a $u_0 \in E_\alpha$ with $\det B(u_0) \neq 0$. Now we have $\rho(h) = B(u_0)^{-1}\rho_\alpha(h)B(u_0)$ for all $h \in H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$, ie. the representations V and V_α of $H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$ are isomorphic already over E_α. This shows that there exists a basis $v_1^{(\alpha)},\dots v_d^{(\alpha)}$ in V_α such that the action of each h in $H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$ is given by a matrix in $\mathrm{GL}_d(\mathbb{F}_p)$ in this basis. We put

$$V_{\Delta\setminus\{\alpha\}} := \mathbb{F}_p v_1^{(\alpha)} + \dots \mathbb{F}_p v_d^{(\alpha)} \subset V_\alpha = \left(E_\alpha^{sep}\otimes_{\mathbb{F}_p} V\right)^{H_{\mathbb{Q}_p,\alpha}} =$$
$$= \left(\bigotimes_{\beta\in\Delta\setminus\{\alpha\}} 1\otimes (E_\alpha^{sep}\otimes_{\mathbb{F}_p} V)\right)^{H_{\mathbb{Q}_p,\alpha}} \subseteq \left(E_{\Delta,\circ}^{sep}\otimes_{\mathbb{F}_p} V\right)^{H_{\mathbb{Q}_p,\alpha}} .$$

By induction we find a basis $v_1,\dots,v_n$ of $E_{\Delta\setminus\{\alpha\}}^{sep}\otimes_{\mathbb{F}_p} V_{\Delta\setminus\{\alpha\}} \subseteq \left(E_{\Delta,\circ}^{sep}\otimes_{\mathbb{F}_p} V\right)^{H_{\mathbb{Q}_p,\alpha}}$ consisting of $H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$-invariant elements which are $H_{\mathbb{Q}_p,\alpha}$-invariant, as well, by construction. Therefore $v_1,\dots,v_n$ is an $H_{\mathbb{Q}_p,\Delta}$-invariant basis of $E_{\Delta,\circ}^{sep}\otimes_{\mathbb{F}_p} V$ as required. □

Lemma 3.5. *We have* $(E_\Delta^{sep})^\times \cap E_\Delta = E_\Delta^\times$.

Proof. Let u be arbitrary in $(E_\Delta^{sep})^\times \cap E_\Delta$. Since u is invariant under the action of $H_{\mathbb{Q}_p,\Delta}$, so is its inverse u^{-1} whence it also lies in E_Δ by Lemma 3.3. □

Lemma 3.6. *We have* $\bigcap_{\alpha\in\Delta}(E_\Delta^{sep})^{\varphi_\alpha=\mathrm{id}} = \mathbb{F}_p$.

Proof. The containment $\mathbb{F}_p \subseteq \bigcap_{\alpha\in\Delta}(E_\Delta^{sep})^{\varphi_\alpha=\mathrm{id}} \subseteq (E_\Delta^{sep})^{\varphi_s=\mathrm{id}}$ is obvious. On the other hand, let $u\in E_\Delta^{sep}$ be arbitrary such that $\varphi_\alpha(u)=u$ for all $\alpha\in\Delta$. Then we also have $u^p=\varphi_s(u)=u$ as φ_s is the absolute Frobenius on E_Δ^{sep}. Since E_Δ^{sep} is defined as an inductive limit, u lies in $E'_\Delta \cong (\bigotimes_{\alpha\in\Delta,\mathbb{F}_p}\mathbb{F}_{q_\alpha})[[X'_\alpha \mid \alpha\in\Delta]][X_\Delta]$ for some collection $E'_\alpha = \mathbb{F}_{q_\alpha}((X'_\alpha))$ ($\alpha\in\Delta$) of finite separable extensions of E_α. Note that $\bigotimes_{\alpha\in\Delta,\mathbb{F}_p}\mathbb{F}_{q_\alpha}$ is a finite étale algebra over $\mathbb{F}_p$, in particular, it is reduced. Therefore we have $|u^p|_{prod} = |u|_{prod}^p$. We deduce $|u|_{prod}=1$ unless $u=0$. In particular, u lies in $E_\Delta'^{+} = (\bigotimes_{\alpha\in\Delta,\mathbb{F}_p}\mathbb{F}_{q_\alpha})[[X'_\alpha \mid \alpha\in\Delta]]$. The constant term $u_0\in\bigotimes_{\alpha\in\Delta,\mathbb{F}_p}\mathbb{F}_{q_\alpha}$ also satisfies $\varphi_\alpha(u_0)=u_0$ for all $\alpha\in\Delta$. For a fixed $\alpha\in\Delta$ we choose an $\mathbb{F}_p$-basis $d_1,\dots,d_n$ of $\bigotimes_{\beta\in\Delta\setminus\{\alpha\},\mathbb{F}_p}\mathbb{F}_{q_\beta}$ and write $u_0=\sum_{i=1}^n c_i\otimes d_i$ with $c_i\in\mathbb{F}_{q_\alpha}$. This decomposition is unique and we compute

$$\sum_{i=1}^n c_i\otimes d_i = u_0 = \varphi_\alpha(u_0) = \sum_{i=1}^n c_i^p\otimes d_i\ .$$

We deduce $c_i=c_i^p$, ie. $c_i\in\mathbb{F}_p$ for all $1\le i\le n$. It follows by induction on $|\Delta|$ that u_0 lies in $\mathbb{F}_p$. Now $u-u_0$ is also fixed by each φ_α ($\alpha\in\Delta$), but we have $|u-u_0|_{prod}<1$. This implies by the discussion above that $u=u_0$ is in $\mathbb{F}_p$ as desired. □

Proposition 3.7. $\mathbb{D}(V)$ *is an étale* $T_{+,\Delta}$*-module over* E_Δ *of rank* $d:=\dim_{\mathbb{F}_p}V$. *Moreover, we have* $E_\Delta^{sep}\otimes_{E_\Delta}\mathbb{D}(V)\cong E_\Delta^{sep}\otimes_{\mathbb{F}_p}V$ *and*

$$V = \bigcap_{\alpha\in\Delta}(E_\Delta^{sep}\otimes_{E_\Delta}\mathbb{D}(V))^{\varphi_\alpha=\mathrm{id}}\ .$$

Proof. By Lemmata 3.3 and 3.4 $\mathbb{D}(V)$ is a free module of rank d over E_Δ. Moreover, the matrix of φ_α in any basis of $\mathbb{D}(V)$ is invertible in E_Δ^{sep}, therefore also in E_Δ by Lemma 3.5. So the action of $T_{+,\Delta}$ on $\mathbb{D}(V)$ is étale. The last statement is a direct consequence of Lemmata 3.4 and 3.6. □

Lemma 3.8. *For objects V, V_1, V_2 in $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta})$ we have $\mathbb{D}(V_1 \otimes_{\mathbb{F}_p} V_2) \cong \mathbb{D}(V_1) \otimes_{E_\Delta} \mathbb{D}(V_2)$ and $\mathbb{D}(V^*) \cong \mathbb{D}(V)^*$.*

Proof. We compute

$$\mathbb{D}(V_1 \otimes_{\mathbb{F}_p} V_2) = \left(E_\Delta^{sep} \otimes_{\mathbb{F}_p} V_1 \otimes_{\mathbb{F}_p} V_2\right)^{H_{\mathbb{Q}_p,\Delta}} \cong \Big((E_\Delta^{sep} \otimes_{\mathbb{F}_p} V_1) \otimes_{E_\Delta^{sep}} (E_\Delta^{sep} \otimes_{\mathbb{F}_p} V_2)\Big)^{H_{\mathbb{Q}_p,\Delta}} \cong$$
$$\Big((E_\Delta^{sep} \otimes_{E_\Delta} \mathbb{D}(V_1)) \otimes_{E_\Delta^{sep}} (E_\Delta^{sep} \otimes_{E_\Delta} \mathbb{D}(V_2))\Big)^{H_{\mathbb{Q}_p,\Delta}} \cong$$
$$\cong \left(E_\Delta^{sep} \otimes_{E_\Delta} (\mathbb{D}(V_1) \otimes_{E_\Delta} \mathbb{D}(V_2))\right)^{H_{\mathbb{Q}_p,\Delta}} \cong \mathbb{D}(V_1) \otimes_{E_\Delta} \mathbb{D}(V_2)\ .$$

For the second statement we have

$$\mathbb{D}(V^*) = \left(E_\Delta^{sep} \otimes_{\mathbb{F}_p} \mathrm{Hom}_{\mathbb{F}_p}(V, \mathbb{F}_p)\right)^{H_{\mathbb{Q}_p,\Delta}} \cong \mathrm{Hom}_{E_\Delta^{sep}}(E_\Delta^{sep} \otimes_{\mathbb{F}_p} V, E_\Delta^{sep})^{H_{\mathbb{Q}_p,\Delta}} \cong$$
$$\cong \mathrm{Hom}_{E_\Delta^{sep}}(E_\Delta^{sep} \otimes_{E_\Delta} \mathbb{D}(V), E_\Delta^{sep})^{H_{\mathbb{Q}_p,\Delta}} \cong \left(E_\Delta^{sep} \otimes_{E_\Delta} \mathrm{Hom}_{E_\Delta}(\mathbb{D}(V), E_\Delta)\right)^{H_{\mathbb{Q}_p,\Delta}} \cong \mathbb{D}(V)^*\ .$$

□

Theorem 3.9. *$\mathbb{D}$ is a fully faithful tensor functor from the category $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta})$ to the category $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, E_\Delta)$.*

Proof. Let $f\colon V_1 \to V_2$ be a nonzero morphism in $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta})$. Then the E_Δ^{sep}-linear map $\mathrm{id} \otimes f\colon E_\Delta^{sep} \otimes_{\mathbb{F}_p} V_1 \to E_\Delta^{sep} \otimes_{\mathbb{F}_p} V_2$ is also nonzero. By the last statement in Prop. 3.7 it follows that $\mathbb{D}(f) \neq 0$ therefore the faithfulness.

Now let V_1 and V_2 be arbitrary objects in $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta})$ and $\theta\colon \mathbb{D}(V_1) \to \mathbb{D}(V_2)$ be a morphism in $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, E_\Delta)$. Then by Prop. 3.7 we obtain a $G_{\mathbb{Q}_p,\Delta}$-equivariant $\mathbb{F}_p$-linear map

$$f\colon V_1 = \bigcap_{\alpha\in\Delta} (E_\Delta^{sep} \otimes_{E_\Delta} \mathbb{D}(V_1))^{\varphi_\alpha = \mathrm{id}} \to \bigcap_{\alpha\in\Delta} (E_\Delta^{sep} \otimes_{E_\Delta} \mathbb{D}(V_2))^{\varphi_\alpha = \mathrm{id}} = V_2$$

induced by θ for which we have $\theta = \mathbb{D}(f)$. Therefore $\mathbb{D}$ is full. The compatibility with tensor products is proven in Lemma 3.8. □

Remark. Note that any étale $T_{+,\Delta}$-module D in the image of the functor $\mathbb{D}$ is free as a module over E_Δ by construction.

Consider the diagonal embedding $\mathrm{diag}\colon G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}_p,\Delta}$ sending $g \in G_{\mathbb{Q}_p}$ to $(g, \ldots, g)$. This defines a functor $\widehat{\mathrm{diag}}\colon \mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta}) \to \mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p})$ via restriction. On the other hand, we have the reduction map $\ell\colon \mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, E_\Delta) \to \mathcal{D}^{et}(\varphi, \Gamma, E)$ to usual (φ, Γ)-modules defined in section 2.4 of [11]. Recall that this is given by taking the quotient by the ideal generated by $(X_\alpha - X_\beta \mid \alpha, \beta \in \Delta)$ and restricting to the diagonal $\varphi = \varphi_s = \prod_{\alpha\in\Delta} \varphi_\alpha$ and $\Gamma := \{(\gamma, \ldots, \gamma)\} \leq \Gamma_\Delta$.

Corollary 3.10. *There is a natural isomorphism $\widehat{\mathrm{diag}} \cong \mathbb{V}_F \circ \ell \circ \mathbb{D}$ of functors $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta}) \to \mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p})$ where $\mathbb{V}_F\colon \mathcal{D}^{et}(\varphi, \Gamma, E) \to \mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p})$ is Fontaine's functor from classical étale (φ, Γ)-modules to Galois representations.*

Proof. We may identify $E_\alpha \xrightarrow{\sim} E = \mathbb{F}_p((X))$ by sending $X_\alpha \to X$ for all $\alpha \in \Delta$. We extend this identification to $E_\alpha^{sep} \to E^{sep}$. So we obtain a map $\ell^{sep}\colon E_\Delta^{sep} \to E^{sep}$ sending each subring E_α^{sep} to E^{sep} via these identifications and completing on the level of each finite extension E'_Δ. We do this in a way so that the diagonal embedding of $G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}_p,\Delta}$ acts on the quotient E^{sep} in the usual way. The restriction of ℓ^{sep} to E_Δ is the map $\ell\colon E_\Delta \to E$ defined above, so the diagram

$$\begin{tikzcd} E_\Delta \arrow[r, hook] \arrow[d, "\ell"'] & E_\Delta^{sep} \arrow[d, "\ell^{sep}"] \\ E \arrow[r, hook] & E^{sep} \end{tikzcd}$$

commutes. Thus for an object V in $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta})$ we compute

$$\begin{aligned} \mathbb{V}_F \circ \ell \circ \mathbb{D}(V) = \mathbb{V}_F(E \otimes_{E_\Delta,\ell} \mathbb{D}(V)) = \mathbb{V}_F((E^{sep})^{H_{\mathbb{Q}_p}} \otimes_{E_\Delta,\ell} \mathbb{D}(V)) = \\ = \mathbb{V}_F((E^{sep} \otimes_{E_\Delta^{sep},\ell^{sep}} E_\Delta^{sep} \otimes_{E_\Delta} \mathbb{D}(V))^{H_{\mathbb{Q}_p}}) = \mathbb{V}_F((E^{sep} \otimes_{E_\Delta^{sep},\ell^{sep}} E_\Delta^{sep} \otimes_{\mathbb{F}_p} V)^{H_{\mathbb{Q}_p}}) = \\ = \mathbb{V}_F((E^{sep} \otimes_{\mathbb{F}_p} V)^{H_{\mathbb{Q}_p}}) = \mathbb{V}_F \circ \mathbb{D}_F(V) = V\mid_{\mathrm{diag}(G_{\mathbb{Q}_p})} = \widehat{\mathrm{diag}}(V)\ . \end{aligned}$$

□

3.2 The functor $\mathbb{V}$

In order to show that the functor $\mathbb{D}$ is essentially surjective, we construct its quasi-inverse $\mathbb{V}$. Let D be an object in $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, E_\Delta)$. The group $G_{\mathbb{Q}_p,\Delta}$ acts on $E_\Delta^{sep} \otimes_{E_\Delta} D$ via the formula $g(\lambda \otimes x) := g(\lambda) \otimes \chi_{cyc}(g)(x)$ ($g \in G_{\mathbb{Q}_p,\Delta}$, $\lambda \in E_\Delta^{sep}$, $x \in D$) where $\chi_{cyc}\colon G_{\mathbb{Q}_p,\Delta} \to \Gamma_\Delta$ is the quotient map. Moreover, each partial Frobenius φ_α ($\alpha \in \Delta$) acts semilinearly on $E_\Delta^{sep} \otimes_{E_\Delta} D$ via the formula $\varphi_\alpha(\lambda \otimes x) := \varphi_\alpha(\lambda) \otimes \varphi_\alpha(x)$. All these actions commute with each other by construction. We define

$$\mathbb{V}(D) := \bigcap_{\alpha \in \Delta} (E_\Delta^{sep} \otimes_{E_\Delta} D)^{\varphi_\alpha = \mathrm{id}}\ .$$

$\mathbb{V}(D)$ is a—a priori not necessarily finite dimensional—representation of $G_{\mathbb{Q}_p,\Delta}$ over $\mathbb{F}_p$.

Lemma 3.11. *For any integer $r > 0$ we have $\bigcap_{\beta\in\Delta}(E_{\Delta\setminus\{\alpha\}}^{sep}[X_\alpha]/(X_\alpha^r))^{\varphi_\beta=\mathrm{id}} = \mathbb{F}_p[X_\alpha]/(X_\alpha^r)$.*

Proof. This follows from Lemma 3.6 noting that $\mathbb{F}_p[X_\alpha]/(X_\alpha^r)$ is a finite dimensional $\mathbb{F}_p$-vector space on which φ_β acts identically for all $\beta \in \Delta \setminus \{\alpha\}$ and we have $E_{\Delta\setminus\{\alpha\}}^{sep}[X_\alpha]/(X_\alpha^r) \cong E_{\Delta\setminus\{\alpha\}}^{sep} \otimes_{\mathbb{F}_p} \mathbb{F}_p[X_\alpha]/(X_\alpha^r)$. □

Lemma 3.12. *For any integer $r > 0$ and finitely generated $E_{\overline{\alpha}}^{+}/(X_\alpha^r)$-module M we have an identification $E_{\Delta\setminus\{\alpha\}}^{sep}[X_\alpha]/(X_\alpha^r) \otimes_{E_{\overline{\alpha}}^{+}/(X_\alpha^r)} M \cong E_{\Delta\setminus\{\alpha\}}^{sep} \otimes_{E_{\Delta\setminus\{\alpha\}}} M$.*

Proof. This follows from the isomorphism $E_{\overline{\alpha}}^{+}/(X_\alpha^r) \cong E_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)$. □

For a subset $S \subseteq \Delta$ we put $E_S^{sep+} := \varinjlim E_S'^{+}$ so we have $E_S^{sep} = E_S^{sep+}[X_S^{-1}]$.

Lemma 3.13. *E_S^{sep} (resp. E_S^{sep+}) is flat as a module over E_S (resp. over E_S^+) for all $S \subseteq \Delta$.*

Proof. By construction, E'_S (resp. E'^+_S) is finite free over E_S (resp. over E^+_S), so E^{sep}_S (resp. E^{sep+}_S) is the direct limit of flat modules hence flat. □

Lemma 3.14. *We have* $(E^{sep+}_{\Delta\setminus\{\alpha\}}[[X_\alpha]][X_\Delta^{-1}])^{H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}} = E_\Delta$.

Proof. We have $E_\Delta = E^+_{\Delta\setminus\{\alpha\}}[[X_\alpha]][X_\Delta^{-1}]$ where $E^+_{\Delta\setminus\{\alpha\}} = (E^{sep+}_{\Delta\setminus\{\alpha\}})^{H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}}$ by Lemma 3.3 and $H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$ acts trivially on both X_α and X_Δ, so acts on the power series ring $E^+_{\Delta\setminus\{\alpha\}}[[X_\alpha]]$ coefficientwise. □

Our main result in this section is the following

Theorem 3.15. *The functors* $\mathbb{D}$ *and* $\mathbb{V}$ *are quasi-inverse equivalences of categories between the Tannakian categories* $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta})$ *and* $\mathcal{D}^{et}(\varphi_\Delta,\Gamma_\Delta,E_\Delta)$.

Corollary 3.16. *Any object* D *in* $\mathcal{D}^{et}(\varphi_\Delta,\Gamma_\Delta,E_\Delta)$ *is a free module over* E_Δ.

Proof. This follows from the essential surjectivity of $\mathbb{D}$ using the remark after Thm. 3.9. □

Proof of Thm. 3.15. This is a long proof that we divide into 5 steps.

Step 1. Reducing the statement to the essential surjectivity of $\mathbb{D}$. By Thm. 3.9 the functor $\mathbb{D}$ is fully faithful and we have $\mathbb{V}\circ\mathbb{D}(V)\cong V$ naturally in V for any object V in $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta})$ by Prop. 3.7. Moreover, by Lemma 3.8 $\mathbb{D}$ is compatible with tensor products and duals. So it remains to show that $\mathbb{D}$ is essentially surjective. We proceed by induction on $|\Delta|$. For $|\Delta|=1$ this is a classical result of Fontaine (see e.g. Thm. 2.21 in [5]). Suppose that $|\Delta|>1$, fix $\alpha\in\Delta$, and pick an object D in $\mathcal{D}^{et}(\varphi_\Delta,\Gamma_\Delta,E_\Delta)$.

Step 2. The goal here is to trivialize the φ_β*-action (*$\beta\in\Delta\setminus\{\alpha\}$*) on* $D^{+*}_{\overline{\alpha}}/X^r_\alpha$ *uniformly in* r *by tensoring up with* $E^{sep}_{\Delta\setminus\{\alpha\}}$. By Prop. 2.10 $D^{+*}_{\overline{\alpha}}$ is an étale $T_{+,\overline{\alpha}}$-module over $E^+_{\overline{\alpha}}$. Reducing mod X^r_α for an integer $r>0$ we deduce that $D^{+*}_{\overline{\alpha},r} := D^{+*}_{\overline{\alpha}}/X^r_\alpha D^{+*}_{\overline{\alpha}}$ is an étale $T_{+,\overline{\alpha}}$-module over $E^+_{\overline{\alpha}}/(X^r_\alpha)\cong E_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X^r_\alpha)$. Since each φ_β ($\beta\in\Delta\setminus\{\alpha\}$) acts trivially on the variable X_α, we have a natural isomorphism of functors

$$E_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X^r_\alpha)\otimes_{E_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X^r_\alpha),\varphi_t}\cdot\cong E_{\Delta\setminus\{\alpha\}}\otimes_{E_{\Delta\setminus\{\alpha\}},\varphi_t}\cdot$$

for all $t\in T_{+,\overline{\alpha}}$. Hence $D^{+*}_{\overline{\alpha},r}$ is an object in $\mathcal{D}^{et}(\varphi_{\Delta\setminus\{\alpha\}},\Gamma_{\Delta\setminus\{\alpha\}},E_{\Delta\setminus\{\alpha\}})$ since $E_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X^r_\alpha)$ is finitely generated as a module over $E_{\Delta\setminus\{\alpha\}}$. By induction, we can trivialize $D^{+*}_{\overline{\alpha},r}$ over $E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X^r_\alpha)$: the natural map

$$E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X^r_\alpha)\otimes_{\mathbb{F}_p[X_\alpha]/(X^r_\alpha)}\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X^r_\alpha)\otimes_{E^+_{\overline{\alpha}}/(X^r_\alpha)}D^{+*}_{\overline{\alpha},r}\right)^{\varphi_\beta=\mathrm{id}}\xrightarrow{\sim}$$
$$\xrightarrow{\sim}E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X^r_\alpha)\otimes_{E^+_{\overline{\alpha}}/(X^r_\alpha)}D^{+*}_{\overline{\alpha},r}\cong E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X^r_\alpha)\otimes_{E^+_{\overline{\alpha}}}D^{+*}_{\overline{\alpha}}\qquad(4)$$

is an isomorphism for all $r>0$ using Lemmata 3.11 and 3.12. Our key Lemma is the following consequence of Prop. 2.10.

Lemma 3.17. *There exists a finitely generated* E^+_Δ*-submodule* $M\le D^{+*}_{\overline{\alpha}}$ *such that*

$$\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X^r_\alpha)\otimes_{E^+_{\overline{\alpha}}}D^{+*}_{\overline{\alpha},}\right)^{\varphi_\beta=\mathrm{id}}\qquad(5)$$

is contained in the image of the map

$$E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_\Delta} M \to E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_\Delta} D^{+*}_{\overline{\alpha}} \cong E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_{\overline{\alpha}}} D^{+*}_{\overline{\alpha}} \quad (6)$$

induced by the inclusion $M \leq D^{+*}_{\overline{\alpha}}$ for all $r>0$. Moreover, M can be chosen in such a way that (6) is injective.

Proof. We show that $M := X^{-k}_{\Delta\setminus\{\alpha\}}(D^+\cap D^{+*}_{\overline{\alpha}})$ will do for k large enough. Since D^+ is finitely generated over E^+_Δ, so is M by noetherianity. Using Lemma 2.11 we choose $k>0$ so that we have $D^{+*}_{\overline{\alpha}} = \bigcup_{l\geq 0} E^+_\Delta \varphi^l_{\overline{\alpha}}(M)$, ie. we put $M := X^{-1}_{\Delta\setminus\{\alpha\}} D_0$. For any fixed $r>0$ there exists an integer $l_r\geq 0$ such that (5) is contained in

$$E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_\Delta} X^{-p^{l_r}+1}_{\Delta\setminus\{\alpha\}} M \subseteq E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_\Delta} E^+_\Delta\varphi^{l_r}_{\overline{\alpha}}(M) =$$
$$= E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\varphi^{l_r}_{\overline{\alpha}}(E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_\Delta} M)\ .$$

Now if x lies in (5), then we have $\varphi^{l_r}_{\overline{\alpha}}(x)=x$. On the other hand, x lies in

$$E'_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\varphi^{l_r}_{\overline{\alpha}}(E'_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_\Delta} M)$$

for some finite separable extensions E'_β/E_β for $\beta\in\Delta\setminus\{\alpha\}$ and $E'_{\Delta\setminus\{\alpha\}} := \widehat{\bigotimes}_{\beta\in\Delta\setminus\{\alpha\},\mathbb{F}_p} E'_\beta$. Therefore x lies in fact in $E'_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_\Delta} M$ by the injectivity of the map

$$\mathrm{id}\otimes\varphi^{l_r}_{\overline{\alpha}}\colon E'_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E'_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r),\varphi^{l_r}_{\overline{\alpha}}} (E'_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_{\overline{\alpha}}} D^{+*}_{\overline{\alpha}}) \to$$
$$\to E'_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_{\overline{\alpha}}} D^{+*}_{\overline{\alpha}}$$

($D^{+*}_{\overline{\alpha}}$ is étale) noting that the absolute Frobenius $\varphi_{\overline{\alpha}}\colon E'_{\Delta\setminus\{\alpha\}}\to E'_{\Delta\setminus\{\alpha\}}$ is injective since the ring $E'_{\Delta\setminus\{\alpha\}}$ is the localization of a power series ring over a finite étale algebra over $\mathbb{F}_p$, in particular, it is reduced.

Finally, by Lemma 2.7 $D^{+*}_{\overline{\alpha}}/M$ has no X_α-torsion as $D^{+*}_{\overline{\alpha}}/M \cong D^{+*}_{\overline{\alpha}}+X^{-k}_{\Delta\setminus\{\alpha\}}D^+/(X^{-k}_{\Delta\setminus\{\alpha\}}D^+)$ is contained in $D^+_{\overline{\alpha}}/(X^{-k}_{\Delta\setminus\{\alpha\}}D^+)\cong D^+_{\overline{\alpha}}/D^+$. Therefore the map (6) is injective. □

Step 3. The goal here is to show the following compatibility of our construction with projective limits with respect to r.

Lemma 3.18. *We have*

$$\varprojlim_r \left(E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_\Delta} M\right) \cong E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{E^+_\Delta} M\ ,$$
$$\varprojlim_r \left(E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_{\overline{\alpha}}} D^{+*}_{\overline{\alpha}}\right) \cong E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{E^+_{\overline{\alpha}}} D^{+*}_{\overline{\alpha}} \quad ,\ and$$
$$\varprojlim_r \left(E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{\mathbb{F}_p[X_\alpha]/(X_\alpha^r)} \bigcap_{\beta\in\Delta\setminus\{\alpha\}} \left(E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_{\overline{\alpha}}/(X_\alpha^r)} D^{+*}_{\overline{\alpha},r}\right)^{\varphi_\beta=\mathrm{id}}\right) \cong$$
$$\cong E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{\mathbb{F}_p[\![X_\alpha]\!]} \bigcap_{\beta\in\Delta\setminus\{\alpha\}} \left(E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!)\otimes_{E_\Delta} D\right)^{\varphi_\beta=\mathrm{id}}\ .$$

Proof. Since M is contained in D, M has no X_α-torsion. In particular, M is flat as a module over the local ring $\mathbb{F}_p[\![X_\alpha]\!]$. Now we deduce that M and $E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)$ are Tor-independent over E^+_Δ by Lemma 3.13 since we have the identification

$$E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_\Delta} M \cong E^{sep+}_{\Delta\setminus\{\alpha\}}\otimes_{E^+_{\Delta\setminus\{\alpha\}}}\left(\mathbb{F}_p[X_\alpha]/(X_\alpha^r)\otimes_{\mathbb{F}_p[\![X_\alpha]\!]} M\right).$$

On the other hand, M is finitely generated over E^+_Δ, so we short exact sequences

$$0\to M_1\to (E^+_\Delta)^{k_0}\overset{f_0}{\to} M\to 0 \qquad\text{and}\qquad 0\to M_2\to (E^+_\Delta)^{k_1}\to M_1\to 0$$

by noetherianity. In order to simplify notation write $(\cdot)_r$ for $E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_\Delta}\cdot$ to obtain an exact sequence

$$(M_2)_r\to (E^+_\Delta)_r^{k_1}\overset{f_{1,r}}{\to}(E^+_\Delta)_r^{k_0}\overset{f_{0,r}}{\to}(M)_r\to 0$$

for all $r>0$ using the Tor-indepence above. Now since the natural map $(N)_{r_1}\to(N)_{r_2}$ is surjective for any E^+_Δ-module N and $r_1\geq r_2>0$ by the right exactness of $\cdot\otimes_{E^+_\Delta}N$, the natural map $\mathrm{Ker}(f_{0,r_1})\to\mathrm{Ker}(f_{0,r_2})$ is also surjective (applying this in case $N=M_1$ and a diagram chasing). So the Mittag-Leffler property is satisfied for these projective systems showing that the map $\varprojlim_r f_{0,r}$ is surjective with kernel $\varprojlim_r\mathrm{Ker}(f_{0,r})=\varprojlim_r\mathrm{Im}(f_{1,r})$. Applying the same trick as above with $N=M_2$ we deduce that the projective system $\mathrm{Ker}(f_{1,r})$ also satisfies the Mittag-Leffler property showing that $\varprojlim_r f_{1,r}$ has image $\varprojlim_r\mathrm{Im}(f_{1,r})$. In particular, $\varprojlim_r(M)_r$ is the cokernel of the map $\varprojlim_r f_{1,r}\colon (E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!])^{k_1}\to(E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!])^{k_0}$ and so is $E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{E^+_\Delta}M$ as claimed. The second statement follows in the exactly same way.

For the third statement note that the isomorphism (4) and the surjectivity of the map $E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^{r_1})\otimes_{E^+_{\overline{\alpha}}}D^{+*}_{\overline{\alpha}}\to E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^{r_2})\otimes_{E^+_{\overline{\alpha}}}D^{+*}_{\overline{\alpha}}$ implies that the map

$$\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^{r_1})\otimes_{E^+_{\overline{\alpha}}/(X_\alpha^{r_1})}D^{+*}_{\overline{\alpha},r}\right)^{\varphi_\beta=\mathrm{id}}\to$$
$$\to\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^{r_2})\otimes_{E^+_{\overline{\alpha}}/(X_\alpha^{r_2})}D^{+*}_{\overline{\alpha},r}\right)^{\varphi_\beta=\mathrm{id}}$$

is also onto for all $r_1\geq r_2$. Therefore the natural map

$$\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{E^+_{\overline{\alpha}}}D^{+*}_{\overline{\alpha}}\right)^{\varphi_\beta=\mathrm{id}}=$$
$$=\varprojlim_r\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^{r})\otimes_{E^+_{\overline{\alpha}}/(X_\alpha^{r})}D^{+*}_{\overline{\alpha},r}\right)^{\varphi_\beta=\mathrm{id}}\to$$
$$\to\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha)\otimes_{E^+_{\overline{\alpha}}/(X_\alpha)}D^{+*}_{\overline{\alpha},r}\right)^{\varphi_\beta=\mathrm{id}}$$

is also onto using the second statement of the Lemma. On the other hand, the kernel of this map equals

$$\bigcap_{\beta\in\Delta\setminus\{\alpha\}} \left(E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!] \otimes_{E^+_{\alpha}} D^{+*}_{\alpha}\right)^{\varphi_\beta=\mathrm{id}} \cap X_\alpha E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!] \otimes_{E^+_{\alpha}} D^{+*}_{\alpha} =$$

$$= X_\alpha \bigcap_{\beta\in\Delta\setminus\{\alpha\}} \left(E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!] \otimes_{E^+_{\alpha}} D^{+*}_{\alpha}\right)^{\varphi_\beta=\mathrm{id}}$$

since X_α is fixed by each φ_β and $E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!] \otimes_{E^+_{\alpha}} D^{+*}_{\alpha}$ has no X_α-torsion. This shows, in particular, that $\bigcap_{\beta\in\Delta\setminus\{\alpha\}} \left(E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!] \otimes_{E^+_{\alpha}} D^{+*}_{\alpha}\right)^{\varphi_\beta=\mathrm{id}}$ is finitely generated over $\mathbb{F}_p[\![X_\alpha]\!]$ by the topological Nakayama Lemma (see [1]). Moreover, it is torsion-free hence free as $E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!] \otimes_{E^+_{\alpha}} D^{+*}_{\alpha}$ has no X_α-torsion either. In particular,

$$E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!] \otimes_{\mathbb{F}_p[\![X_\alpha]\!]} \bigcap_{\beta\in\Delta\setminus\{\alpha\}} \left(E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!) \otimes_{E_\Delta} D\right)^{\varphi_\beta=\mathrm{id}}$$

is X_α-adically complete and the result follows. $\square$

Step 4. The goal here is to obtain a $(\varphi_\alpha, \Gamma_\alpha)$-module D_α over E_α (by trivializing the action of each φ_β, $\beta \in \Delta \setminus \{\alpha\}$) which is at the same time a linear representation of the group $G_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$. We take projective limits of the inclusions in Lemma 3.17 with respect to r to conclude (using Lemma 3.18) that

$$\bigcap_{\beta\in\Delta\setminus\{\alpha\}} \left(E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!] \otimes_{E^+_{\alpha}} D^{+*}_{\alpha}\right)^{\varphi_\beta=\mathrm{id}}$$

is contained in the image of the map

$$E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!] \otimes_{E^+_\Delta} M \to E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!] \otimes_{E^+_{\alpha}} D^{+*}_{\alpha}\ .$$

Note that $M[X_\Delta^{-1}] = D^{+*}_{\alpha}[X_\Delta^{-1}] = D^{+*}_{\alpha}[X_\alpha^{-1}] = D$ and φ_β acts trivially on X_α. So inverting X_Δ above we deduce that

$$D_\alpha := \bigcap_{\beta\in\Delta\setminus\{\alpha\}} \left(E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!) \otimes_{E_\Delta} D\right)^{\varphi_\beta=\mathrm{id}}$$

is contained in the image of the map

$$E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!][X_\Delta^{-1}] \otimes_{E_\Delta} D \hookrightarrow E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!) \otimes_{E_\Delta} D\ .$$

On the other hand, by (4) and the third statement of Lemma 3.18 we have an isomorphism

$$E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!) \otimes_{\mathbb{F}_p(\!(X_\alpha)\!)} D_\alpha \xrightarrow{\sim} E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!) \otimes_{E_\Delta} D\ . \tag{7}$$

Lemma 3.19. *The finite dimensional $\mathbb{F}_p(\!(X_\alpha)\!)$-vector space D_α has the structure of an étale $(\varphi_\alpha, \Gamma_\alpha)$-module. At the same time it is a (linear) representation of the group $G_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$. These two actions commute with each other.*

Proof. The operator φ_α and the groups Γ_α and $G_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$ act naturally on D_α. For the étaleness of the action of φ_α on D_α note that we have $\mathbb{F}_p(\!(X_\alpha)\!)\otimes_{\mathbb{F}_p(\!(X_\alpha)\!),\varphi_\alpha} D\cong D$ by the étale property of φ_α on D and that φ_β acts trivially on $\mathbb{F}_p(\!(X_\alpha)\!)$. So we compute

$$\begin{aligned}\mathbb{F}_p(\!(X_\alpha)\!)\otimes_{\mathbb{F}_p(\!(X_\alpha)\!),\varphi_\alpha} D_\alpha = \mathbb{F}_p(\!(X_\alpha)\!)\otimes_{\mathbb{F}_p(\!(X_\alpha)\!),\varphi_\alpha} \bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!)\otimes_{E_\Delta} D\right)^{\varphi_\beta=\mathrm{id}} = \\ = \bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(\mathbb{F}_p(\!(X_\alpha)\!)\otimes_{\mathbb{F}_p(\!(X_\alpha)\!),\varphi_\alpha} E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!)\otimes_{E_\Delta} D\right)^{\varphi_\beta=\mathrm{id}} = \\ = \bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!)\otimes_{E_\Delta} \mathbb{F}_p(\!(X_\alpha)\!)\otimes_{\mathbb{F}_p(\!(X_\alpha)\!),\varphi_\alpha} D\right)^{\varphi_\beta=\mathrm{id}} \cong \\ \cong \bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!)\otimes_{E_\Delta} D\right)^{\varphi_\beta=\mathrm{id}} = D_\alpha\ .\end{aligned}$$

□

Step 5. We show the essential surjectivity of $\mathbb{D}$ here. Now we apply $\mathbb{V}_{F,\alpha}=(E_\alpha^{sep}\otimes_{\mathbb{F}_p(\!(X_\alpha)\!)}\cdot)^{\varphi_\alpha=\mathrm{id}}$ on D_α to obtain a finite dimensional $\mathbb{F}_p$-representation V of $G_{\mathbb{Q}_p,\Delta}$. Moreover, we have $\dim_{\mathbb{F}_p} V=\dim_{\mathbb{F}_p(\!(X_\alpha)\!)} D_\alpha=\mathrm{rk}_{E_\Delta} D$ by the isomorphism (7) since $\mathbb{V}_{F,\alpha}$ is rank-preserving by Fontaine's classical result. Using again the isomorphism (7) and the containment $D_\alpha\subset E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!][X_\Delta^{-1}]\otimes_{E_\Delta} D$ we conclude an injective map

$$E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!][X_\Delta^{-1}]\otimes_{\mathbb{F}_p(\!(X_\alpha)\!)} D_\alpha\hookrightarrow E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!][X_\Delta^{-1}]\otimes_{E_\Delta} D$$

and applying $E_\alpha^{sep}\otimes_{\mathbb{F}_p(\!(X_\alpha)\!)}\cdot$ another injective composite map

$$\begin{aligned}E_\Delta^{sep}\otimes_{\mathbb{F}_p} V\hookrightarrow \\ \hookrightarrow\left(E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!][X_\Delta^{-1}]\otimes_{\mathbb{F}_p(\!(X_\alpha)\!)} E_\alpha^{sep}\right)\otimes_{\mathbb{F}_p} V\cong \\ \cong E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!][X_\Delta^{-1}]\otimes_{\mathbb{F}_p(\!(X_\alpha)\!)} E_\alpha^{sep}\otimes_{\mathbb{F}_p(\!(X_\alpha)\!)} D_\alpha = \\ = E_\alpha^{sep}\otimes_{\mathbb{F}_p(\!(X_\alpha)\!)} E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!][X_\Delta^{-1}]\otimes_{\mathbb{F}_p(\!(X_\alpha)\!)} D_\alpha\hookrightarrow \\ \hookrightarrow\left(E_\alpha^{sep}\otimes_{\mathbb{F}_p(\!(X_\alpha)\!)} E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!][X_\Delta^{-1}]\right)\otimes_{E_\Delta} D\ .\end{aligned}$$

Taking $G_{\mathbb{Q}_p,\Delta}$-invariants of this inclusion we deduce an inclusion $\mathbb{D}(V)\hookrightarrow D$ using Lemma 3.14. However, this is an isomorphism by Prop. 2.1 in [11] as $\mathbb{D}(V)$ and D have the same rank. □

Remarks. 1. Even though we have constructed V in the proof of the above theorem by a different procedure from just putting $V:=\mathbb{V}(D)$, we still have an isomorphism $V\cong\mathbb{V}(\mathbb{D}(V))\cong\mathbb{V}(D)$ by Prop. 3.7.

2. If κ is a finite extension of $\mathbb{F}_p$, then we have an equivalence of categories between $\mathrm{Rep}_\kappa(G_{\mathbb{Q}_p,\Delta})$ and $\mathcal{D}^{et}(\varphi_\Delta,\Gamma_\Delta,\kappa\otimes_{\mathbb{F}_p}E_\Delta)$. Indeed, we have a natural isomorphism $(\kappa\otimes_{\mathbb{F}_p}E_\Delta^{sep})\otimes_\kappa\cdot\cong E_\Delta^{sep}\otimes_{\mathbb{F}_p}\cdot$ as functors on $\mathrm{Rep}_\kappa(G_{\mathbb{Q}_p,\Delta})$.

4 The case of p-adic representations

4.1 Cohomological preliminaries

We will need the following multivariable analogue of Hilbert's Theorem 90 (additive form).

Proposition 4.1. *The continuous group cohomology $H^1_{cont}(H_{\mathbb{Q}_p,\Delta}, E_\Delta^{sep})$ vanishes.*

Proof. By Prop. 3.3 it suffices to show that for finite Galois extensions E'_α/E_α (for all $\alpha \in \Delta$) with Galois group $H'_\alpha := \mathrm{Gal}(E'_\alpha/E_\alpha)$ we have $H^1(H', E'_\Delta) = \{1\}$ where we put $H' := \prod_{\alpha\in\Delta} H'_\alpha$. Choose a normal basis $e_1, \dots, e_{n_\alpha} \in E'_\alpha$ over E_α for each $\alpha \in \Delta$. By Lemma 3.2 the set $\{\prod_{\alpha\in\Delta} e_{i_\alpha} \mid 1 \leq i_\alpha \leq n_\alpha,\ \alpha \in \Delta\}$ is a basis of the free E_Δ-module E'_Δ. In particular, $E'_\Delta \cong E_\Delta[H']$ is induced as an H'-module whence the cohomology group $H^1(H', E'_\Delta)$ is trivial. $\square$

Let D be an abelian group admitting an action of the commutative monoid $\prod_{\alpha\in\Delta} \varphi_\alpha^{\mathbb{N}}$. Fix a total ordering $<$ on Δ and consider the complex

$$\Phi^\bullet(D)\colon 0 \to D \to \bigoplus_{\alpha\in\Delta} D \to \cdots \to \bigoplus_{\{\alpha_1,\dots,\alpha_r\}\in\binom{\Delta}{r}} D \to \cdots \to D \to 0$$

where for all $0 \leq r \leq |\Delta| - 1$ the map $d^{\beta_1,\dots,\beta_{r+1}}_{\alpha_1,\dots,\alpha_r}\colon D \to D$ from the component in the rth term corresponding to $\{\alpha_1, \dots, \alpha_r\} \subseteq \Delta$ to the component corresponding to the $(r+1)$-tuple $\{\beta_1, \dots, \beta_{r+1}\} \subseteq \Delta$ is given by

$$d^{\beta_1,\dots,\beta_{r+1}}_{\alpha_1,\dots,\alpha_r} = \begin{cases} 0 & \text{if } \{\alpha_1,\dots,\alpha_r\} \not\subseteq \{\beta_1,\dots,\beta_{r+1}\} \\ (-1)^\varepsilon(\mathrm{id} - \varphi_\beta) & \text{if } \{\beta_1,\dots,\beta_{r+1}\} = \{\alpha_1,\dots,\alpha_r\} \cup \{\beta\}\ , \end{cases}$$

where $\varepsilon = \varepsilon(\alpha_1, \dots, \alpha_r, \beta)$ is the number of elements in the set $\{\alpha_1, \dots, \alpha_r\}$ smaller than β. Since the operators $(\mathrm{id} - \varphi_\beta)$ commute with each other, $\Phi^\bullet(D)$ is a chain complex of abelian groups. Note that for each $\alpha \in \Delta$ we have a complex

$$\Phi^\bullet_\alpha(D)\colon 0 \to D \overset{\mathrm{id}-\varphi_\alpha}{\to} D \to 0$$

such that $\Phi^\bullet(E_\Delta^{sep})$ is a kind of completed tensor product of the complexes $\Phi^\bullet_\alpha(E_\alpha^{sep})$. More precisely, the tensor product over $\mathbb{F}_p$ of the complexes $\Phi^\bullet(E_\alpha^{sep})$ is the complex $\Phi^\bullet(E_{\Delta,\circ}^{sep})$ which is therefore acyclic in nonzero degrees with 0th cohomology equal to $\mathbb{F}_p$ by the Künneth formula. Note that there are no higher Tor's as the tensor product is taken over the field $\mathbb{F}_p$. We need the following completed version of this observation.

Proposition 4.2. *The complex $\Phi^\bullet(E_\Delta^{sep})$ is acyclic in nonzero degrees with 0th cohomology equal to $\mathbb{F}_p$.*

The following Lemma is well-known.

Lemma 4.3. *For any finite separable extension E'_α/E_α the map $\mathrm{id} - \varphi_\alpha\colon X'_\alpha E'^+_\alpha \to X'_\alpha E'^+_\alpha$ is bijective.*

Proof. The kernel of $\mathrm{id} - \varphi_\alpha$ is $\mathbb{F}_p$ which is not contained in $X'_\alpha E'^+_\alpha$. On the other hand, $\sum_{n=0}^\infty \varphi_\alpha^n$ converges on this set and is therefore an inverse to $\mathrm{id} - \varphi_\alpha$ by formal reasons. $\square$

Our key is the following

Lemma 4.4. *For all $\alpha \in S \subseteq \Delta$ the map* $\mathrm{id} - \varphi_\alpha \colon E_S^{sep} \to E_S^{sep}$ *is surjective with kernel* $E_{S\setminus\{\alpha\}}^{sep}$.

Proof. We may assume $S = \Delta$. The inclusion $E_{\Delta\setminus\{\alpha\}}^{sep} \subseteq \mathrm{Ker}(\mathrm{id} - \varphi_\alpha)$ is clear. For a collection $E_\beta \leq E'_\beta = \mathbb{F}_{q_\beta}((X'_\beta))$ $(\beta \in \Delta)$ of finite separable extensions the ring E'_Δ is embedded into $(E'_{\Delta\setminus\{\alpha\}} \otimes_{\mathbb{F}_p} \mathbb{F}_{q_\alpha})((X'_\alpha))$. By comparing the coefficients we find that $(E'_{\Delta\setminus\{\alpha\}} \otimes_{\mathbb{F}_p} \mathbb{F}_{q_\alpha})((X'_\alpha))^{\varphi_\alpha = \mathrm{id}} = E'_{\Delta\setminus\{\alpha\}}$.

For the surjectivity pick an element c in $E'_\Delta \subset E_\Delta^{sep}$ for some collection of finite separable extensions $E_\beta \leq E'_\beta = \mathbb{F}_{q_\beta}((X'_\beta))$ $(\beta \in \Delta)$. There exists an integer $k \geq 0$ such that c lies in $X_\Delta^{-k} E_\Delta'^{+} = \widehat{\bigotimes}_{\beta\in\Delta,\mathbb{F}_p} X_\beta^{-k} E_\beta'^{+}$. So we may write c as a convergent sum $c = \sum_{n=1}^\infty c_{\overline{\alpha},n} \otimes c_{\alpha,n}$ such that $c_{\overline{\alpha},n} \in X_{\Delta\setminus\{\alpha\}}^{-k} E_{\Delta\setminus\{\alpha\}}'^{+}$ with $c_{\overline{\alpha},n} \to 0$ and $c_{\alpha,n} \in X_\alpha^{-k} E_\alpha'^{+}$. Now the images of the elements $c_{\alpha,n}$ $(n \geq 1)$ under the map $E'_\alpha / X'_\alpha E_\alpha'^{+}$ are contained in the finite set $X_\alpha^{-k} E_\alpha'^{+} / X'_\alpha E_\alpha'^{+}$, so by Lemma 4.3 there exists an finite separable extension $E'_\alpha \leq E''_\alpha$ such that $c_{\alpha,n} = d_{\alpha,n} - \varphi_\alpha(d_{\alpha,n})$ for some $d_{\alpha,n} \in E''_\alpha$ for all $n \geq 1$. Moreover, the X_α-adic valuation of $d_{\alpha,n}$ is bounded by that of the X_α-adic valuation of $c_{\alpha,n}$ showing that the sum $d := \sum_{n=1}^\infty c_{\overline{\alpha},n} \otimes d_{\alpha,n}$ defines an element in E_Δ^{sep} with $c = d - \varphi_\alpha(d)$. □

Proof of Prop. 4.2. We proceed by induction on $|\Delta|$. The case $|\Delta| = 1$ is clear, so suppose $n := |\Delta| > 1$ and we have proven the statement for any proper subset $S \subsetneq \Delta = \{\alpha_1, \ldots, \alpha_n\}$. Let $c = (c_S)_{S \in \binom{\Delta}{r}} \in \bigoplus_{S\in\binom{\Delta}{r}} E_\Delta^{sep}$ be a cocycle in degree r. By Lemma 4.4 we find an element $x = (x_U)_{U\in\binom{\Delta}{r-1}}$ with $d_U = 0$ for all U with $\alpha_n \nmid U$ such that $(c - d^{r-1}(x))_S = 0$ for all $S \in \binom{\Delta}{r}$ with $\alpha_n \in S$. Indeed, the map $\cdot \cup \{\alpha_n\} \colon \binom{\Delta\setminus\{\alpha_n\}}{r-1} \to \{S \in \binom{\Delta}{r} \mid \alpha_n \in S\}$ is a bijection and by our assumption that x is concentrated into $\binom{\Delta\setminus\{\alpha_n\}}{r-1} \subset \binom{\Delta}{r-1}$ only the $S \setminus \{\alpha\}$-component of x contributes to the S component of $d^{r-1}(x)$ for $\alpha_n \in S$. So by replacing c with $c - d^{r-1}(x)$ we may assume without loss of generality that $c_S = 0$ for all S containing α_n. In particular, for $S' \in \binom{\Delta\setminus\{\alpha_n\}}{r}$ we compute

$$0 = (d^r(c))_{S'\cup\{\alpha_n\}} = (-1)^r(\mathrm{id} - \varphi_{\alpha_n})(c_{S'}) + \sum_{\beta\in S'} (-1)^{\varepsilon(\beta,S)}(\mathrm{id} - \varphi_\beta)(c_{S'\cup\{\alpha_n\}\setminus\{\beta\}}) =$$

$$= (-1)^r(\mathrm{id} - \varphi_{\alpha_n})(c_{S'}) \ .$$

Using Lemma 4.4 again this yields $c_{S'} \in E_{\Delta\setminus\{\alpha_n\}}^{sep}$ for all $S' \in \binom{\Delta}{r}$. Now the statement follows by induction. □

The association $D \mapsto \Phi^\bullet(D)$ is an exact functor from the category of abelian groups with an action of $\prod_{\alpha\in\Delta} \varphi_\alpha^{\mathbb{N}}$ to the category of chain complexes of abelian groups. In particular, for any short exact sequence $0 \to D_1 \to D_2 \to D_3 \to 0$, we have a short exact sequence $0 \to \Phi^\bullet(D_1) \to \Phi^\bullet(D_2) \to \Phi^\bullet(D_3) \to 0$ of chain complexes. This yields a long exact sequence

$$0 \to h^0\Phi^\bullet(D_1) \to h^0\Phi^\bullet(D_2) \to h^0\Phi^\bullet(D_3) \to h^1\Phi^\bullet(D_1) \to h^1\Phi^\bullet(D_2) \to h^1\Phi^\bullet(D_3) \to \cdots$$

of abelian groups.

4.2 The multivariable p-adic coefficient ring

Our goal in this section is to lift E_Δ and E_Δ^{sep} to characteristic 0 so we can classify p-adic representations of $G_{\mathbb{Q}_p,\Delta}$. Recall [5] that $\mathcal{O}_{\mathcal{E}} \cong \varprojlim_h \mathbb{Z}/(p^h)((X))$ is constructed as a Cohen ring of $E \cong \mathbb{F}_p((X))$. Via the embedding $X \mapsto [\varepsilon]-1$ these are subrings of $\tilde{B}$ which is defined as $\tilde{B} := W(\widehat{E^{sep}})[p^{-1}]$ where $W(\widehat{E^{sep}})$ is the ring of p-typical Witt vectors of the completion $\widehat{E^{sep}}$ (with respect to the X-adic topology) of the separable closure E^{sep}. Here $[\varepsilon]$ denotes the Teichmüller representative of the sequence $\varepsilon = (\varepsilon_n)_n \in \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p} \cong \widehat{E^{sep}}^{+}$ of p-power roots of unity with $\varepsilon_1 \neq 1$. Note that $\widehat{E^{sep}}$ is an algebraically closed field of characteristic p which is, in fact, isomorphic to the tilt $\mathbb{C}_p^\flat = \mathrm{Frac}(\varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/(p))$ of $\mathbb{C}_p$ in the modern terminology. Further, for any finite extension E'/E contained in E^{sep} there exists a unique finite unramified extension $\mathcal{E}'$ of $\mathcal{E} = \mathcal{O}_{\mathcal{E}}[p^{-1}]$ contained in $\tilde{B}$ with residue field E' (Prop. 4.20 in [5]).

We define the ring $\mathcal{O}_{\mathcal{E}_\Delta}$ as the projective limit $\varprojlim_h \left(\mathbb{Z}/(p^h)[[X_\alpha \mid \alpha \in \Delta]][X_\Delta^{-1}]\right)$ and put $\mathcal{E}_\Delta := \mathcal{O}_{\mathcal{E}_\Delta}[p^{-1}]$ so we have $\mathcal{O}_{\mathcal{E}_\Delta}/(p) \cong E_\Delta$. The Iwasawa algebra $\mathcal{O}^+_{\mathcal{E}_\Delta} = \mathbb{Z}_p[[X_\alpha \mid \alpha \in \Delta]] \leq \mathcal{O}_{\mathcal{E}_\Delta}$ is isomorphic to the completed tensor product of the one-variable Iwasawa algebras $\mathcal{O}^+_{\mathcal{E}_\alpha} := \mathbb{Z}_p[[X_\alpha]]$ ($\alpha \in \Delta$) over $\mathbb{Z}_p$. This motivates the way we can lift E'_Δ to characteristic 0 for a collection E'_α/E_α ($\alpha \in \Delta$) of finite separable extensions. We define

$$\mathcal{O}^+_{\mathcal{E}'_\Delta} := \widehat{\bigotimes_{\alpha\in\Delta,\mathbb{Z}_p}} \mathcal{O}_{\mathcal{E}'_\alpha}$$

as a completed tensor product. If we write $E'_\alpha = \mathbb{F}_{q_\alpha}((X'_\alpha))$ ($\alpha \in \Delta$) then we may identify $\mathcal{O}^+_{\mathcal{E}'_\Delta}$ with the power series ring $\left(\bigotimes_{\alpha\in\Delta,\mathbb{Z}_p} W(\mathbb{F}_{q_\alpha})\right)[[X'_\alpha \mid \alpha \in \Delta]]$ over the finite étale $\mathbb{Z}_p$-algebra $\bigotimes_{\alpha\in\Delta,\mathbb{Z}_p} W(\mathbb{F}_{q_\alpha})$. We define $\mathcal{O}_{\mathcal{E}'_\Delta}$ as the p-adic completion $\widehat{\mathcal{O}^+_{\mathcal{E}'_\Delta}[X_\Delta^{-1}]} = \varprojlim_h \mathcal{O}^+_{\mathcal{E}'_\Delta}[X_\Delta^{-1}]/(p^h)$ and put $\mathcal{E}'_\Delta := \mathcal{O}_{\mathcal{E}'_\Delta}[p^{-1}]$. We have the following alternative characterization of $\mathcal{O}_{\mathcal{E}'_\Delta}$.

Lemma 4.5. *Writing $\Delta = \{\alpha_1, \dots, \alpha_n\}$ we have*

$$\mathcal{O}_{\mathcal{E}'_\Delta} \cong \mathcal{O}_{\mathcal{E}'_{\alpha_1}} \otimes_{\mathcal{O}_{\mathcal{E}_{\alpha_1}}} (\cdots (\mathcal{O}_{\mathcal{E}'_{\alpha_n}} \otimes_{\mathcal{O}_{\mathcal{E}_{\alpha_n}}} \mathcal{O}_{\mathcal{E}_\Delta}))\ .$$

In particular, $\mathcal{O}_{\mathcal{E}'_\Delta}$ is a free module of rank $\prod_{i=1}^n |E'_{\alpha_i} : E_{\alpha_i}|$ over $\mathcal{O}_{\mathcal{E}_\Delta}$.

Proof. Each $\mathcal{O}_{\mathcal{E}'_{\alpha_i}}$ is naturally a subring in $\mathcal{O}_{\mathcal{E}'_\Delta}$ and so is $\mathcal{O}_{\mathcal{E}_\Delta}$. Therefore there is a ring homomorphism from the right hand side to the left hand side which is an isomorphism modulo p by Lemma 3.2. The first statement follows from the p-adic completeness of both sides.

Since $\mathcal{O}_{\mathcal{E}_{\alpha_i}}$ is a complete discrete valuation ring, $\mathcal{O}_{\mathcal{E}'_{\alpha_i}}$ is finite free over $\mathcal{O}_{\mathcal{E}_{\alpha_i}}$ of rank $|E'_{\alpha_i} : E_{\alpha_i}|$ ($i = 1, \dots, n$). Therefore the second statement. □

Now we define $\mathcal{E}_\Delta^{ur} := \varinjlim \mathcal{E}'_\Delta$ and $\mathcal{O}_{\mathcal{E}_\Delta^{ur}} := \varinjlim \mathcal{O}_{\mathcal{E}'_\Delta}$ where E'_α runs over the finite subextensions of E_α in E_α^{sep} for all $\alpha \in \Delta$. Further, we denote by $\widehat{\mathcal{E}_\Delta^{ur}}$ (resp. by $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}$) the p-adic completion of $\mathcal{E}_\Delta^{ur}$ (resp. of $\mathcal{O}_{\mathcal{E}_\Delta^{ur}}$). We have $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}/(p) \cong E_\Delta^{sep}$ by construction. The group $G_{\mathbb{Q}_p,\Delta}$ acts naturally on $\widehat{\mathcal{E}_\Delta^{ur}}$ (resp. on $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}$). Moreover, for each $\alpha \in \Delta$ we have the Frobenius lift φ_α on $\tilde{B}_\alpha$ (the copy of $\tilde{B}$ indexed by α) which acts on $[\varepsilon]$ by raising to the pth power (as it is a Teichmüller representative). So we have $\varphi_\alpha(X_\alpha) = (X_\alpha+1)^p - 1$. For each finite extension E'_α/E_α we have $\varphi(\mathcal{E}'_\alpha) \subset \mathcal{E}'_\alpha$, so this defines an action of φ_α on the rings $\mathcal{E}_\Delta^{ur}$, $\mathcal{O}_{\mathcal{E}_\Delta^{ur}}$, $\widehat{\mathcal{E}_\Delta^{ur}}$, and $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}$ for all $\alpha\in\Delta$. These operators commute with each other and with the action of the group $G_{\mathbb{Q}_p,\Delta}$.

Proposition 4.6. *We have*

$$\widehat{\mathcal{E}_\Delta^{ur}}^{H_{\mathbb{Q}_p,\Delta}} = \mathcal{E}_\Delta\ , \qquad \bigcap_{\alpha\in\Delta} \widehat{\mathcal{E}_\Delta^{ur}}^{\varphi_\alpha=\mathrm{id}} = \mathbb{Q}_p\ ,\ \textit{and}$$
$$\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}^{H_{\mathbb{Q}_p,\Delta}} = \mathcal{O}_{\mathcal{E}_\Delta}\ , \qquad \bigcap_{\alpha\in\Delta} \mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}^{\varphi_\alpha=\mathrm{id}} = \mathbb{Z}_p\ .$$

Proof. The statements on $\widehat{\mathcal{E}_\Delta^{ur}}$ follow from those on $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}$ as p is φ_α- and $H_{\mathbb{Q}_p,\Delta}$-invariant for all $\alpha\in\Delta$. Moreover, the latter statements are consequences of Prop. 3.3, resp. Lemma 3.6 using devissage. □

4.3 The equivalence of categories

We denote by $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p,\Delta})$ (resp. by $\mathrm{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p,\Delta})$) the category of continuous representations of $G_{\mathbb{Q}_p,\Delta}$ on finitely generated $\mathbb{Z}_p$-modules (resp. on finite dimensional $\mathbb{Q}_p$-vector spaces). Let T (resp. V) be an object in $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p,\Delta})$ (resp. in $\mathrm{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p,\Delta})$). We define

$$\mathbb{D}(T) := \left(\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} T\right)^{H_{\mathbb{Q}_p,\Delta}} \qquad \left(\text{resp. } \mathbb{D}(V) := \left(\widehat{\mathcal{E}_\Delta^{ur}} \otimes_{\mathbb{Q}_p} V\right)^{H_{\mathbb{Q}_p,\Delta}}\right)\ .$$

By Prop. 4.6 $\mathbb{D}(T)$ (resp. $\mathbb{D}(V)$) is a module over $\mathcal{O}_{\mathcal{E}_\Delta}$ (resp. over $\mathcal{E}_\Delta$). Moreover, it admits an action of the monoid $T_{+,\Delta}$: the action of φ_α ($\alpha\in\Delta$) is trivial on T (resp. on V) and therefore comes from the action on $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}$ (resp. on $\widehat{\mathcal{E}_\Delta^{ur}}$) defined above. The action of $\Gamma_\Delta = G_{\mathbb{Q}_p,\Delta}/H_{\mathbb{Q}_p,\Delta}$ comes from the diagonal action of $G_{\mathbb{Q}_p,\Delta}$ on $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} T$ (resp. on $\widehat{\mathcal{E}_\Delta^{ur}} \otimes_{\mathbb{Q}_p} V$).

Proposition 4.7. *Let T be an object in $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p,\Delta})$. The natural map*

$$\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} \mathbb{D}(T) \to \mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} T$$

is an isomorphism.

Proof. This is very similar to the proof of Prop. 2.30 in [5]. We proceed in two steps. Assume first that T is killed by a power p^h of p. We use induction on h. The case $h=1$ is done in Prop. 3.7. Now for $h>1$ we have a short exact sequence $0\to T_1\to T\to T_2\to 0$ of objects in $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p,\Delta})$ such that $pT_1=0$ and $p^{h-1}T_2$. Since $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}$ has no p-torsion, it is flat as $\mathbb{Z}_p$-module. Therefore we obtain a short exact sequence

$$0\to \mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} T_1 \to \mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} T \to \mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} T_2 \to 0\ .$$

Now we have an identification $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} T_1 \cong E_\Delta^{sep} \otimes_{\mathbb{F}_p} T_1 \cong E_\Delta^{sep} \otimes_{E_\Delta} \mathbb{D}(T_1)$. In particular, as a representation of $H_{\mathbb{Q}_p,\Delta}$ we have $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} T_1 \cong (E_\Delta^{sep})^{\dim_{\mathbb{F}_p} T_1}$. In particular, Prop. 4.1 yields

$H^1_{cont}(H_{\mathbb{Q}_p,\Delta}, \mathcal{O}_{\widehat{\mathcal{E}^{ur}_\Delta}} \otimes_{\mathbb{Z}_p} T_1) = \{1\}$. By the long exact sequence of continuous $H_{\mathbb{Q}_p,\Delta}$-cohomology we deduce the exactness of the sequence

$$0 \to \mathbb{D}(T_1) \to \mathbb{D}(T) \to \mathbb{D}(T_2) \to 0 \ .$$

Now we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\widehat{\mathcal{E}^{ur}_\Delta}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} \mathbb{D}(T_1) \arrow[r] \arrow[d, "\sim"] & \mathcal{O}_{\widehat{\mathcal{E}^{ur}_\Delta}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} \mathbb{D}(T) \arrow[r] \arrow[d] & \mathcal{O}_{\widehat{\mathcal{E}^{ur}_\Delta}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} \mathbb{D}(T_2) \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & \mathcal{O}_{\widehat{\mathcal{E}^{ur}_\Delta}} \otimes_{\mathbb{Z}_p} T_1 \arrow[r] & \mathcal{O}_{\widehat{\mathcal{E}^{ur}_\Delta}} \otimes_{\mathbb{Z}_p} T \arrow[r] & \mathcal{O}_{\widehat{\mathcal{E}^{ur}_\Delta}} \otimes_{\mathbb{Z}_p} T_2 \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Thus the vertical map in the middle is an isomorphism by induction using the 5-lemma.

The general case follows from this by taking the projective limit of the isomorphisms above for T/p^hT as h tends to infinity. □

An étale $T_{+,\Delta}$-module over $\mathcal{O}_{\mathcal{E}_\Delta}$ is a finitely generated $\mathcal{O}_{\mathcal{E}_\Delta}$-module D together with a semilinear action of the monoid $T_{+,\Delta}$ such that for all $\varphi_t \in T_{+,\Delta}$ the map

$$\mathrm{id} \otimes \varphi_t \colon \varphi_t^* D := \mathcal{O}_{\mathcal{E}_\Delta} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}, \varphi_t} D \to D$$

is an isomorphism. We denote by $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{O}_{\mathcal{E}_\Delta})$ the category of étale $T_{+,\Delta}$-modules over $\mathcal{O}_{\mathcal{E}_\Delta}$. As in the mod p case, $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{O}_{\mathcal{E}_\Delta})$ has the structure of a neutral Tannakian category. If D is finitely generated $\mathcal{O}_{\mathcal{E}_\Delta}$ module that is killed by a power p^h of p we define the generic length of D as $\mathrm{length}_{gen} D := \sum_{i=1}^h \mathrm{rk}_{E_\Delta}\, p^{i-1}D/p^iD$ where rk_{E_Δ} denotes the generic rank (ie. dimension over $\mathrm{Frac}(E_\Delta)$ of the localisation at (0)).

Corollary 4.8. *The functor $\mathbb{D}$ is exact. $\mathbb{D}(T)$ is an object in $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{O}_{\mathcal{E}_\Delta})$ for any T in $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p,\Delta})$. Moreover, if T is killed by a power of p then the we have $\mathrm{length}_{gen}\mathbb{D}(T) = \mathrm{length}_{\mathbb{Z}_p} T$.*

Proof. If T is an object in $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p,\Delta})$ such that $p^hT = 0$, then we have $H^1(H_{\mathbb{Q}_p,\Delta}, \mathcal{O}_{\widehat{\mathcal{E}^{ur}_\Delta}} \otimes_{\mathbb{Z}_p} T) = \{1\}$ by induction on h using the long exact sequence of continuous $H_{\mathbb{Q}_p,\Delta}$-cohomology. So the exactness of $\mathbb{D}$ on finite length objects in $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p,\Delta})$ follows the same way as in the proof of Prop. 4.7 in the special case when $pT_1 = 0$. Now if $0 \to T_1 \to T_2 \to T_3 \to 0$ is an arbitrary short exact sequence in $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p,\Delta})$ then we have an exact sequence

$$0 \to T_1[p^h] \to T_2[p^h] \to T_3[p^h] \xrightarrow{\partial_h} T_1/p^hT_1 \to T_2/p^hT_2 \to T_3/p^hT_3 \to 0$$

of finite length objects for all $h \geq 1$. Applying $\mathbb{D}$ yields an exact sequence

$$0 \to \mathbb{D}(T_1[p^h]) \to \mathbb{D}(T_2[p^h]) \to \mathbb{D}(T_3[p^h]) \to \mathbb{D}(T_1/p^hT_1) \to \mathbb{D}(T_2/p^hT_2) \to \mathbb{D}(T_3/p^hT_3) \to 0$$

for all $h \geq 1$. Since T_i is finitely generated over $\mathbb{Z}_p$, we have $T_i[p^h] = (T_i)_{tors}$ for $h \geq h_0$ large enough ($i = 1, 2, 3$). In particular, the connecting map $T_i[p^{(n+1)h}] \xrightarrow{p^h\cdot} T_i[p^{nh}]$ is the zero map for $h \geq h_0$ and $i = 1, 2, 3$. Thus the Mittag–Leffler property is satisfied for both $\mathrm{Im}(\partial_h)_h$ and

$\mathrm{Coker}(\partial_h)_h$ as the map $T_1/p^{h+1}T_1 \to T_1/p^hT_1$ is surjective for all $h \geq 1$. Hence taking the projective limit we obtain an exact sequence $0 \to \mathbb{D}(T_1) \to \mathbb{D}(T_2) \to \mathbb{D}(T_3) \to 0$ as claimed.

The statement on the generic length follows from the exactness using Prop. 3.7 and induction on h such that $p^hT = 0$. In particular, $\mathbb{D}(T)$ is finitely generated over $\mathcal{O}_{\mathcal{E}_\Delta}$ if T has finite length. Now if T is not necessarily of finite length then we apply the exactness of $\mathbb{D}$ on the exact sequence $0 \to T[p] \to T \xrightarrow{p\cdot} T \to T/pT \to 0$ we obtain that $\mathbb{D}(T/pT) = \mathbb{D}(T)/p\mathbb{D}(T)$ which is finitely generated over E_Δ. Therefore $\mathbb{D}(T)$ is finitely generated over $\mathcal{O}_{\mathcal{E}_\Delta}$ by the p-adic completeness of $\mathbb{D}(T)$ (by definition we have $\varprojlim_h \mathbb{D}(T/p^hT) = \mathbb{D}(T)$).

Finally, the étale property for finite length modules follows by induction on the length from the case $h = 1$ (Prop. 3.7) and in general by taking the projective limit. $\square$

Conversely, let D be an object in $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{O}_{\mathcal{E}_\Delta})$. We define

$$\mathbb{T}(D) := \bigcap_{\alpha\in\Delta} \left(\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} D\right)^{\varphi_\alpha = \mathrm{id}} .$$

This is a $\mathbb{Z}_p$-module admitting a diagonal action of $G_{\mathbb{Q}_p,\Delta}$ via the formula $g(\lambda \otimes d) := g(\lambda) \otimes \chi(g)(d)$ where $\chi\colon G_{\mathbb{Q}_p,\Delta} \twoheadrightarrow \Gamma_\Delta$ is the quotient map.

Proposition 4.9. *For any object D in $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{O}_{\mathcal{E}_\Delta})$, the natural map*

$$\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} \mathbb{T}(D) \to \mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} D$$

is an isomorphism.

Proof. This is completely analogous to the proof of Prop. 2.31 in [5]. We proceed in two steps. At first assume that $p^hD = 0$ for some integer $h \geq 1$. Consider the exact sequence $0 \to D[p] \to D \to D/D[p] \to 0$ and apply the exact functor $\Phi^\bullet \circ (\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} \cdot)$ to obtain an exact sequence

$$0 \to \Phi^\bullet(\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} D[p]) \to \Phi^\bullet(\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} D) \to \Phi^\bullet(\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} D/D[p]) \to 0 .$$

By Thm. 3.15 $D[p]$ is in the image of the functor $\mathbb{D}$ whence $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} D[p]$ is isomorphic to $(E_\Delta^{sep})^{\mathrm{rk}_{E_\Delta} D[p]}$ as a $\prod_{\alpha\in\Delta} \varphi_\alpha^{\mathbb{N}}$-module using Prop. 3.7. In particular, $h^1\Phi^\bullet(\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} D[p]) = 0$ by Prop. 4.2. This yields an exact sequence

$$0 \to \mathbb{T}(D[p]) \to \mathbb{T}(D) \to \mathbb{T}(D/D[p]) \to 0 ,$$

and the statement follows the same way as in the proof of Prop. 4.7.

The general case follows by taking the limit. $\square$

Now note that $\mathbb{T}(D)$ is finitely generated over $\mathbb{Z}_p$: this is obvious in the case when $p^hD = 0$ using induction on h and in the general case by Nakayama's lemma as we have $\mathbb{T}(D) = \varprojlim_h \mathbb{T}(D/p^hD)$ by construction. So we deduce

Theorem 4.10. *The functors $\mathbb{D}$ and $\mathbb{T}$ are quasi-inverse equivalences of categories between the Tannakian categories $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p,\Delta})$ and $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{O}_{\mathcal{E}_\Delta})$.*

Finally, an étale $T_{+,\Delta}$-module over $\mathcal{E}_\Delta$ is a finitely generated $\mathcal{E}_\Delta$-module D together with a semilinear action of the monoid $T_{+,\Delta}$ such that there exists an object D_0 in $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{O}_{\mathcal{E}_\Delta})$ with an isomorphism $D \cong D_0[p^{-1}] = \mathcal{E}_\Delta \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} D_0$. We denote by $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{E}_\Delta)$ the category of étale $T_{+,\Delta}$-modules over $\mathcal{E}_\Delta$. As before, $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{E}_\Delta)$ has the structure of a neutral Tannakian category. We have the following characteristic 0 version of the category equivalence:

Theorem 4.11. *The functors*

$$\begin{aligned} V &\mapsto \mathbb{D}(V) := \left(\widehat{\mathcal{E}_\Delta^{ur}} \otimes_{\mathbb{Q}_p} V\right)^{H_{\mathbb{Q}_p,\Delta}} \\ D &\mapsto \mathbb{V}(D) := \bigcap_{\alpha\in\Delta} \left(\widehat{\mathcal{E}_\Delta^{ur}} \otimes_{\mathcal{E}_\Delta} D\right)^{\varphi_\alpha = \mathrm{id}} \end{aligned}$$

are quasi-inverse equivalences of categories between the Tannakian categories $\mathrm{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p,\Delta})$ *and* $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{E}_\Delta)$.

Proof. Since $G_{\mathbb{Q}_p,\Delta}$ is compact, any finite dimensional $\mathbb{Q}_p$-representation V contains a $G_{\mathbb{Q}_p,\Delta}$-invariant lattice T. The statement follows from Thm. 4.10 by inverting p on both sides. The compatibility with tensor products and duals follows the same way as in characteristic p. $\square$

Remarks.

1. If A is a $\mathbb{Z}_p$-algebra which is finitely generated as a module over $\mathbb{Z}_p$, then we have an equivalence of categories between $\mathrm{Rep}_A(G_{\mathbb{Q}_p,\Delta})$ and $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, A \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{E}_\Delta})$. Indeed, we have a natural isomorphism $(A \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}) \otimes_A \cdot \cong \mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} \cdot$ as functors on $\mathrm{Rep}_A(G_{\mathbb{Q}_p,\Delta})$. Similarly, if K is a finite extension of $\mathbb{Q}_p$, then we have an equivalence of categories between $\mathrm{Rep}_K(G_{\mathbb{Q}_p,\Delta})$ and $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, K \otimes_{\mathbb{Q}_p} \mathcal{E}_\Delta)$.

2. It is expected that there is a similar equivalence of categories for representations of the $|\Delta|$th direct power of the group $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ for a finite extension $F/\mathbb{Q}_p$. However, at this point it is not clear what type of (φ, Γ)-modules one should consider. The usual cyclotomic (φ, Γ)-modules do not seem to be well-suited for the purpose of the p-adic and mod p Langlands programme. On the other hand, the Lubin–Tate setting may not work properly in characteristic p due to the non-existence of the distinguished left inverse ψ of φ. To work over the character variety of the group $\mathcal{O}_F$ [2] seems, however, to be a good candidate.

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