Annihilator design for linear parameter varying systems ⋆

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Abstract: Computation of the annihilator of a linear parameter-dependent (LPV) dynamical system is an important subtask in several fields of control theory, such as fault detection design or reconfigurable control design. While for LTI systems the problem can be efficiently handled, for LPV systems is not available a single algorithm that would provide the desired annihilator. Thus alternative solutions are needed to handle the cases that might appear in applications. This paper provides new insights and additional technical details for the authors' previously proposed inversion based technique. As an alternative, for a special class of LPV systems where the parameter variation is affine, this paper provides an extension of the LTI techniques that computes the annihilator based on the geometric framework. A comparison of the two approaches gives an opportunity to illustrate some of the intricacies of the LPV modeling.

Keywords: LPV system, null space, geometric method

1. INTRODUCTION AND MOTIVATION

The image and the kernel of a linear map are well-known notions. More formally, given a bounded linear operator $T : U \to Y$ it is standard to introduce the notion of the image-space $\mathcal{R}(T) = \{y \in Y \,|\, \exists u \in U, T(u) = y\}$ and null-space (or kernel) $\mathcal{N}(T) = \{u \in U \,|\, T(u) = 0\}$, respectively. Having an input output description of a linear dynamical system Σ, i.e., a transfer function in the linear time invariant (LTI) context, it is natural to consider the set $\mathcal{N}(\Sigma)$ and to search for a stable linear system Γ such that $\mathcal{R}(\Gamma) = \mathcal{N}(\Sigma)$, i.e., $\Sigma\Gamma = 0$. This system Γ will be called the right-annihilator of Σ. Analogously one can define as $\Lambda\Sigma = 0$ the left-annihilator system associated to Σ. In what follows, we will concentrate on right-annihilators only.

A traditional application of annihilators can be found in the fault detection filter design and the fault tolerant reconfiguration based control design approaches of LTI systems, see, e.g., Patton and Hou [1998], Varga [2009, 2013] and Cristofaro and Galeani [2014], Galeani et al. [2015]. For this class of systems the annihilator is determined based on (both polynomial or rational) matrix pencil methods, see Forney [1975], Kailath [1980] and Varga [2008]. Although this approach is computationally efficient, it highly depends on the frequency domain formulation, which prevents its extension to general time varying systems.

Linear parameter varying (LPV) modelling has proven to be an efficient approach in many areas of control and filtering in treating nonlinear problems in the past years. A broad class of nonlinear system models can be converted into a quasi-linear form, obtaining the system:

$$\dot{x}(t) = A(\rho)x(t) + B(\rho)u(t), \quad x(0) = 0, \tag{1}$$

$$y(t) = C(\rho)x(t) + D(\rho)u(t), \tag{2}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the input and output functions, respectively, while ρ is the scheduling function, which is determined by the measured variables. This means that its values are known in operational time by measurement. The approach is particularly appealing when a natively nonlinear problem, embedded in the LPV framework, can be solved by using traditional linear techniques.

It is a standard assumption that each parameter ρ_i ranges between its known extremal values $\rho_i(t) \in [\underline{\rho}_i, \overline{\rho}_i]$. In control design problems even the derivative of the parameters are supposed to be bounded. The given parameter set $(\rho_1(t), \cdots, \rho_N(t))$, fulfilling the modelling assumptions, will be denoted by $\mathcal{P}$. We assume that $0 \in \mathcal{P}$ and if $\mathcal{P}_i \subseteq \mathbb{R}^{n_\rho}$ denote the sets of parameter derivatives such that $\rho^{(i)}(t) \in \mathcal{P}_i$, then $0 \in \mathcal{P}_i$. It is also assumed that the parameters are sufficiently smooth, i.e., all the derivatives that might enter in the formulae exists. Note that this assumption out-rules the switching systems, that sometimes are also cast as an LPV system.

It assumed that the matrices are given as a well-defined linear fractional transform (LFT). Often the model takes the form of a polytopic qLPV system, i.e., affine parameter dependence of the form

$$M(\rho(t)) = M_0 + \rho_1(t)M_1 + \ldots + \rho_N(t)M_N \tag{3}$$

is assumed for the system matrices. None of these assumptions is restrictive. In contrast to the misbelieve often encountered in the literature, by merely defining some LTI systems on a given parameter grid does not define an LPV

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system. For a sound definition of the system a rule is also necessary, that uniquely provides the system matrices in every frozen parameter point. Since this rule is often a linear interpolation, our assumptions are justified.

While the LPV system is actually a set of time varying systems determined by the assumptions on the parameters, by a slight abuse of the notation, we identify and denote it as $\Sigma(\rho)$:

$$\Sigma(\rho) \sim \left[\begin{array}{c|c} A(\rho) & B(\rho) \\ \hline C(\rho) & D(\rho) \end{array}\right], \tag{4}$$

i.e., as a linear system that obeys to (1)-(2) viewed at fixed parameter ρ. In what follows we will use greek upper case letters to denote systems and latin upper case letters for the matrices of a state space representation. Note that we consider the rank of the parameter varying matrices as being independent on the parameter, i.e., we assume tacitly that the rank is constant on $\mathcal{P}$. A standing assumption throughout this paper is that the system is stable. To ease the notation we will suppress the parameter dependence when there is no risk of confusion.

At a fixed parameter ρ we can define the corresponding (right) annihilator system $\Gamma(\rho)$ as $\Sigma(\rho)\Gamma(\rho) = 0$ and the notion can be trivially extended to the LPV system itself. In Szabó et al. [2015] we already proposed a general method for the computation of the annihilator based on an inversion technique. The core of the algorithm assumes a full row-rank $D(\rho)$ and, if it is not the case, a considerable amount of work is necessary to produce an equivalent system with the desired property by a successive derivation of the outputs. In other words, it is supposed that the LPV system has a well defined vector relative degree, see, e.g., Isidori [1989], Edelmayer et al. [2004] for the definition.

In contrast to the LTI case, however, the computation of the LPV annihilators is not a straightforward task in general and there is an interest in alternative approaches to comply the task. As an extension of the LTI construction detailed in Galeani et al. [2015], this paper proposes a geometric method to obtain the annihilator based on the extension of the invariant spaces to the LPV case. The new algorithm can be applied for affinely parametrized LPV systems and it is intended mainly for the case when the system does not have a direct feedthrough term ($D = 0$).

Since the new algorithm, compared to the inversion based one, can be applied only to a smaller class of LPV systems, it could be questioned its usefulness. Actually both approaches have their merit and the paper provides some explanation on their role and applicability. Despite the fact that LPV systems neither have a coherent input-output theory nor a transfer function description, the inversion based approach is related to such a description while the geometric approach is related to a state transformation approach performed with a constant (not time varying) state transformation.

We emphasise that this paper does not revolve around conditions for the existence of an LPV annihilator or around issues concerning the minimal state space representation for the annihilator. Both topics goes well beyond the possibilities of this paper.

For the sake of completeness in Section 2 we recall the inversion based strategy for the annihilator design. Then, in addition to the previous results contained in Szabó et al. [2015], we provide more insights and give additional details that facilitates the necessary computations. Section 3 is dedicated to the main result of the paper, i.e., the construction of an annihilator for LPV systems that have affine parameter dependence based on parameter-invariant subspaces. This newly introduced algorithm is illustrated on a simple LPV example. The paper is concluded by a comparison of the two approaches.

2. INVERSION BASED ANNIHILATORS

The aim of this section is to recall the basic setting for the inversion based approach of the annihilator design. Since a detailed discussion was already provided in Szabó et al. [2015], here we only provide a sketch of the algorithm by pointing out the main issues that one might encounter during the implementation.

2.1 A particular case

The starting point of the approach is the observation that if one has a system Σ with a full row rank matrix in the form $D = [I\ 0]$, the computation of a proto-annihilator is immediate: partition the inputs according to the structure of D as $B = [B_i\ B_s]$, i.e.,

$$\Sigma \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|cc} A & B_i & B_s \\ \hline C & I_i & 0 \end{array}\right], \tag{5}$$

and augment the outputs with the lines $y_a = [0\ I_s]\, u$. Here I_i and I_s are identity matrices with appropriate size. The term "proto" refers to the fact that the annihilator is not necessarily stable. Thus we obtain an invertible system

$$\Sigma_a \sim \left[\begin{array}{c|c} A & B \\ \hline C_a & D_a \end{array}\right]. \tag{6}$$

By an application of the matrix inversion lemma, see, e.g., Zhou et al. [1996], we have in general that

$$\Sigma_a^{-1} \sim \left[\begin{array}{c|c} A - BD_a^{-1}C_a & BD_a^{-1} \\ \hline -D_a^{-1}C_a & D_a^{-1} \end{array}\right], \tag{7}$$

i.e., in our particular case

$$\Sigma_a^{-1} \sim \left[\begin{array}{c|cc} A - B_iC & B_i & B_s \\ \hline -C & I_i & 0 \\ 0 & 0 & I_s \end{array}\right]. \tag{8}$$

Thus, a candidate for the annihilator of Σ is the system

$$\Gamma^0 \sim \left[\begin{array}{c|c} A - B_iC & B_s \\ \hline -C & 0 \\ 0 & I_s \end{array}\right]. \tag{9}$$

In general we need an additional step here: our task is to obtain a stable annihilator but Γ^0 is not necessarily stable. A remedy would be a suitable (not necessarily coprime) factorization $\Gamma^0 = \Gamma\Omega^{-1}$ with stable terms Γ and Ω. Then Γ would be the desired annihilator. We note here, that such a factorization is highly non unique and there is a certain freedom in imposing the stable dynamics of Γ, that can be exploited in the solution of different design problems, e.g., in the design of a robust FDI filter in order to obtain a required detection performance and to have a desired

disturbance rejection property, see, e.g. Varga [2009]. In the LTI case the coprime factorization provides a solution which always exists after obtaining a minimal realization, if necessary. It is a standard fact that in the LTI case a right coprime factorization is obtained by computing a stabilizing state feedback, see, e.g., Zhou et al. [1996].

The same idea can be applied for LPV systems by using a quadratic stability notion and by using either constant or parameter varying Lyapunov matrices for the design of the parameter-varying state feedback gain, which can be cast as a set of linear matrix inequality (LMI) feasibility problems, see, e.g., Apkarian and Gahinet [1995], Wu [2001], Scherer [2001]. The LPV design might fail, however, at this step due to different reasons: either the representation is not minimal (stabilizable) or it is not quadratically stabilizable. Recall that in the LPV case even to decide wether the realization is minimal or not is highly nontrivial. This fact is a fundamental limitation of the technique.

Remark 1. It is an easy exercise to check directly that $\Sigma\Gamma^0 = 0$: after a change of variables one has

$$\Sigma\Gamma^0 \sim \left[\begin{array}{cc|c} A & 0 & 0 \\ 0 & A - B_i C & B_s \\ \hline C & 0 & 0 \end{array}\right]. \tag{10}$$

Thus, while with the assumption $x(0) = 0$ we have the annihilation, for dynamical systems we only have an asymptotic nullity provided Σ is stable. Recall that the concept of the annihilator is an open-loop concept: Γ does not depend on the current state. If such a feedback is allowed one arrive to the notion of the zero-dynamics, Edelmayer et al. [2004]. In this particular example, however, one can immediately see that the dynamics of the proto-annihilator (9) is related to the zero-dynamics of Σ (using the constraint $y = 0$ eliminate u_i from the output equation). But, in contrast to the dynamic inverse tasks encountered, e.g., in input reconstruction tasks, this fact does not introduce any additional constraints. Observe that here the factorization is related to feedback stabilization and not to output injection, as for the applications related to the dynamic inverse.

Starting from the special case of this subsection is not hard to compile a general algorithm to provide a sufficiently rich set of annihilators: the question here is the dimension of the annihilator as a MIMO system. As (9) this dimension is $(m - p) \times m$. For a more formal proof see Lemma 7 of the Appendix.

2.2 The general scheme

The main observation that is needed for the general case, i.e., $D(\rho)$ does not have full row-rank, is that the null-space of the strictly proper system is equal to the null space of the "shifted" system associated to the time derivative of the output, defined by

$$\left[\begin{array}{c|c} A(\rho) & B(\rho) \\ \hline [\frac{d}{dt}C + CA](\rho) & [CB](\rho) \end{array}\right] \tag{11}$$

for LPV systems. Thus the preparatory phase of the algorithm is a procedure that finally ends up in an equivalent system with a full row rank feedthrough term. Once that we arrive to that point it is almost trivial to reduce the system with a suitable input mixing to the special case of (5). Note that the core of the applied procedure, i.e., a successive application of the "shifts" for certain outputs, is the same as the one that defines the vector relative degree of a MIMO system.

Thus the skeleton of the inversion based annihilator design algorithm is the following:

Step. 1 If D is of full row rank compute the input mixing map that brings the system to the form (5), compute (9) and stabilize the system if necessary by a suitable factorization. Finally apply the necessary transformations at the output in order to have the desired annihilator.

Step. 2 If D is not of full row rank compute the input and output mixing map that splits the system into a full row rank part and a strictly proper part. For the strictly proper part repeat Step 2 and 3 until the strictly proper part vanishes. Then apply Step 1 and finally apply the necessary transformations at the input and output in order to have the desired annihilator.

Step. 3 If the system is strictly proper consider the time derivatives of the outputs until a nonzero feedthrough term appears by applying recursively (11). Note that at this step time derivatives of the parameters might also occur in the expressions and thus a reparametrization of the LPV system might be needed.

Since a thorough description of the algorithm has been given in Szabó et al. [2015] we skip its detailed presentation. In that paper we also point out how to apply the algorithm for memoryless parameter varying matrices given in an LFT form. In what follows we make some additional technical remarks that might facilitate the implementation and enhance the method. Moreover we provide some insight that emphasises the input-output nature of the approach.

2.3 Comments to the general scheme

While the conceptual structure of the annihilator construction algorithm is simple, the quest for an intensive manipulation of parameter varying matrices might make it practically challenging even if there are available tools, e.g., the MATLAB LFR toolbox, Magni [2006], to comply such tasks. In the parameter varying context it is highly nontrivial to determine the rank of a matrix, the domain of a rational matrix and the the full rank factorization of a matrix, which are the basic operations involved in the algorithm. It goes well beyond the scope of this paper to tackle these topics in detail. In what follows only some hints are given that might facilitate the execution of the method.

Step 1. If D is of full row rank we should augment it to an invertible D_a. This task is trivial for constant matrices: taking a full SVD and setting $D = U\,[S\ 0]\,V^*$ one has

$$D_a = \begin{bmatrix} US & 0 \\ 0 & I \end{bmatrix} V^*, \text{ i.e., } D_a^{-1} = V \begin{bmatrix} S^{-1}U^* & 0 \\ 0 & I \end{bmatrix}. \tag{12}$$

The parameter-varying case is more intricate in general, even if we have an LFT representation

$$D(\rho) = \mathfrak{F}_u(M, \Delta(\rho)) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}.$$

Since M_{22} should be of full row rank by assumption we can complete it to an invertible matrix $M_{22,a}$, e.g., as in (12), to obtain a $D_a(\rho)$ with the symbol matrix

$$M_a = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22,a} \end{array}\right], \quad \bar{M}_{21} = \begin{bmatrix} M_{21} \\ 0 \end{bmatrix}.$$

Its inverse is $D_a^{-1}(\rho) = \mathfrak{F}_u(M_{a,i}, \Delta(\rho))$ with

$$M_{a,i} = \left[\begin{array}{c|c} M_{11} - M_{12}M_{22,a}^{-1}\bar{M}_{21} & M_{12}M_{22,a}^{-1} \\ \hline -M_{22,a}^{-1}\bar{M}_{21} & M_{22,a}^{-1} \end{array}\right]. \tag{13}$$

The problem is that by this construction is not granted that $D_a(\rho)$ is invertible on the entire parameter set $\mathcal{P}$. Note that on the level memoryless matrices this is the same problem that make us to stabilize the proto-annihilator: since causality is granted, in the memoryless context stability is equivalent to well-posed.

As an illustrative example take

$$D(\rho) = [1-\rho \ \ \rho \ \ 0], \text{ with}$$
$$D_a(\rho) = \begin{bmatrix} 1-\rho & \rho & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

being singular at $\rho = 1$. In contrast, the choice

$$D_a(\rho) = \begin{bmatrix} 1-\rho & \rho & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is everywhere invertible.

As a general strategy one can perform a gausian elimination strategy line by line to obtain a full rank factorization, in our example

$$[1-\rho \ \ \rho \ \ 0] = [1 \ \ \rho \ \ 0] \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The aim is that the transformed matrix, say $\bar{D}$ to be in the form $\bar{D} = [\ \bar{D}_1 \ \ \bar{D}_2\ ]$, with $\bar{D}_1$ invertible on $\mathcal{P}$.

Step 2. If D is not of full row rank we need to perform a full rank factorization, i.e., to obtain a setting with $T_y D T_u^{-1} = \begin{bmatrix} \bar{D} \\ 0 \end{bmatrix}$, where $\bar{D}$ is of full row rank and by applying suitable nonsingular input and output mixing. Again, the case when D is constant can be trivially solved by applying an SVD factorization.

If $D(\rho)$ is parameter varying and we have an LFT representation

$$D(\rho) = \mathfrak{F}_u(M, \Delta(\rho)) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12},$$

then with $M_{22} = U \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} V^*$ and by applying a Schur complement based factorization we have

$$U^* D(\rho) V = \begin{bmatrix} \bar{D}_{11}(\rho) & \bar{D}_{12}(\rho) \\ \bar{D}_{21}(\rho) & \bar{D}_{22}(\rho) \end{bmatrix}$$

Recall that by our standing assumption $D(\rho)$ has a constant rank on $\mathcal{P}$. If $\bar{D}_{11}(\rho)$ is not invertible, as we have already seen in the previous step, we need to perform an additional transformation that brings $\begin{bmatrix} \bar{D}_{11}(\rho) & \bar{D}_{12}(\rho) \end{bmatrix}$ to the desired form. Thus, for the sake of simplicity let us suppose that $\bar{D}_{11}(\rho)$ is invertible. Then

$$U^* D(\rho) V =$$
$$\bar{T}_y(\rho) \begin{bmatrix} \bar{D}_{11}(\rho) & 0 \\ 0 & \bar{D}_{22}(\rho) - \bar{D}_{21}(\rho)\bar{D}_{11}^{-1}(\rho)\bar{D}_{12}(\rho) \end{bmatrix} \bar{T}_u(\rho)$$

with

$$\bar{T}_y(\rho) = \begin{bmatrix} I & 0 \\ \bar{D}_{21}(\rho)\bar{D}_{11}^{-1}(\rho) & I \end{bmatrix}, \quad \bar{T}_u(\rho) = \begin{bmatrix} I & \bar{D}_{11}^{-1}(\rho)\bar{D}_{12}(\rho) \\ 0 & I \end{bmatrix}.$$

Due to the assumption made on the rank of $D(\rho)$ we have that $\bar{D}_{22}(\rho) - \bar{D}_{21}(\rho)\bar{D}_{11}^{-1}(\rho)\bar{D}_{12}(\rho) = 0$, by, e.g., the Guttman rank addition formula, see Tian [2004].

Thus, the desired transformation can be obtained as

$$T_y(\rho) = \bar{T}_y^{-1}(\rho)U^*, \text{ and } T_u(\rho) = V\bar{T}_u^{-1}(\rho).$$

Further details are left out for brevity.

Step 3. This is the most problematic part of the algorithm. While we can assume that for the original system C and D does not depend on the parameter it is very likely that after the second derivation, if needed, the new readout map will be littered by the derivatives of the scheduling variable. At one hand side this is a technical difficulty because a reparametrization is needed to continue the algorithm. Moreover, the obtained annihilator will also depend on these derivatives, that might be unmeasurable, making the implementation impossible. Recall that in the controller design based on parameter varying Lyapunov techniques the controller also depends on the derivatives of the scheduling variable, Wu et al. [1996], but that dependency can be eliminated at the expense of some conservativeness. In this case this is not possible.

Remark 2. We conclude this section with an observation related to the Schur complement formula: we have seen that by a possible input and output mixing any system can be reduced to the following particular form:

$$\left[\begin{array}{c|cc} A & B_i & B_s \\ \hline C_i & I & 0 \\ C_s & 0 & 0 \end{array}\right] \sim \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}. \tag{14}$$

It follows that its annihilator is the same as for the system

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ 0 & \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \end{bmatrix},$$

i.e.,

$$\Gamma^0 = \begin{bmatrix} -\Sigma_{11}^{-1}\Sigma_{12} \\ I \end{bmatrix} \Gamma_{sch} = \Gamma_{fr}\Gamma_{sch}, \tag{15}$$

where Γ_{fr} is the annihilator for the full row rank part while Γ_{sch} is the annihilator of the Schur complement – a strictly proper plant. This formula does not give any advantage for LTI systems over existing formulas. For LPV systems, however, it might provide a viable method for annihilator computation if one can obtain Γ_{sch}.

To conclude this section we stress the fact that the inversion based approach to compute the annihilator is basically an input-output approach. Through the algorithm we only operate on the D matrix and apply suitable memoryless input and output mixing maps without requiring to modify the state of the system. However, being in the time varying case, we actually perform all the manipulations on the available state matrices. As a consequence representation of the resulting proto-annihilator will not necessarily be minimal which will result in additional nontrivial task when searching for a stable annihilator.

In the next section we provide a geometry based method that might be successfully applied for LPV systems with affine parameter dependence in A and B (C constant).

3. GEOMETRY BASED ANNIHILATORS

Embedding the basic concepts of control in the system of geometry and the interpretation (and re-interpretation) of the results of mathematical system theory by using a geometric approach was initiated in the beginning of the 1970's by Basile, Marro and Wonham. By now, the approach has proved to be an effective means to the analysis and design of control systems and the idea gained some popularity that was followed by many authors successfully. Good summaries of the subject can be found in the classical books of Wonham [1985] and Basile and Marro [1992].

The linear geometric systems theory was extended to nonlinear systems in the 1980's, see, e.g., Isidori [1989], De Persis and Isidori [2000]. In the nonlinear theory, the underlying fundamental concepts are almost the same, but the mathematics is different. For nonlinear systems the tools from differential geometry and Lie-theory are primarily used. Due to the computational complexity involved, these general nonlinear methods have limited applicability in practice.

The concept of invariant subspace known from the geometric theory of LTI systems were extended to LPV dynamics by introducing the notion of parameter-varying invariant subspace, see Balas et al. [2003]. We emphasise that despite to its names these subspaces are not parameter dependent! In introducing the various parameter-varying invariant subspaces an important goal was to set notions that lead to computationally tractable algorithms for the case when the parameter dependency of the system matrices is affine. These invariant subspaces play the same role in the solution of the fundamental problems, such as disturbance decoupling, unknown input observer design, fault detection, as their counterparts in the time invariant context, see Szabo et al. [2003], Bokor and Balas [2004].

3.1 Parameter varying invariant subspaces

The invariant subspace concept, which is a cornerstone of the classical LTI geometric framework, can be extended to LPV systems as follows:

Definition 1. A subspace $\mathcal{V}$ is called parameter-varying invariant subspace for the family of the linear maps $A(\rho)$ (or shortly $\mathcal{A}$-invariant subspace) if

$$A(\rho)\mathcal{V} \subset \mathcal{V} \qquad \text{for all } \rho \in \mathcal{P}. \tag{16}$$

Definition 2. Let $\mathcal{B}(\rho)$ denote Im $B(\rho)$. Then a subspace $\mathcal{V}$ is called a parameter-varying (A,B)-invariant subspace (or shortly $(\mathcal{A},\mathcal{B})$-invariant subspace) if for all $\rho \in \mathcal{P}$:

$$A(\rho)\mathcal{V} \subset \mathcal{V} + \mathcal{B}(\rho). \tag{17}$$

As in the classical case this definition is equivalent to the existence of a mapping $F \circ \rho : [0,T] \to \mathbb{R}^{m\times n}$ such that $(A(\rho) + B(\rho)F(\rho))\mathcal{V} \subset \mathcal{V}$. The set of $(\mathcal{A},\mathcal{B})$-invariant subspaces contained in a given set is an upper semilattice with respect to the subspace addition, hence it has a maximal element. As in the LTI case we denote by $\mathcal{V}^*$ the maximal $(\mathcal{A},\mathcal{B})$-invariant subspace contained in $\ker C$.

Let us denote the maximal $\mathcal{A}$-invariant subspace contained in a constant subspace $\mathcal{K}$ by $\langle \mathcal{K}|A(\rho)\rangle$. For the LPV case one can get the following definition:

Definition 3. A subspace $\mathcal{R}$ is called parameter-varying controllability subspace if there exists a constant matrix K and a parameter–varying matrix $F : [0,T] \to \mathbb{R}^{m\times n}$ such that

$$\mathcal{R} = \langle \mathcal{A} + B\mathcal{F} | \text{Im } BK\rangle, \tag{18}$$

where $\mathcal{A} + B\mathcal{F}$ denotes the system $A(\rho) + BF(\rho)$.

As in the classical case, the family of controllability subspaces contained in a given subspace $\mathcal{K}$ has a maximal element. We denote by $\mathcal{R}^*$ the maximal controllability subspace that corresponds to $\mathcal{K} = \ker C$.

From a practical point of view it is an important question to characterize these subspaces by a finite number of conditions. If the parameter dependence is affine these subspaces can be computed by efficient algorithms in a finite number of steps, for details see Balas et al. [2003], Bokor and Szabo [2009].

If certain conditions are fulfilled, e.g., if the parameter functions are differential algebraically independent, then the parameter invariant subspaces defined above coincide with the corresponding invariant distribution or codistribution, respectively. To give sufficient conditions for the solution of observer filter design problems, however, it is enough that some decompositions of the state equations could be performed. The parameter–varying versions of these invariant spaces are suitable objects to define the required decompositions, therefore they can play the same role as their counterparts in the time invariant context.

3.2 Annihilator construction

In this section we would like to construct an annihilator for systems of the type

$$\dot{x}(t) = A(\rho)x(t) + Bu(t)$$
$$y(t) = Cx(t),$$

where $A(\rho)$ depends affinely on the parameters. Note that the results can be also extended to the case where B is also parameter varying.

Let us denote by $\mathcal{V}^*$ the weakly unobservable subspace, i.e., the set of initial conditions for which there exists an input function such that the ensuing output is identically zero. $\mathcal{V}^*$ is the largest subspace such that there exists a static feedback gain $F(\rho)$ ensuring

$$(A(\rho) + BF(\rho))\mathcal{V} \subset \mathcal{V}, \quad (C + DF(\rho))\mathcal{V} = 0, \ \rho \in \mathcal{P}.$$

These gains F are called friends of $\mathcal{V}$.

Recall that the controllable weakly unobservable subspace $\mathcal{R}^* \subset \mathcal{V}^*$, i.e., the set of initial conditions for which there exists an input function able to steer the state to zero in finite time while keeping the output identically zero, has the same friends as $\mathcal{V}^*$. Moreover, it is known that in the LTI case the spectrum on $\mathcal{R}^*$ is freely assignable.

Let us chose the invertible input mixing map

$$T_u = [V_1 \ V_2 \ V_3]$$

such that

$$\text{im}V_1 = B^{(-1)}\mathcal{R}^*, \quad \text{im}\,[V_1 \ V_2] = \ker D$$

and the invertible output mixing map $T_y = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$ such that $W_1 D V_3 = 0$ and $W_2 D V_3 = I$, respectively.

Chose a state transformation ($\xi = T^{-1}x$) matrix

$$T = [T_1 \ T_2 \ T_3]$$

such that

$$\mathrm{im} T_1 = \mathcal{R}^*, \quad \mathrm{im}\,[T_1 \ T_2] = \mathcal{V}^*.$$

Note that these matrices do not depend on the parameters.

Applying all these transformations one has, see Aling and Schumacher [1984], Basile and Marro [1992]

$$\dot{\xi} = \bar{A}(\rho)\xi + \bar{B}\bar{u}, \quad \bar{u} = T_u^{-1}u, \tag{19}$$

$$\bar{y} = \bar{C}\xi + \bar{D}\bar{u}, \quad \bar{y} = T_y y \tag{20}$$

with

$$\bar{A}(\rho) = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} \\ \bar{A}_{21} & \bar{A}_{22} & \bar{A}_{23} \\ \bar{A}_{31} & \bar{A}_{32} & \bar{A}_{33} \end{bmatrix}(\rho), \quad \bar{B} = \begin{bmatrix} \bar{B}_{11} & \bar{B}_{12} & \bar{B}_{13} \\ 0 & \bar{B}_{22} & \bar{B}_{23} \\ 0 & \bar{B}_{32} & \bar{B}_{33} \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} 0 & 0 & \bar{C}_{13} \\ \bar{C}_{21} & \bar{C}_{22} & \bar{C}_{23} \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}.$$

One should take a $\bar{F}_1(\rho)$ (e.g., by considering the first block column of a friend of $\mathcal{V}^*$) that fulfils conditions

$$\bar{C}_{21} + \bar{F}_{13}(\rho) = 0, \tag{21}$$

$$\bar{A}_{21}(\rho) + \bar{B}_{22}\bar{F}_{12}(\rho) + \bar{B}_{23}\bar{F}_{13}(\rho) = 0, \tag{22}$$

$$\bar{A}_{31}(\rho) + \bar{B}_{32}\bar{F}_{12}(\rho) + \bar{B}_{33}\bar{F}_{13}(\rho) = 0, \tag{23}$$

and renders the parameter varying matrix $\bar{A}_0(\rho) = \bar{A}_{11}(\rho) + \bar{B}_{11}\bar{F}_{11}(\rho) + \bar{B}_{12}\bar{F}_{12}(\rho) + \bar{B}_{13}\bar{F}_{13}(\rho)$ stable (e.g., quadratically). Note that for strictly proper systems condition (21) vanishes.

LPV annihilator: after all these preparatory steps we are ready to design the LPV annihilator Γ:

$$\dot{\zeta} = \bar{A}_0(\rho)\zeta + \bar{B}_{11}v, \quad \zeta(0) = 0 \tag{24}$$

$$u = T_u(\bar{F}_1(\rho)\zeta + Ev), \tag{25}$$

with $E = \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T$ and $\bar{F}_1(\rho)$ the first block column of $\bar{F}(\rho)$. Note that while the dimensions of the matrices follows from the dimensions of the invariant subspaces, one can figure out that $v(t) \in \mathbb{R}^{m-p}$, as for the inversion based approach.

One can check that this Γ defined by (24) is indeed an annihilator. Plug in (25) into (19) and take the new state as $e = \begin{bmatrix} \xi_1^T - \zeta^T & \xi_2^T & \xi_3^T \end{bmatrix}^T$ to get

$$\dot{e} = \bar{A}e, \quad e(0) = 0,$$

$$\dot{\zeta} = \bar{A}_0\zeta + \bar{B}_{11}v$$

$$\bar{y} = \begin{bmatrix} 0 & 0 & \bar{C}_{13} \\ 0 & \bar{C}_{22} & \bar{C}_{23} \end{bmatrix} e,$$

i.e., $\bar{y} = 0$. This result has the same structure as (10).

Remark 3. Even if $D \neq 0$ one can compute $\mathcal{R}^*$ and $\mathcal{V}^*$ by using an equivalent strictly proper system, Aling and Schumacher [1984], Basile and Marro [1992]. Then $\mathcal{V}^*$ is the maximal $(\mathcal{A}, \mathcal{B})$-invariant subspace contained in $\ker C$ and $\mathcal{R}^*$ is the corresponding maximal controllability subspace that can be computed by applying the algorithms described in Balas et al. [2003], Bokor and Szabo [2009]. This fact and also Remark 2 reveals that the most relevant application case for the geometry based method (if applicable) is for the case when $D = 0$.

Remark 4. In the control design of LPV systems it is quite popular to work with a pointwise approach (gridding) by applying some sort of - usually linear - interpolation. The expressions for the annihilators for both method reveals that the gridding approach cannot be applied, in general. However, if the conditions for the geometry based design are fulfilled and either B is constant or one can chose a parameter independent friend F, then the annihilator will be linear in the original data. Hence, in that case one can apply a pointwise approach for the implementation.

3.3 Example

For illustrative purposes we show a simple academic example. Consider an LPV system

$$\dot{x}(t) = A(\rho)x(t) + Bu(t)$$

$$y(t) = Cx(t),$$

with $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$ and $|\rho_i| \leq 1$. The state matrices are:

$$A_0 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Applying the algorithms which compute the parameter-varying invariant subspaces one has

$$\mathcal{V}^* = \mathrm{Im} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$$

and

$$\mathcal{R}^* = \mathrm{Im}\,[\,0 \ 0 \ 1 \ 0 \ 0\,]^T,$$

respectively. Then the corresponding state transform and input mixing is given by:

$$T = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_u = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Taking $\bar{F}_1 = \begin{bmatrix} -\rho_1 \\ \rho_1 \\ -\rho_1 \end{bmatrix}$ one has the annihilator

$$\dot{\zeta} = -\zeta + v,$$

$$u = \begin{bmatrix} -\rho_1 \\ 0 \\ 0 \end{bmatrix} \zeta + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} v.$$

It is not hard to figure out that by following the inversion based principle one will necessarily have derivatives in that description: after taking the first two derivatives we have $CB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $CA(\rho)B = \begin{bmatrix} 1+\rho_1 & \rho_2 & 1+\rho_1+\rho_2 \\ 0 & -1 & -1 \end{bmatrix}$. Thus, a second derivation is needed ($\ddot{y}_1$ in our case) that will introduce the derivatives of the parameters into the state matrices through $CA^2(\rho) + \frac{d}{dt}CA(\rho)$. Moreover, selecting $\ddot{y}_1$ and $\dot{y}_2$, the final full row rank D matrix will be

$$D = \begin{bmatrix} 1+\rho_1 & \rho_2 & 1+\rho_1+\rho_2 \\ 0 & 1 & 1 \end{bmatrix}$$

which needs some further manipulation to be augmented to an invertible one. Thus even this simple example can show the benefit of the geometric approach for the annihilator computation.

4. CONCLUSION

In this paper we have revisited the topic of the computation of (stable) annihilators based entirely on time domain techniques. Concerning the inversion based approach, additional details has been provided over the existing results that leads to some simplification of the algorithm.

The main result of the paper is that it provides an extension of the LTI techniques that allows to compute the annihilator based on the geometric framework for a special class of LPV systems where the parameter variation is affine.

We conclude the paper by making a short comparison of the two approaches: it is obvious that the geometry based approach has serious limitations in that the assumption on the parameter dependence and the fact that the parameter-invariant subspaces are robust counterparts of the classical ones, i.e., the methods based on these concepts provides only sufficient conditions. Observe that the key element of the approach is that state transformation is defined by a constant matrix.

If the design can be performed, we can obtain a very economic description of the null space by using a "one shot" algorithm, avoiding the intricacies of the inversion based approach. Moreover, in view of Remark 2 the two approaches can be also applied in a complementary way.

There is a more subtle issue, however, revealed by these two approaches. The inversion based method relies on techniques that consider the system from an external (input-output) perspective even if we represent the system by using a particular state space description (this is indispensable for LPV systems!).

The problem appears at the point where this system is replaced with an "equivalent" one and a reparametrization is needed (the derivatives of the scheduling variables might occur). It is apparent that this issue is not present in the geometry based approach which always works on the same system and, which is very important in this context, uses only parameter independent transformations. Thus derivatives of the parameters cannot occur. Thus the general question, that needs more research and goes far beyond the scope of this paper, is that how can be two LPV systems related from an input-output perspective (e.g., when the two system are equal?).

Appendix

If one has a fine rank linear operator L its kernel can be computed based on the following lemmas, see Wemhoff [2003].

Lemma 5. Suppose L is a finite rank linear operator of the form $L((x,y)) = L_1x + L_2y$ with $L_1 : \mathbb{R}^m \to \mathbb{R}^m$ an invertible linear operator and $L_2 : \mathbb{R}^{p-m} \to \mathbb{R}^m$. Define $N : \mathbb{R}^{p-m} \to \mathbb{R}^p$ by $N(f) := (-L_1^{-1}L_2f, f)$. Then N is a basis for the nullspace of L, i.e. $\mathcal{R}(N) = \mathcal{N}(L)$ and $\mathcal{N}(N) = \{0\}$, i.e. N is a bijection between $\mathbb{R}^{p-m}$ and $\mathcal{N}(L)$.

Lemma 6. For U, V invertible let $\mathcal{R}(B) = \mathcal{N}(U^{-1}AV)$. Then $\mathcal{R}(VB) = \mathcal{N}(A)$.

These lemmas imply the following strategy: for a given L find a linear transformation, which splits L into the sum of an invertible L_1 and a non-invertible L_2 operators. Lemma 5 can be applied then to compute the kernel of the transformed operator. Finally, Lemma (6) is used to transform this nullspace back to the kernel of L.

The next lemma determines the kernel space from an invertible operator: instead of finding the invertible part it completes to be invertible.

Lemma 7. Suppose L is a finite rank linear operator. Let Q be chosen such that $\left[\begin{smallmatrix} L \\ Q \end{smallmatrix}\right]$ is invertible. Then $\mathcal{N}(L) = \mathcal{R}\left(\left[\begin{smallmatrix} L \\ Q \end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]\right)$.

Proof:

a) Let $z \in \mathcal{R}\left(\begin{bmatrix} L \\ Q \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix}\right)$, i.e. $z = \begin{bmatrix} L \\ Q \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix} r$, with some r. If $[M\ N] = \begin{bmatrix} L \\ Q \end{bmatrix}^{-1}$, then

$$Lz = L[M\ N]\begin{bmatrix} 0 \\ I \end{bmatrix} r = [I\ 0]\begin{bmatrix} 0 \\ I \end{bmatrix} r = 0$$

b) Let z s.t. $Lz = 0$. Then with $r = Qz$

$$\begin{bmatrix} L \\ Q \end{bmatrix} z = \begin{bmatrix} 0 \\ Qz \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} r \Longrightarrow z \in \mathcal{R}\left(\begin{bmatrix} L \\ Q \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix}\right)$$

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