# Strong converse exponent for classical-quantum channel coding

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We determine the exact strong converse exponent of classical-quantum channel coding, for every rate above the Holevo capacity. Our form of the exponent is an exact analogue of Arimoto's, given as a transform of the Rényi capacities with parameters  $\alpha > 1$ . It is important to note that, unlike in the classical case, there are many inequivalent ways to define the Rényi divergence of states, and hence the Rényi capacities of channels. Our exponent is in terms of the Rényi capacities corresponding to a version of the Rényi divergences that has been introduced recently in [Müller-Lennert, Dupuis, Szehr, Fehr and Tomamichel, J. Math. Phys. **54**, 122203, (2013)], and [Wilde, Winter, Yang, Commun. Math. Phys., **331**, (2014)]. Our result adds to the growing body of evidence that this new version is the natural definition for the purposes of strong converse problems.

# I. INTRODUCTION

Reliable transmission of information through a noisy channel is one of the central problems in both classical and quantum information theory. In quantum information theory, a memoryless classicalquantum channel is a map that assigns to every input signal from an input alphabet a state of a quantum system, and repeated use of the channel maps every sequence of input signals into the tensor product of the output states corresponding to the elements of the sequence. This is a direct analogue of a memoryless classical channel, where the outputs are probability distributions on some output alphabet; in fact, classical channels can be seen as a special subclass of classical-quantum channels where all possible output states commute with each other.

To transmit information through n uses of the channel, the sender and the receiver have to agree on a code, i.e., an assignment of a sequence of input signals and a measurement operator on the output system to each possible message, such that the measurement operators form a valid quantum measurement, normally described by a POVM (positive operator valued measure). The maximum rate (the logarithm of the number of messages divided by the number of channel uses) that can be attained by such coding schemes in the asymptotics of large n, with an asymptotically vanishing probability of erroneous decoding, is the capacity of the channel. The classical-quantum channel coding theorem, due to Holevo [37] and Schumacher and Westmoreland [65], identifies this operational notion of capacity with an entropic quantity, called the Holevo capacity, that is the maximum mutual information in a classical-quantum state between the input and the output of the channel. This is one of the cornerstones of quantum information theory, and is a direct analogue of Shannon's classic channel coding theorem, which in turn can be considered as the starting point of modern information theory.

Clearly, there is a trade-off between the coding rate and the error probability, and the Holevo-Schumacher-Westmoreland (HSW) theorem identifies a special point on this trade-off curve, marked by the Holevo capacity of the channel. The direct part of the theorem [37, 65] states that for any rate below the Holevo capacity, a sequence of codes with asymptotically vanishing error probability exists, while the converse part (also known as Holevo bound [35, 36]) says that for any rate above the Holevo capacity, the error probability cannot go to zero. In fact, more is true: for any rate above the capacity, the error probability inevitably goes to one asymptotically. This is known as the strong converse theorem, and it is due to Wolfowitz [77, 78] in the case of classical channels. The strong converse theorem for classical-quantum channels has been shown indepently by Winter [76] and Ogawa and Nagaoka [56]. Winter's proof follows Wolfowitz's approach, based on the method of types, while the proof of Ogawa and Nagaoka follows Arimoto's proof for classical channels [5]. A much simplifed approach has been found later by Nagaoka [51], based on the monotonicity of Rényi divergences.

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Thus, if one plots the optimal asymptotic error against the coding rate, one sees a sharp jump from zero below the Holevo capacity to one above the Holevo capacity. However, to understand the trade-off between the error and the coding rate, one has to plot also the error on the logarithmic scale. Indeed, it is known that in the direct domain, i.e., for any rate below the Holevo capacity, the optimal error probability vanishes with an exponential speed (see, e.g., [26] for the classical-quantum case), and in the converse domain, i.e., for rates above the capacity, the convergence of the optimal success probability to zero is also exponential [51, 56, 76]. The value of these exponents as a function of the coding rate gives a quantification of the trade-off between the error rate and the coding rate. In the direct domain, it is called the strong converse exponent, and a lower bound on its value has been given in Arimoto's work [5]. This was later complemented by Dueck and Körner [19], who obtained an upper bound on the strong converse exponent in the form of a variational expression in terms of the relative entropy. Despite their very different forms, the bounds of Arimoto and of Dueck and Körner turn out to coincide, and hence together they give the exact strong converse exponent for classical channels.

In this paper we determine the exact strong converse exponent for classical-quantum channels. Our form of the exponent is an exact analogue of Arimoto's, given as a transform of the Rényi capacities with parameters  $\alpha > 1$ . If sc(R, W) denotes the strong converse exponent of the classical-quantum channel at coding rate R, and  $\chi^*_{\alpha}(W)$  is the Rényi  $\alpha$ -capacity of the channel, then our result tells that

$$sc(R,W) = \sup_{\alpha>1} \frac{\alpha-1}{\alpha} \left\{ R - \chi_{\alpha}^{*}(W) \right\}.$$
(1)

It is important to note that, unlike in the classical case, there are many inequivalent ways to define the Rényi divergences of states, and hence the Rényi capacities of channels. Based on results in hypothesis testing [15, 29, 48], it seems that there are two different families of Rényi divergences that appear naturally in the quantification of trade-off relations for quantum information theoretic problems: one for the direct domains, and another one, introduced recently in [50] and [75], for the converse domains. We denote these families by  $D_{\alpha}$  and  $D_{\alpha}^{*}$ , respectively, and give their definitions in Section III. Our expression for the strong converse exponent is in terms of the  $D_{\alpha}^{*}$  divergences, and hence it gives an operational interpretation to the capacities derived from these divergences. This shows that the  $D_{\alpha}^{*}$  divergences are the natural trade-off quantifiers in the converse domain not only for hypothesis testing but also for classical-quantum channel coding. This in turn provides further evidence to the expectation that the picture should be the same for other - most probably all - coding problems where a direct and a converse domain can be defined.

In classical information theory, the direct and the strong converse exponents are typically expressed in two very different-looking forms: the Gallager- or Arimoto-type exponents, which are in terms of Rényi divergences, or by an optimization formula in terms of the Kullback-Leibler divergence (relative entropy), like the Dueck-Körner exponent. Despite their very different forms, the bounds of Arimoto and of Dueck and Körner turn out to coincide, which is stated in the paper of Dueck and Körner without proof [19]; see also [60] for an argument. For an early discussion about the conversion between the two, see e.g., the work of Blahut [11]. Interestingly, the same optimization formulas with the quantum relative entropy result in suboptimal exponents, as has been observed in [57]. In fact, they give rise to the same formulas as the classical Rényi divergence expressions, but with a family of quantum Rényi divergences (denoted by  $D^{\flat}_{\alpha}$  in this paper) that is different from both  $D_{\alpha}$  and  $D^{*}_{\alpha}$ . One of the main contributions of our paper is the observation that these suboptimal exponents may be converted into the correct ones by the method of pinching [24, 33]. Thus, even though the  $D^{\flat}_{\alpha}$  divergences are not expected to have a direct operational interpretation like the  $D_{\alpha}$  and the  $D_{\alpha}^*$  divergences, they turn out to be an important intermediate quantity when extending classical results into the quantum domain. We remark that these quantities already appeared in the context of matrix analysis in [34], and it has been shown in [7] that they arise as a limiting case of a two-parameter family of quantum Rényi divergences. However, their properties and their application in quantum information theory have been largely unexplored so far. For this reason, and because we need many of their mathematical properties to obtain our main result, we give a detailed exposition of them in Sections III and IV.

The structure of the paper is as follows. In Section II, we summarize the necessary mathematical preliminaries. Section III is devoted to Rényi divergences. The new contribution here is the investigation of the properties of the  $D^{\flat}_{\alpha}$  divergences. For completeness and for comparison, we state most results for all the three families of Rényi divergences mentioned above, but we give most of the proofs only for the  $D^{\flat}_{\alpha}$  quantities, as the properties of the other two families have been investigated in detail

elsewhere [8, 21, 48, 50, 75]. In Section IV we first give an overview of the different notions of mutual information and capacity formulas derived from Rényi divergences, and then, in Section IVB, we give one of the main technical contributions of the paper, the connection of the mutual informations of an n-fold product channel and its pinched version. This will be the key tool in converting the suboptimal exponent obtained in Section VC into the correct exponent. The main result of the paper is given in Section V. After an overview of the problem of classical-quantum channel coding in Section VA, we give a lower bound on the strong converse exponent in Section VB. This follows by the monotonicity of the  $D^*_{\alpha}$  divergences by a standard argument due to Nagaoka [52]. In Sections VC-VD we show that this lower bound is also an upper bound. We first extend the result of Dueck and Körner to classical-quantum channels in Section VC, and obtain an upper bound on the strong converse exponent in the form of a relative entropy optimization expression, which is then turned into an Arimoto-type expression involving the  $D^{\flat}_{\alpha}$  capacities of the channel. We obtain the correct expression for the strong converse exponent in Section VD by applying block pinching to the Arimoto-type expression of the preceding section, for increasing blocklengths.

Finally, in Section VI we apply our results to classical-quantum channels to obtain the exact strong converse exponent for some classes of quantum channels studied in [41, 75]. In particular, we show that the strong converse exponent is given by (1) also for entanglement breaking channels and group covariant channels with additive minimum output  $\alpha$ -entropy.

#### II. PRELIMINARIES

#### Notations and basic lemmas Α.

For a finite-dimensional Hilbert space  $\mathcal{H}$ , we use the notation  $\mathcal{L}(\mathcal{H})$  for the set of linear operators on  $\mathcal{H}$ , and we denote by  $\mathcal{L}(\mathcal{H})_+$  and  $\mathcal{L}(\mathcal{H})_{++}$  the set of non-zero positive semidefinite and positive definite linear operators on  $\mathcal{H}$ , respectively. The set of density operators on  $\mathcal{H}$  is denoted by  $\mathcal{S}(\mathcal{H})$ , i.e.,

$$\mathcal{S}(\mathcal{H}) = \{ \tau \in \mathcal{L}(\mathcal{H})_+ \mid \operatorname{Tr} \tau = 1 \},\$$

and  $\mathcal{S}(\mathcal{H})_{++}$  stands for the set of invertible density operators. For any  $\rho \in \mathcal{L}(\mathcal{H})_{+}$ , we use the notation

$$\mathcal{S}_{\varrho}(\mathcal{H}) = \{ \tau \in \mathcal{S}(\mathcal{H}) \mid \operatorname{supp} \tau \leq \operatorname{supp} \varrho \}.$$

We use the notation  $SU(\mathcal{H})$  for the special unitary group on  $\mathcal{H}$ .

For a self-adjoint operator  $A \in \mathcal{L}(\mathcal{H})$ , let  $\{A \geq 0\}$  denote the spectral projection of A corresponding to all non-negative eigenvalues. The notations  $\{A > 0\}$ ,  $\{A \le 0\}$  and  $\{A < 0\}$  are defined similarly. The positive part  $A_+$  of A is then defined as

$$A_{+} := A\{A > 0\}.$$

It is easy to see that for any  $D \in \mathcal{L}(\mathcal{H})_+$  such that  $D \leq I$ , we have

$$\operatorname{Tr} A_+ \ge \operatorname{Tr} AD.$$
 (2)

We will use the convention that powers of a positive semidefinite operator are only taken on its support and defined to be 0 on the orthocomplement of its support. That is, if  $a_1, \ldots, a_r$  are the eigenvalues of  $A \in \mathcal{L}(\mathcal{H})_+$ , with corresponding eigenprojections  $P_1, \ldots, P_r$ , then  $A^p := \sum_{i: a_i > 0} a_i^p P_i$  for any  $p \in \mathbb{R}$ . In particular,  $A^0 = \sum_{i:a_i>0} P_i$  is the projection onto the support of A. We denote the natural logarithm by log, and use the standard conventions of information theory

$$\log 0 := -\infty, \qquad \log +\infty := +\infty. \tag{3}$$

We will also use the following extension of the logarithm function:

$$\widehat{\log}: [0, +\infty) \mapsto \mathbb{R}, \qquad \widehat{\log}(x) := \log x, \ x \in (0, +\infty), \quad \text{and} \quad \widehat{\log} 0 := 0.$$
(4)

We use  $\widehat{\log}$  to define the logarithm of a positive semidefinite operator only on its support, and to be 0 on its orthocomplement, analogously to the definition of powers above; that is,  $\log A = \sum_{i: a_i > 0} \log(a_i) P_i$ . Note, however, that numbers can be considered as one-dimensional operators, and (3) gives a different extension of the logarithm function, that we will also use frequently in later sections, e.g., when expressing the Rényi divergences of states with orthogonal or disjoint supports (see (26), Remark III.4).

Measurements with finitely many outcomes on a quantum system with Hilbert space  $\mathcal{H}$  can be identified with (completely) positive trace-preserving maps  $M : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$  with some finite-dimensional Hilbert space  $\mathcal{K}$ , such that  $M(\mathcal{L}(\mathcal{H}))$  is commutative. We denote the set of all such maps by  $\mathcal{M}(\mathcal{H})$ .

For an operator  $\sigma \in \mathcal{L}(\mathcal{H})$ , we denote by  $v(\sigma)$  the number of different eigenvalues of  $\sigma$ . If  $\sigma$  is selfadjoint with spectral projections  $P_1, \ldots, P_r$ , then the *pinching* by  $\sigma$  is the map  $\mathcal{E}_{\sigma} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ , defined as

$$\mathcal{E}_{\sigma}: X \mapsto \sum_{i=1}^{r} P_i X P_i, \qquad X \in \mathcal{L}(\mathcal{H}).$$
 (5)

The pinching inequality [24, 25] tells that if X is positive semidefinite then

$$X \le v(\sigma)\mathcal{E}_{\sigma}(X). \tag{6}$$

For self-adjoint operators  $A, B \in \mathcal{L}(\mathcal{H}), A \leq B$  is understood in the sense of positive semidefinite (PSD) ordering, i.e., it means that B - A is positive semidefinite. The following lemma is standard.

**Lemma II.1** Let  $f : J \to \mathbb{R}$  be a monotone function, where J is some interval in  $\mathbb{R}$ , and let  $\mathcal{L}(\mathcal{H})_{\mathrm{sa},J}$  be the set of self-adjoint operators with their spectra in J.

If f is monotone then 
$$A \mapsto \operatorname{Tr} f(A)$$
 is monotone on  $\mathcal{L}(\mathcal{H})_{\operatorname{sa},J}$ , (7)

if f is convex then 
$$A \mapsto \operatorname{Tr} f(A)$$
 is convex on  $\mathcal{L}(\mathcal{H})_{\operatorname{sa},J}$ . (8)

We say that a function  $f: (0, +\infty) \to \mathbb{R}$  is operator monotone increasing if  $A, B \in \mathcal{L}(\mathcal{H})_+, A \geq B$ implies that  $f(A) \geq f(B)$ . We say that f is operator monotone decreasing if -f is operator monotone increasing. The following lemma is from [3, Proposition 1.1]:

**Lemma II.2** Let f be a nonnegative operator monotone decreasing function on  $(0, +\infty)$ , and  $\omega$  be a positive linear functional on  $\mathcal{L}(\mathcal{H})$ . Then the functional

$$A \mapsto \log \omega(f(A))$$
 is convex on  $\mathcal{L}(\mathcal{H})_{++}$ 

### B. Convexity

We will use the following lemma, and its equivalent version for concavity, without further notice:

**Lemma II.3** Let X be a convex set in a vector space and Y be an arbitrary set, and let  $f : X \times Y \to \mathbb{R}$  be such that for every  $y \in Y$ ,  $x \mapsto f(x, y)$  is convex. Then

$$x \mapsto \sup_{y \in Y} f(x, y) \quad is \ convex. \tag{9}$$

If, moreover, Y is also a convex set in a vector space, and  $(x, y) \mapsto f(x, y)$  is convex, then

$$x \mapsto \inf_{y \in Y} f(x, y)$$
 is convex. (10)

**Proof** The assertion in (9) is trivial from the definition of convexity. The proof of (10) is also quite straightforward; see, e.g., [13, Section 3.2.5].  $\Box$ 

**Definition II.4** A multi-variable function on the product of convex sets is called jointly convex.

**Lemma II.5** Let  $f: J \to \mathbb{R}$  be a convex function on some interval  $J \subseteq (0, +\infty)$ . Then for any affine function  $\varphi: J' \to J$  on an interval  $J' \subseteq \mathbb{R}$ , the function  $t \mapsto \varphi(t) f\left(\frac{1}{\varphi(t)}\right)$  is convex on any subinterval of J' where it is well-defined and f is continuous on  $\operatorname{ran} \varphi$  (this is only important if an endpoint of J is in the range of  $\varphi$ ).

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**Proof** Since f is convex, it can be written as  $f(x) = \sup_{i \in \mathcal{I}} \{a_i x + b_i\}$ , where  $\mathcal{I}$  is some index set, and  $a_i, b_i \in \mathbb{R}$ . Thus,

$$\varphi(t)f\left(\frac{1}{\varphi(t)}\right) = \varphi(t)\sup_{i}\left\{a_{i}\frac{1}{\varphi(t)} + b_{i}\right\} = \sup_{i}\left\{a_{i} + b_{i}\varphi(t)\right\}$$

which, as the supremum of affine functions, is convex.

#### C. Minimax theorems

Let X, Y be non-empty sets and  $f : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$  be a function. Minimax theorems provide sufficient conditions under which

$$\inf_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \inf_{x \in X} f(x, y).$$

$$\tag{11}$$

The following minimax theorem is from [47, Corollary A.2].

**Lemma II.6** Let X be a compact topological space, Y be an ordered set, and let  $f : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$  be a function. Assume that

(i) f(., y) is lower semicontinuous for every  $y \in Y$  and

(ii) f(x,.) is monotonic increasing for every  $x \in X$ , or f(x,.) is monotonic decreasing for every  $x \in X$ .

Then (11) holds, and the infima in (11) can be replaced by minima.

The following lemma combines a special version of the minimax theorems due to Kneser [40] and Fan [20] (with conditions (i) and (ii)), and Sion's minimax theorem [42, 69] (with conditions (i') and (ii')). Recall that a function  $f: C \to \mathbb{R}$  on a convex set C is quasi-convex if

$$f(tx_1 + (1-t)x_2) \le \max\{f(x_1), f(x_2)\}, \quad x_1, x_2 \in C, \ t \in (0,1),$$

and it is *quasi-concave* if -f is quasi-convex.

**Lemma II.7** Let X be a compact convex set in a topological vector space V and Y be a convex subset of a vector space W. Let  $f: X \times Y \to \mathbb{R}$  be such that

(i) f(x, .) is concave on Y for each  $x \in X$ , and

(ii) f(.,y) is convex and lower semi-continuous on X for each  $y \in Y$ .

or

(i') f(x, .) is quasi-concave and upper semi-continuous on Y for each  $x \in X$ , and

(ii') f(.,y) is quasi-convex and lower semi-continuous on X for each  $y \in Y$ .

Then (11) holds, and the infima in (11) can be replaced by minima.

### D. Universal symmetric states

For every  $n \in \mathbb{N}$ , let  $\mathfrak{S}_n$  denote the symmetric group, i.e., the group of permutations of n elements. For every finite-dimensional Hilbert space  $\mathcal{H}$ ,  $\mathfrak{S}_n$  has a natural unitary representation on  $\mathcal{H}^{\otimes n}$ , defined by

$$\pi_{\mathcal{H}}: |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle \longmapsto |\psi_{\pi^{-1}(1)}\rangle \otimes \cdots \otimes |\psi_{\pi^{-1}(n)}\rangle \qquad |\psi_i\rangle \in \mathcal{H}, \pi \in \mathfrak{S}_n.$$

Let  $\mathcal{L}_{sym}(\mathcal{H}^{\otimes n})$  denote the set of symmetric, or permutation-invariant, operators, i.e.,

$$\mathcal{L}_{\rm sym}(\mathcal{H}^{\otimes n}) := \{ A \in \mathcal{L}(\mathcal{H}^{\otimes n}) : \, \pi_{\mathcal{H}} A = A \pi_{\mathcal{H}} \; \; \forall \pi \in \mathfrak{S}_n \} = \{ \pi_{\mathcal{H}} | \, \pi \in \mathfrak{S}_n \}',$$

where for  $\mathcal{A} \subset \mathcal{L}(\mathcal{K})$ ,  $\mathcal{A}'$  denotes the commutant of  $\mathcal{A}$ . Likewise, we denote by  $\mathcal{S}_{sym}(\mathcal{H}^{\otimes n})$  the set of symmetric states, i.e.,  $\mathcal{S}_{sym}(\mathcal{H}^{\otimes n}) := \mathcal{L}_{sym}(\mathcal{H}^{\otimes n}) \cap \mathcal{S}(\mathcal{H}^{\otimes n})$ .

**Lemma II.8** For every finite-dimensional Hilbert space  $\mathcal{H}$  and every  $n \in \mathbb{N}$ , there exists a symmetric state  $\sigma_{u,n} \in \mathcal{S}_{sym}(\mathcal{H}^{\otimes n}) \bigcap \mathcal{S}_{sym}(\mathcal{H}^{\otimes n})'$  such that every symmetric state  $\omega \in \mathcal{S}_{sym}(\mathcal{H}^{\otimes n})$  is dominated as

$$\omega \le v_{n,d} \,\sigma_{u,n}, \qquad v_{n,d} \le (n+1)^{\frac{(d+2)(d-1)}{2}},$$
(12)

where  $d = \dim \mathcal{H}$ . Moreover, the number of different eigenvalues of  $\sigma_{u,n}$  is upper bounded by  $v_{n,d}$ . We call every such state  $\sigma_{u,n}$  a universal symmetric state.

A construction for a universal symmetric state has been given in [27], which we briefly review in Appendix A for readers' convenience. See also [29, Lemma 1] for a different argument for the existence of a universal symmetric state, with  $(n+1)^{d^2-1}$  in place of  $(n+1)^{\frac{(d+2)(d-1)}{2}}$  in (12).

The crucial property for us is that

$$\lim_{n \to +\infty} \frac{1}{n} \log v_{n,d} = 0.$$
(13)

#### E. **Classical-quantum channels**

By a *classical-quantum channel* (or *channel*, for short) we mean a map

$$W: \mathcal{X} \to \mathcal{S}(\mathcal{H})$$

where  $\mathcal{X}$  is an arbitrary set (called the *input alphabet*), and  $\mathcal{H}$  is a finite-dimensional Hilbert space. That is, W maps input signals in  $\mathcal{X}$  into quantum states on  $\mathcal{H}$ . We denote the set of classical-quantum channels with input space  $\mathcal{X}$  and output Hilbert space  $\mathcal{H}$  by  $C(\mathcal{H}|\mathcal{X})$ .

For every channel  $W \in C(\mathcal{H}|\mathcal{X})$ , we define the lifted channel

$$\mathbb{W}: \mathcal{X} \to \mathcal{S}(\mathcal{H}_{\mathcal{X}} \otimes \mathcal{H}), \qquad \mathbb{W}(x) := |x\rangle \langle x| \otimes W(x).$$

Here,  $\mathcal{H}_{\mathcal{X}}$  is an auxiliary Hilbert space, and  $\{|x\rangle : x \in \mathcal{X}\}$  is an orthonormal basis in it. As a canonical choice, one can use  $\mathcal{H}_{\mathcal{X}} = l^2(\mathcal{X})$ , the L<sup>2</sup>-space on  $\mathcal{X}$  with respect to the counting measure, and choose  $|x\rangle$ to be the characteristic function (indicator function) of the singleton  $\{x\}$ . Note that this is well-defined irrespectively of the cardinality of  $\mathcal{X}$ .

Let  $\mathcal{P}_f(\mathcal{X})$  denote the set of finitely supported probability measures on  $\mathcal{X}$ . We identify every  $P \in$  $\mathcal{P}_f(\mathcal{X})$  with the corresponding probability mass function, and hence write P(x) instead of  $P(\{x\})$  for every  $x \in \mathcal{X}$ . We can redefine every channel W with input alphabet  $\mathcal{X}$  as a channel on the set of Dirac measures  $\{\delta_x : x \in \mathcal{X}\} \subset \mathcal{P}_f(\mathcal{X})$  by defining  $W(\delta_x) := W(x)$ . W then admits a natural affine extension to  $\mathcal{P}_f(\mathcal{X})$ , given by

$$W(P) := \sum_{x \in \mathcal{X}} P(x)W(x).$$

In particular, the extension of the lifted channel W outputs classical-quantum states of the form

$$\mathbb{W}(P) = \sum_{x \in \mathcal{X}} P(x) |x\rangle \langle x| \otimes W(x).$$

Note that the marginals of  $\mathbb{W}(P)$  are

$$\operatorname{Tr}_{\mathcal{H}} \mathbb{W}(P) = \sum_{x \in \mathcal{X}} P(x) |x\rangle \langle x|, \qquad \operatorname{Tr}_{\mathcal{H}_{\mathcal{X}}} \mathbb{W}(P) = \sum_{x \in \mathcal{X}} P(x) W(x) = W(P).$$
(14)

With a slight abuse of notation, we will also denote  $\sum_{x \in \mathcal{X}} P(x) |x\rangle \langle x|$  by P. The *n*-fold *i.i.d.* extension of a channel  $W : \mathcal{X} \to \mathcal{S}(\mathcal{H})$  is defined as  $W^{\otimes n} : \mathcal{X}^n \to \mathcal{S}(\mathcal{H}^{\otimes n})$ ,

$$W^{\otimes n}(\underline{x}) := W(x_1) \otimes \ldots \otimes W(x_n), \qquad \underline{x} = x_1 \dots x_n \in \mathcal{X}^n.$$

Given  $\mathcal{X}$ , we will always choose the auxiliary Hilbert space  $\mathcal{H}_{\mathcal{X}^n}$  to be  $\mathcal{H}_{\mathcal{X}}^{\otimes n}$  and  $|\underline{x}\rangle := |x_1\rangle \otimes \ldots \otimes |x_n\rangle$ ,  $\underline{x} = x_1 \ldots x_n \in \mathcal{X}^n$ . With this convention, the lifted channel of  $W^{\otimes n}$  is equal to  $\mathbb{W}^{\otimes n}$ . Moreover, for every probability distribution  $P \in \mathcal{P}_f(\mathcal{X})$ ,

$$W^{\otimes n}(P^{\otimes n}) = W(P)^{\otimes n}$$
 and  $\mathbb{W}^{\otimes n}(P^{\otimes n}) = \mathbb{W}(P)^{\otimes n}$ 

where  $P^{\otimes n} \in \mathcal{P}(\mathcal{X}^n)$ ,  $P^{\otimes n}(\underline{x}) := P(x_1) \cdot \ldots \cdot P(x_n)$ ,  $\underline{x} = x_1 \ldots x_n \in \mathcal{X}^n$ , denotes the *n*-th i.i.d. extension of P.

# III. QUANTUM RÉNYI DIVERGENCES

#### A. Definitions and basic properties

For classical probability distributions p, q on a finite set  $\mathcal{X}$ , their Rényi divergence with parameter  $\alpha \in [0, +\infty) \setminus \{1\}$  is defined as

$$D_{\alpha}(p||q) := \frac{1}{\alpha - 1} \log Q_{\alpha}(p||q), \qquad Q_{\alpha}(p||q) := \sum_{x \in \mathcal{X}} p(x)^{\alpha} q(x)^{1 - \alpha},$$

when  $\operatorname{supp} p \subseteq \operatorname{supp} q$  or  $\alpha \in [0, 1)$ , and it is defined to be  $+\infty$  otherwise. For non-commuting states, various inequivalent generalizations of the Rényi divergences have been proposed. Here we consider the following quantities, defined for every pair of positive definite operators  $\varrho, \sigma \in \mathcal{L}(\mathcal{H})_{++}$  and every  $\alpha \in (0, +\infty)$ :

$$Q_{\alpha}(\varrho \| \sigma) := \operatorname{Tr} \varrho^{\alpha} \sigma^{1-\alpha}, \tag{15}$$

$$Q_{\alpha}^{*}(\varrho \| \sigma) := \operatorname{Tr}\left(\varrho^{\frac{1}{2}} \sigma^{\frac{1-\alpha}{\alpha}} \varrho^{\frac{1}{2}}\right)^{\alpha}, \tag{16}$$

$$Q^{\flat}_{\alpha}(\varrho \| \sigma) := \operatorname{Tr} e^{\alpha \log \varrho + (1-\alpha) \log \sigma}.$$
(17)

The expression in (15) is a quantum f-divergence, or quasi-entropy, corresponding to the power function  $x^{\alpha}$  [30, 61]. Its concavity [44] and convexity [2] properties are of central importance to quantum information theory [45, 54], and the corresponding Rényi divergences have an operational significance in the direct part of binary quantum state discrimination as quantifiers of the trade-off between the two types of error probabilities [6, 26, 52]. The Rényi divergences corresponding to (16) have been introduced recently in [50] and [75]; in the latter paper, they were named "sandwiched Rényi relative entropy". They have been shown to have an operational significance in the converse part of various discrimination problems as quantifiers of the trade-off between the type I success and the type II error probability [15, 29, 48, 49].  $Q^{\flat}_{\alpha}$  has been studied in information geometry [1], and its logarithm appeared in [34] in connection to the Golden-Thompson inequality. It is the natural quantity appearing in classical divergence-sphere optimization representations of various information quantities, as pointed out in [57, Section VI, [55, Section V], [25] and [52, Remark 1]. The corresponding Rényi divergence was shown to be a limiting case of a two-parameter family of Rényi divergences in [7]. A closely related quantity appears as a free energy functional in quantum statistical physics [58]. Note that for commuting  $\rho$  and  $\sigma$ , all three definitions (15)–(17) coincide, and are equal to the classical expression  $\sum_{x} \varrho(x)^{\alpha} \sigma(x)^{1-\alpha}$ , where  $\rho(x)$  and  $\sigma(x)$  are the diagonal elements of  $\rho$  and  $\sigma$ , respectively, in a joint eigenbasis.

We extend the above definitions for general, not necessarily invertible positive semidefinite operators  $\rho, \sigma \in \mathcal{L}(\mathcal{H})_+$  as

$$Q_{\alpha}^{(t)}(\varrho \| \sigma) := \lim_{\varepsilon \searrow 0} Q_{\alpha}^{(t)}(\varrho + \varepsilon I \| \sigma + \varepsilon I)$$
(18)

$$= \lim_{\varepsilon \searrow 0} Q_{\alpha}^{(t)}(\varrho + \varepsilon (I - \varrho^0) \| \sigma + \varepsilon (I - \sigma^0)).$$
(19)

Here and henceforth (t) stands for one of the three possible values  $(t) = \{ \}, (t) = * \text{ or } (t) = \flat$ , where  $\{ \}$  denotes the empty string, i.e.,  $Q_{\alpha}^{(t)}$  with  $(t) = \{ \}$  is simply  $Q_{\alpha}$ .

**Lemma III.1** For every  $\rho, \sigma \in \mathcal{L}(\mathcal{H})_+$ , and every  $\alpha \in (0, +\infty) \setminus \{1\}$ , the limits in (18) and (19) exist and are equal to each other. Moreover, if  $\alpha \in (0, 1)$  or  $\rho^0 \leq \sigma^0$ ,

$$Q_{\alpha}(\varrho \| \sigma) = \operatorname{Tr} \varrho^{\alpha} \sigma^{1-\alpha}, \tag{20}$$

$$Q_{\alpha}^{*}(\varrho \| \sigma) = \operatorname{Tr}\left(\varrho^{\frac{1}{2}} \sigma^{\frac{1-\alpha}{\alpha}} \varrho^{\frac{1}{2}}\right)^{\alpha}, \qquad (21)$$

$$Q_{\alpha}^{\flat}(\varrho \| \sigma) = \operatorname{Tr} P e^{\alpha P(\widehat{\log} \varrho) P + (1-\alpha)P(\widehat{\log} \sigma)P},$$
(22)

where  $P := \rho^0 \wedge \sigma^0$  is the projection onto the intersection of the supports of  $\rho$  and  $\sigma$ , and for all three values of t,

$$Q_1^{(t)}(\varrho \| \sigma) = \operatorname{Tr} \varrho,$$

and

$$Q_{\alpha}^{(t)}(\varrho \| \sigma) = +\infty \qquad \text{when} \qquad \alpha > 1 \quad \text{and} \quad \varrho^{0} \nleq \sigma^{0}$$

In particular, the extension in (18)–(19) is consistent in the sense that for invertible  $\rho$  and  $\sigma$  we recover the formulas in (15)–(17).

**Proof** We only prove the assertions for  $Q^{\flat}_{\alpha}$ , as the proofs for the other quantities follow similar lines, and are simpler. For  $\alpha \in (0, 1)$ , (22) has been proved in [34, Lemma 4.1]. Next, assume that  $\alpha > 1$  and  $\rho^0 \leq \sigma^0$ . Then we can assume without loss of generality that  $\sigma$  is invertible. Hence,

$$Q_{\alpha}^{\flat}(\varrho + \varepsilon(I - \varrho^{0}) \| \sigma + \varepsilon(I - \sigma^{0})) = Q_{\alpha}^{\flat}(\varrho + \varepsilon(I - \varrho^{0}) \| \sigma) = \operatorname{Trexp}\left(\alpha \log(\varrho + \varepsilon(I - \varrho^{0})) + (\alpha - 1) \log \sigma^{-1}\right),$$

and applying again [34, Lemma 4.1], we see that the limit as  $\varepsilon \searrow 0$  is equal to

$$\operatorname{Tr} P \exp\left(\alpha P(\widehat{\log} \varrho)P + (\alpha - 1)P(\widehat{\log} \sigma^{-1})P\right) = \operatorname{Tr} P \exp\left(\alpha P(\widehat{\log} \varrho)P + (1 - \alpha)P(\widehat{\log} \sigma)P\right)$$

This shows that the limit in (19) exists and is equal to (22). Showing that the limit in (18) also exists, and is equal to (22), follows by a trivial modification.

Hence, we are left to prove the case where  $\alpha > 1$  and  $\varrho^0 \nleq \sigma^0$ . By the latter assumption, there exists an eigenvector  $\psi$  of  $\varrho$ , with eigenvalue  $\lambda > 0$  such that  $c := \langle \psi, (I - \sigma^0)\psi \rangle > 0$ . Then we have

$$\begin{aligned} &\operatorname{Ir} \exp\left(\alpha \log(\varrho + \varepsilon I) + (1 - \alpha) \log(\sigma + \varepsilon I)\right) \\ &\geq \langle \psi, \exp\left(\alpha \log(\varrho + \varepsilon I) + (1 - \alpha) \log(\sigma + \varepsilon I)\right) \psi \rangle \\ &\geq \exp\left(\alpha \langle \psi, \log(\varrho + \varepsilon I) \psi \rangle + (1 - \alpha) \langle \psi, \log(\sigma + \varepsilon I) \psi \rangle\right) \\ &= \exp\left(\alpha \log(\lambda + \varepsilon) + (1 - \alpha) \langle \psi, \widehat{\log}(\sigma + \varepsilon \sigma^{0}) \psi \rangle + (1 - \alpha) (\log \varepsilon) \langle \psi, (I - \sigma^{0}) \psi \rangle\right) \\ &= \varepsilon^{(1 - \alpha)c} (\lambda + \varepsilon)^{\alpha} \exp((1 - \alpha) \langle \psi, \widehat{\log}(\sigma + \varepsilon \sigma^{0}) \psi \rangle), \end{aligned}$$
(23)

where the first inequality is obvious, and the second one is due to the convexity of the exponential function. The expression in (23) goes to  $+\infty$  as  $\varepsilon \searrow 0$ , and hence the limit in (19) is equal to  $+\infty$ , as required. The proof for the limit in (18) goes the same way.

**Remark III.2** When  $\varrho^0 \leq \sigma^0$ , (22) can be written as

$$Q^{\flat}_{\alpha}(\varrho \| \sigma) = \operatorname{Tr} \varrho^{0} e^{\alpha \, \widehat{\log} \, \varrho + (1-\alpha) \varrho^{0}(\widehat{\log} \, \sigma) \varrho^{0}}$$
(24)

$$= \operatorname{Tr} e^{\alpha \log \varrho + (1-\alpha)\varrho^0 (\log \sigma)\varrho^0} - \operatorname{Tr}(I-\varrho^0).$$
(25)

The quantum Rényi divergences corresponding to the Q quantities are defined as

$$D_{\alpha}^{(t)}(\varrho\|\sigma) := \frac{1}{\alpha - 1} \log \frac{Q_{\alpha}^{(t)}(\varrho\|\sigma)}{\operatorname{Tr} \varrho} = \frac{1}{\alpha - 1} \log Q_{\alpha}^{(t)}(\varrho\|\sigma) - \frac{1}{\alpha - 1} \log \operatorname{Tr} \varrho = \frac{\psi_{\alpha}^{(t)}(\varrho\|\sigma) - \psi_{1}^{(t)}(\varrho\|\sigma)}{\alpha - 1},$$
(26)

for any  $\alpha \in (0, +\infty) \setminus \{1\}$ , where t is any of the three possible values, and we use the notation

$$\psi_{\alpha}^{(t)}(\varrho \| \sigma) := \log Q_{\alpha}^{(t)}(\varrho \| \sigma), \qquad \alpha \in (0, +\infty).$$
<sup>(27)</sup>

By definition,

$$\psi_{\alpha}^{(t)}(\varrho \| \sigma) = \lim_{\varepsilon \searrow 0} \psi_{\alpha}^{(t)}(\varrho + \varepsilon I \| \sigma + \varepsilon I), \qquad \alpha \in (0, +\infty).$$
<sup>(28)</sup>

With these definitions we have

**Lemma III.3** If  $\varrho^0 \leq \sigma^0$  then  $\psi_{\alpha}^{(t)}(\varrho \| \sigma)$  is continuous in  $\alpha$  on  $(0, +\infty)$ . If  $\varrho^0 \nleq \sigma^0$  then  $\psi_{\alpha}^{(t)}(\varrho \| \sigma)$  is continuous in  $\alpha$  on (0, 1), it has a jump at 1 as

$$\lim_{\alpha \nearrow 1} \psi_{\alpha}^{(t)}(\varrho \| \sigma) < \log \operatorname{Tr} \varrho = \psi_{1}^{(t)}(\varrho \| \sigma),$$
(29)

and it is  $+\infty$  on  $(1, +\infty)$ .

**Proof** The only non-trivial claim is (29) for  $t = \flat$ , which can be seen the following way. Let  $r_{\min}$  denote the smallest positive eigenvalue of  $\varrho$ , let  $\tilde{\varrho} := \varrho/r_{\min}$ , let  $P := \varrho^0 \wedge \sigma^0$  and  $P^{\perp} := I - P$ . If P = 0 then  $\psi^{\flat}_{\alpha}(\varrho \| \sigma) = -\infty$  for all  $\alpha \in (0, 1)$ , from which (29) is immediate. Assume now that  $0 \neq P \neq \varrho^0$ . Then

$$\begin{split} \lim_{\alpha \nearrow 1} Q^{\flat}_{\alpha}(\varrho \| \sigma) &= \operatorname{Tr} P e^{P(\widehat{\log} \varrho)P} = \operatorname{Tr} P e^{P(\widehat{\log} \tilde{\varrho})P + (\log r_{\min})P} = r_{\min} \operatorname{Tr} P e^{P(\widehat{\log} \tilde{\varrho})P} \\ &= r_{\min} \left[ \operatorname{Tr} e^{P(\widehat{\log} \tilde{\varrho})P} - \operatorname{Tr}(I - P) \right] \le r_{\min} \left[ \operatorname{Tr} e^{P(\widehat{\log} \tilde{\varrho})P + P^{\perp}(\widehat{\log} \tilde{\varrho})P^{\perp}} - \operatorname{Tr}(I - P) \right] \\ &\leq r_{\min} \left[ \operatorname{Tr} e^{\widehat{\log} \tilde{\varrho}} - \operatorname{Tr}(I - P) \right] = r_{\min} \left[ \operatorname{Tr} \tilde{\varrho} + \operatorname{Tr}(I - \varrho^{0}) - \operatorname{Tr}(I - P) \right] \\ &= \operatorname{Tr} \varrho - r_{\min} \operatorname{Tr}(\varrho^{0} - P) \\ &< \operatorname{Tr} \varrho, \end{split}$$

where the first inequality is due to (7), and the second inequality is due to (8) and the fact that the pinching by  $(P, P^{\perp})$  can be written as a convex combination of unitaries [10, Problem II.5.4]. Taking the logarithm gives (29).

Remark III.4 Note that

for 
$$\alpha > 1$$
,  $D_{\alpha}^{(t)}(\varrho \| \sigma) = +\infty \iff Q_{\alpha}^{(t)}(\varrho \| \sigma) = +\infty \iff \varrho^0 \nleq \sigma^0$ .

for all three values of (t). On the other hand,

$$for \ \alpha \in (0,1), \ D_{\alpha}^{(t)}(\varrho \| \sigma) = +\infty \iff Q_{\alpha}^{(t)}(\varrho \| \sigma) = 0 \iff \begin{cases} \varrho^0 \sigma^0 = 0, & (t) = \{ \}, (t) = *, \\ \varrho^0 \wedge \sigma^0 = 0, & (t) = \flat. \end{cases}$$

Here,  $\varrho^0 \sigma^0 = 0$  is equivalent to the supports of  $\varrho$  and  $\sigma$  being orthogonal to each other, while  $\varrho^0 \wedge \sigma^0 = 0$ is equivalent to the supports being disjoint in the sense that  $\operatorname{supp} \varrho \cap \operatorname{supp} \sigma = \{0\}$ . In all other cases,  $D_{\alpha}^{(t)}(\varrho \| \sigma)$  is finite.

The relative entropy of a pair of positive semidefinite operators  $\varrho, \sigma \in \mathcal{L}(\mathcal{H})_+$  is defined [74] as

$$D(\varrho \| \sigma) := \operatorname{Tr} \varrho(\widehat{\log} \varrho - \widehat{\log} \sigma)$$
(30)

when  $\rho^0 \leq \sigma^0$ , and  $+\infty$  otherwise. It is easy to verify that

$$D(\varrho \| \sigma) = \lim_{\varepsilon \searrow 0} D(\varrho + \varepsilon I \| \sigma + \varepsilon I).$$
(31)

For any  $\rho \in \mathcal{L}(\mathcal{H})_+$ , its von Neumann entropy  $H(\rho)$  is defined as

$$H(\varrho) := -D(\varrho \| I) = -\operatorname{Tr} \varrho \log \varrho.$$

The same way as in [70, Lemma 2.1] (see also [62, Proposition 3]), we see that

$$D(\varrho \| \sigma) - \operatorname{Tr}(\varrho - \sigma) \ge \frac{1}{2 \max\{\|\varrho\|, \|\sigma\|\}} \operatorname{Tr}(\varrho - \sigma)^2 \ge 0$$
(32)

for any  $\rho, \sigma \in \mathcal{L}(\mathcal{H})_+$ . As it was pointed out in [73, Lemma 6], this implies that for any  $A \in \mathcal{L}(\mathcal{H})_+$ ,

$$\operatorname{Tr} A = \max_{\tau \in \mathcal{L}(\mathcal{H})_{+}} \{ \operatorname{Tr} \tau - D(\tau \| A) \},$$
(33)

and the maximum is attained uniquely at  $\tau = A$ . Strictly speaking, (32) and (33) were shown in the above references for invertible operators; they can be obtained in the general case by using (31).

**Lemma III.5** For every  $\rho, \sigma \in \mathcal{L}(\mathcal{H})_+$ , and all three possible values of t,

$$D_1^{(t)}(\varrho \| \sigma) := \lim_{\alpha \to 1} D_\alpha^{(t)}(\varrho \| \sigma) = D_1(\varrho \| \sigma) := \frac{1}{\operatorname{Tr} \varrho} D(\varrho \| \sigma).$$
(34)

**Proof** The case  $(t) = \{ \}$  follows by a straightforward computation, and the case (t) = \* have been proved by various methods in [46, 50, 75]. Hence, we only have to prove the case  $(t) = \flat$ . Assume first that  $\varrho^0 \leq \sigma^0$ . Then (24) holds for every  $\alpha \in (0, +\infty)$ , and thus

$$\begin{split} \lim_{\alpha \to 1} D^{\flat}_{\alpha}(\varrho \| \sigma) &= \lim_{\alpha \to 1} \frac{\psi^{\flat}_{\alpha}(\varrho \| \sigma) - \psi^{\flat}_{1}(\varrho \| \sigma)}{\alpha - 1} = \frac{d}{d\alpha} \Big|_{\alpha = 1} \psi^{\flat}_{\alpha}(\varrho \| \sigma) \\ &= \frac{1}{Q^{\flat}_{\alpha}(\varrho \| \sigma)} \operatorname{Tr} \varrho^{0} e^{\alpha \widehat{\log} \varrho + (1 - \alpha) \varrho^{0} (\widehat{\log} \sigma) \varrho^{0}} \left[ \widehat{\log} \varrho - \varrho^{0} (\widehat{\log} \sigma) \varrho^{0} \right] \Big|_{\alpha = 1} \\ &= \frac{1}{\operatorname{Tr} \varrho} \operatorname{Tr} \varrho (\widehat{\log} \varrho - \widehat{\log} \sigma), \end{split}$$

as required. Now assume that  $\varrho^0 \nleq \sigma^0$ . Then  $\lim_{\alpha \nearrow 1} \psi^{\flat}_{\alpha}(\varrho \| \sigma) < \psi^{\flat}_{1}(\varrho \| \sigma)$  by (29), and thus

$$\lim_{\alpha \nearrow 1} D^{\flat}_{\alpha}(\varrho \| \sigma) = \lim_{\alpha \nearrow 1} \frac{\psi^{\flat}_{\alpha}(\varrho \| \sigma) - \psi^{\flat}_{1}(\varrho \| \sigma)}{\alpha - 1} = +\infty = \frac{1}{\operatorname{Tr} \varrho} D(\varrho \| \sigma),$$

while  $D^{\flat}_{\alpha}(\varrho \| \sigma) = +\infty, \, \alpha > 1$ , which implies

$$\lim_{\alpha \searrow 1} D^{\flat}_{\alpha}(\varrho \| \sigma) = \lim_{\alpha \searrow 1} +\infty = +\infty = \frac{1}{\operatorname{Tr} \varrho} D(\varrho \| \sigma).$$

By (28) and (31) we have, for every  $\rho, \sigma \in \mathcal{L}(\mathcal{H})_+$  and all three values of t,

$$D_{\alpha}^{(t)}(\varrho \| \sigma) = \lim_{\varepsilon \searrow 0} D_{\alpha}(\varrho + \varepsilon I \| \sigma + \varepsilon I), \qquad \alpha \in (0, +\infty).$$
(35)

The importance of the  $\flat$  quantities stems from the following variational representations in Theorem III.6. Let

$$s(\alpha) := \begin{cases} 1, & \alpha \ge 1, \\ -1, & \alpha < 1. \end{cases}$$
(36)

**Theorem III.6** For every  $\varrho, \sigma \in \mathcal{L}(\mathcal{H})_+$  such that  $P := \varrho^0 \wedge \sigma^0 \neq 0$ , and for every  $\alpha \in (0, +\infty) \setminus \{1\}$ ,

$$Q^{\flat}_{\alpha}(\varrho \| \sigma) = \max_{\tau \in \mathcal{L}(\mathcal{H})_{+}, \tau^{0} \leq \varrho^{0}} \{ \operatorname{Tr} \tau - \alpha D(\tau \| \varrho) - (1 - \alpha) D(\tau \| \sigma) \},$$
(37)

$$\psi_{\alpha}^{\flat}(\varrho \| \sigma) = -\min_{\tau \in \mathcal{S}_{\varrho}(\mathcal{H})} \left\{ \alpha D(\tau \| \varrho) + (1 - \alpha) D(\tau \| \sigma) \right\},$$
(38)

$$D_{\alpha}^{\flat}(\varrho \| \sigma) = s(\alpha) \max_{\tau \in \mathcal{S}_{\varrho}(\mathcal{H})} s(\alpha) \left\{ D(\tau \| \sigma) - \frac{\alpha}{\alpha - 1} D(\tau \| \varrho) \right\}.$$
(39)

Moreover, (38)–(39) are valid even if  $\varrho^0 \wedge \sigma^0 = 0$  and  $\alpha \in (0,1)$ , and (37)–(38) hold also for  $\alpha = 1$ . When  $D^{\flat}_{\alpha}(\varrho \| \sigma)$  is finite and  $\alpha \in (0, +\infty) \setminus \{1\}$ , the optima in (38)–(39) are reached at the unique state

$$\tau_{\alpha} := P e^{\alpha P(\widehat{\log}\varrho)P + (1-\alpha)P(\widehat{\log}\sigma)P} / Q_{\alpha}^{\flat}(\varrho \| \sigma), \tag{40}$$

and at  $Q^{\flat}_{\alpha}(\varrho \| \sigma) \tau_{\alpha}$  in (37).

**Proof** First, note that (38) and (39) only differ in a constant multiplier, and hence we only prove (38). For  $\alpha = 1$  we have

$$\max_{\tau \in \mathcal{L}(\mathcal{H})_+, \tau^0 \le \varrho^0} \{ \operatorname{Tr} \tau - D(\tau \| \varrho) \} = \operatorname{Tr} \varrho = Q_1^{\flat}(\varrho \| \sigma),$$

due to (33), and

$$\min_{\tau \in \mathcal{S}_{\varrho}(\mathcal{H})} \{ D(\tau \| \varrho) \} = \min_{\tau \in \mathcal{S}_{\varrho}(\mathcal{H})} \{ D(\tau \| \varrho / \operatorname{Tr} \varrho) - \log \operatorname{Tr} \varrho \} = -\log \operatorname{Tr} \varrho = -\psi_1^{\flat}(\varrho \| \sigma).$$

Next, consider the case where  $\alpha > 1$  and  $\varrho^0 \leq \sigma^0$ . Then the choice  $\tau = \tilde{\varrho} := \varrho / \operatorname{Tr} \varrho \in \mathcal{S}_{\varrho}(\mathcal{H})$  yields

$$\operatorname{Tr} \tau - \alpha D(\tilde{\varrho} \| \varrho) - (1 - \alpha) D(\tilde{\varrho} \| \sigma) = 1 + \alpha \log \operatorname{Tr} \varrho - (1 - \alpha) \cdot (+\infty) = +\infty = Q_{\alpha}^{\flat}(\varrho \| \sigma),$$
$$\alpha D(\tilde{\varrho} \| \varrho) + (1 - \alpha) D(\tilde{\varrho} \| \sigma) = -\alpha \log \operatorname{Tr} \varrho + (1 - \alpha) \cdot (+\infty) = -\infty = -\psi_{\alpha}^{\flat}(\varrho \| \sigma),$$

proving (37)–(38). If  $\alpha \in (0,1)$  and  $\varrho^0 \wedge \sigma^0 = 0$  then for any state  $\tau$ ,  $D(\tau \| \varrho)$  or  $D(\tau \| \sigma)$  is equal to  $+\infty$ , and thus

$$\min_{\tau \in \mathcal{S}_{\varrho}(\mathcal{H})} \{ (1 - \alpha) D(\tau \| \sigma) + \alpha D(\tau \| \varrho) \} = +\infty = -\psi_{\alpha}^{\flat}(\varrho \| \sigma).$$

Hence, for the rest we assume that  $P = \rho^0 \wedge \sigma^0 \neq 0$ , and  $\alpha \in (0, 1)$  or  $\rho^0 \leq \sigma^0$ , in which case we can use (22). Note that if  $\rho^0 \leq \sigma^0$  then  $P = \rho^0$ , and if  $\alpha \in (0, 1)$  and  $\tau^0 \nleq \sigma^0$  then  $(1 - \alpha)D(\tau \| \sigma) + \alpha D(\tau \| \rho) = +\infty$ . Hence, in both cases the optimization can be restricted to  $S_P(\mathcal{H})$ , i.e., we have to prove that

$$Q^{\flat}_{\alpha}(\varrho \| \sigma) = \max_{\tau \in \mathcal{L}(\mathcal{H})_{+}, \tau^{0} \leq P} \{ \operatorname{Tr} \tau - \alpha D(\tau \| \varrho) - (1 - \alpha) D(\tau \| \sigma) \},$$
(41)

$$\psi_{\alpha}^{\flat}(\varrho \| \sigma) = -\min_{\tau \in \mathcal{S}_{P}(\mathcal{H})} \left\{ \alpha D(\tau \| \varrho) + (1 - \alpha) D(\tau \| \sigma) \right\}.$$
(42)

For every  $\tau \in \mathcal{L}(\mathcal{H})_+, \tau^0 \leq P$ , let  $c(\tau) := \alpha D(\tau \| \varrho) + (1 - \alpha) D(\tau \| \sigma)$ , and  $\tilde{\tau} := \tau / \operatorname{Tr} \tau$ . Then

$$\max_{\tau \in \mathcal{L}(\mathcal{H})_+, \tau^0 \le P} \{\operatorname{Tr} \tau - c(\tau)\} = \max_{\tau \in \mathcal{L}(\mathcal{H})_+, \tau^0 \le P} \{\operatorname{Tr} \tau - (\operatorname{Tr} \tau) \log(\operatorname{Tr} \tau) - (\operatorname{Tr} \tau)c(\tilde{\tau})\}$$
(43)

$$= \max_{\tau \in \mathcal{S}_P(\mathcal{H})} \max_{t>0} \{t - t \log t - tc(\tau)\} = \max_{\tau \in \mathcal{S}_P(\mathcal{H})} \exp\left(-c(\tau)\right)$$
(44)

$$= \exp\left(-\min_{\tau \in \mathcal{S}_{P}(\mathcal{H})} c(\tau)\right),\tag{45}$$

i.e., (41) and (42) are equivalent to each other. A straightforward computation shows that (22) and (33) yield (41), proving (37)–(39). The assertion about the unique optimizers then follows from the uniqueness of the optimizer in (33).  $\Box$ 

**Remark III.7** The family of states in (40) is a quantum generalization of the Hellinger arc.

Next, we give a refinement of (42), which also yields an alternative proof of (42).

**Proposition III.8** Let  $\varrho, \sigma \in \mathcal{L}(\mathcal{H})_+$  be such that  $P := \varrho^0 \wedge \sigma^0 \neq 0$ , and let  $\tau_\alpha$  be as in (40). For every  $\tau \in \mathcal{S}_P(\mathcal{H})$ , and every  $\alpha \in (0, +\infty) \setminus \{1\}$ ,

$$\alpha D(\tau \| \varrho) + (1 - \alpha) D(\tau \| \sigma) = D(\tau \| \tau_{\alpha}) - \psi_{\alpha}^{\flat}(\varrho \| \sigma),$$
(46)

or equivalently,

$$D^{\flat}_{\alpha}(\varrho \| \sigma) = \frac{\alpha}{1-\alpha} D(\tau \| \varrho) + D(\tau \| \sigma) - \frac{1}{1-\alpha} D(\tau \| \tau_{\alpha}).$$
(47)

In particular, (42) holds, with  $\tau_{\alpha}$  being the unique minimizer, and

$$D_{\alpha}^{\flat}(\varrho \| \sigma) = \frac{\alpha}{1-\alpha} D(\tau_{\alpha} \| \varrho) + D(\tau_{\alpha} \| \sigma).$$

**Proof** Let  $\alpha \in (0, +\infty) \setminus \{1\}$ . The quantum relative entropy admits the following simple identity, related to the triangular relation in information geometry [1, Theorems 3.7, 7.1]: for any  $r, s \in \mathcal{S}(\mathcal{H})$  and any  $t \in \mathcal{L}(\mathcal{H})_+$  such that  $r^0 \leq s^0 \leq t^0$ ,

$$D(r||t) = D(r||s) + D(s||t) + \text{Tr}(r-s)(\widehat{\log s} - \widehat{\log t}).$$
(48)

Hence for any  $\tau \in \mathcal{S}_P(\mathcal{H})$ , we have

$$D(\tau \| \varrho) = D(\tau \| \tau_{\alpha}) + D(\tau_{\alpha} \| \varrho) + \operatorname{Tr}(\tau - \tau_{\alpha})(\widehat{\log} \tau_{\alpha} - \widehat{\log} \varrho)$$
  
$$= D(\tau \| \tau_{\alpha}) + D(\tau_{\alpha} \| \varrho) + (\alpha - 1)\operatorname{Tr}(\tau - \tau_{\alpha})(\widehat{\log} \varrho - \widehat{\log} \sigma),$$
  
$$D(\tau \| \sigma) = D(\tau \| \tau_{\alpha}) + D(\tau_{\alpha} \| \sigma) + \operatorname{Tr}(\tau - \tau_{\alpha})(\widehat{\log} \tau_{\alpha} - \widehat{\log} \sigma)$$
  
$$= D(\tau \| \tau_{\alpha}) + D(\tau_{\alpha} \| \sigma) + \alpha \operatorname{Tr}(\tau - \tau_{\alpha})(\widehat{\log} \varrho - \widehat{\log} \sigma).$$

Combining these relations, it holds for any  $\tau \in \mathcal{S}_P(\mathcal{H})$  that

$$\alpha D(\tau \| \varrho) + (1 - \alpha) D(\tau \| \sigma) = D(\tau \| \tau_{\alpha}) + \alpha D(\tau_{\alpha} \| \varrho) + (1 - \alpha) D(\tau_{\alpha} \| \sigma)$$
(49)

By the definition of  $\tau_{\alpha}$ ,

$$\widehat{\log} \tau_{\alpha} = \alpha P(\widehat{\log} \varrho) P + (1 - \alpha) P(\widehat{\log} \sigma) P - \psi_{\alpha}^{\flat}(\varrho \| \sigma),$$
(50)

and thus

$$D(\tau_{\alpha} \| \varrho) = \operatorname{Tr} \tau_{\alpha}(\widehat{\log} \tau_{\alpha} - \widehat{\log} \varrho) = (\alpha - 1) \operatorname{Tr} \tau_{\alpha}(\widehat{\log} \varrho - \widehat{\log} \sigma) - \psi_{\alpha}^{\flat}(\varrho \| \sigma),$$
  
$$D(\tau_{\alpha} \| \sigma) = \operatorname{Tr} \tau_{\alpha}(\widehat{\log} \tau_{\alpha} - \widehat{\log} \sigma) = \alpha \operatorname{Tr} \tau_{\alpha}(\widehat{\log} \varrho - \widehat{\log} \sigma) - \psi_{\alpha}^{\flat}(\varrho \| \sigma).$$

Hence,

$$\alpha D(\tau_{\alpha} \| \varrho) + (1 - \alpha) D(\tau_{\alpha} \| \sigma) = -\psi_{\alpha}^{\flat}(\varrho \| \sigma).$$
(51)

Combining this with (49) yields (46). By the strict positivity of the relative entropy, (46) yields (42) and that  $\tau_{\alpha}$  is the unique minimizer.

**Remark III.9** The following variational formulas were shown in [73] and [34], respectively: For any self-adjoint operator H, and any positive definite operator A,

$$\operatorname{Tr} e^{H + \log A} = \max_{\tau \in \mathcal{L}(\mathcal{H})_{++}} \{ \operatorname{Tr} \tau + \operatorname{Tr} \tau H - D(\tau \| A) \},$$
(52)

$$\log \operatorname{Tr} e^{H + \log A} = \max_{\tau \in \mathcal{S}(\mathcal{H})_{++}} \{ \operatorname{Tr} \tau H - D(\tau \| A) \}.$$
(53)

With the substitution  $H := \alpha \log \rho$ ,  $A := \sigma^{1-\alpha}$ , we can recover (37)–(38) for invertible  $\rho$  and  $\sigma$ . What is new in Theorem III.6, apart from extending the variational representations for non-invertible  $\rho$  and  $\sigma$ , is making the connection between the variational expressions (37) and (38) (equivalently, between (52) and (53)) in (43)–(45). This shows that proving either of (37) or (38) yields immediately the other variational expression as well. In the proof of Theorem III.6 we followed Tropp's argument [73] based on (33) to obtain (37), and from it (38). The proof based on Proposition III.8 proceeds the other way around: we first prove (38), which then yields (37). Note that this alternative proof gives a new proof of (52) and (53) through the choice  $\rho := e^{H/\alpha}$ ,  $\sigma := A^{\frac{1}{1-\alpha}}$ .

**Remark III.10** For classical random variables (corresponding to commuting density operators), the expression (47) seems to have first appeared in [17]. This yields the variational expressions (38) and (39) for classical random variables; an alternative proof for these have appeared in [67].

Most of the relevant properties of  $D^{\flat}_{\alpha}$  can be derived from the variational formula in Theorem III.6. The following Lemma has the same importance for  $D^{\ast}_{\alpha}$ :

**Lemma III.11** For any  $\rho, \sigma \in \mathcal{L}(\mathcal{H})_+$ , we have

$$D^*_{\alpha}(\varrho \| \sigma) = \lim_{n \to \infty} \frac{1}{n} D_{\alpha}(\mathcal{E}_{\sigma^{\otimes n}} \varrho^{\otimes n} \| \sigma^{\otimes n}), \qquad \alpha \in (0, +\infty), \tag{54}$$

$$D_{\alpha}^{*}(\varrho \| \sigma) = \lim_{n \to \infty} \frac{1}{n} \max_{M_{n} \in \mathcal{M}(\mathcal{H}^{\otimes n})} D_{\alpha} \left( M_{n}(\varrho^{\otimes n}) \| M_{n}(\sigma^{\otimes n}) \right), \qquad \alpha \in [1/2, +\infty),$$
(55)

where  $\mathcal{E}_{\sigma^{\otimes n}}$  is the pinching (5) by  $\sigma^{\otimes n}$ , and the maximization in the second line is over finite-outcome measurements on  $\mathcal{H}^{\otimes n}$  (see section IIA.)

Both (54) and (55) tells that  $D^*_{\alpha}$  can be recovered as the limit of the Rényi divergences of commuting operators. The first identity (54) was proved in [48, Corollary III.8] for  $\alpha \in (1, +\infty)$ , and later extended to  $\alpha \in (0, +\infty)$  in [29, Corollary 3]. The second identity (55) tells that  $D^*_{\alpha}$  can be recovered as the largest post-measurement Rényi divergence in the asymptotics of many copies, and it follows immediately from (54) and the monotonicity of  $D^*_{\alpha}$  under measurements for  $\alpha \in [1/2, +\infty)$  [21]. **Lemma III.12** Let  $\rho, \sigma \in \mathcal{L}(\mathcal{H})_+$ , and t be any of the three possible values. Then the functions

$$\alpha \mapsto \psi_{\alpha}^{(t)}(\varrho \| \sigma) \quad and \quad \alpha \mapsto Q_{\alpha}^{(t)}(\varrho \| \sigma) \quad are \ convex \ on \qquad (0, +\infty), \tag{56}$$

and the function

$$\alpha \mapsto D_{\alpha}^{(t)}(\varrho \| \sigma) \quad is \text{ monotone increasing on } (0, +\infty).$$
(57)

**Proof** It is enough to prove (56) for invertible  $\rho$  and  $\sigma$  due to (28) and the fact that the limit of convex functions is convex. The second derivative of  $\alpha \mapsto \psi_{\alpha}(\rho \| \sigma)$  can be seen to be non-negative by a straightforward computation, proving (56) for  $(t) = \{\}$ . Combining this with (54) shows that  $\alpha \mapsto \psi_{\alpha}^*(\rho \| \sigma)$  is the limit of convex functions, and hence is itself convex. For  $(t) = \flat$ , (38) yields

$$\psi_{\alpha}^{\flat}(\varrho \| \sigma) = \sup_{\tau \in \mathcal{S}(\mathcal{H})} \left\{ -\alpha D(\tau \| \varrho) - (1 - \alpha) D(\tau \| \sigma) \right\}.$$
(58)

Thus,  $\alpha \mapsto \psi_{\alpha}^{\flat}(\varrho \| \sigma)$  is the supremum of convex functions in  $\alpha$ , and hence is itself convex. Since  $Q_{\alpha}^{(t)}(\varrho \| \sigma) = \exp(\psi_{\alpha}^{(t)}(\varrho \| \sigma))$ , and the exponential function is monotone increasing and convex, convexity of  $\alpha \mapsto Q_{\alpha}^{(t)}(\varrho \| \sigma)$  follows from the above. Since

$$D_{\alpha}^{(t)}(\varrho \| \sigma) = \frac{\psi_{\alpha}^{(t)}(\varrho \| \sigma) - \psi_{1}^{(t)}(\varrho \| \sigma)}{\alpha - 1}$$

(57) follows immediately from the convexity of  $\alpha \mapsto \psi_{\alpha}^{(t)}(\varrho \| \sigma)$ .

**Remark III.13** Monotonicity of  $\alpha \mapsto D^*_{\alpha}(\varrho \| \sigma)$  has been shown in [50, Theorem 7] by a different method.

**Remark III.14** Convexity of  $\alpha \mapsto \psi_{\alpha}^{(t)}$  easily yields the concavity of the so-called auxiliary function. We comment on it in more detail in Appendix B.

Monotonicity in  $\alpha$  ensures that the limits

$$D_0^{(t)}(\varrho \| \sigma) := \lim_{\alpha \searrow 0} D_\alpha^{(t)}(\varrho \| \sigma) = \inf_{\alpha \in (0, +\infty)} D_\alpha^{(t)}(\varrho \| \sigma),$$
(59)

$$D_{\infty}^{(t)}(\varrho \| \sigma) := \lim_{\alpha \to +\infty} D_{\alpha}^{(t)}(\varrho \| \sigma) = \sup_{\alpha \in (0, +\infty)} D_{\alpha}^{(t)}(\varrho \| \sigma)$$
(60)

exist. For  $\alpha = 0$ , a straightforward computation verifies that

$$D_0(\varrho \| \sigma) = \log \operatorname{Tr} \varrho - \log \operatorname{Tr} \varrho^0 \sigma$$
 and  $D_0^{\flat}(\varrho \| \sigma) = \log \operatorname{Tr} \varrho - \log \operatorname{Tr} P e^{P(\log \sigma)P}$ ,

where  $P = \rho^0 \wedge \sigma^0$ . For (t) = \*, a procedure to compute  $D_0^*(\rho \| \sigma)$  was given in [7, Section 5] for the case  $\rho^0 \leq \sigma^0$ .

For  $\alpha = +\infty$ , we get

$$D_{\infty}(\varrho \| \sigma) = \log \max \left\{ \frac{r}{s} : \operatorname{Tr} P_{r} Q_{s} > 0 \right\},$$
(61)

$$D_{\infty}^{*}(\varrho \| \sigma) = D_{\max}(\varrho \| \sigma) := \log \inf \left\{ \lambda : \varrho \le \lambda \sigma \right\},$$
(62)

$$D^{\flat}_{\infty}(\varrho \| \sigma) = \log \inf \{ \lambda : \widehat{\log} \, \varrho \le \varrho^0(\widehat{\log}(\lambda \sigma)) \varrho^0 \}$$
(63)

when  $\rho^0 \leq \sigma^0$ , and  $D_{\infty}^{(t)}(\rho \| \sigma) = +\infty$  otherwise. In (61),  $P_r$  and  $Q_s$  denote the spectral projections of  $\rho$  and  $\sigma$ , corresponding to the eigenvalues r and s, respectively, and the equality follows by a straightforward computation. In (62),  $D_{\max}$  is the max-relative entropy [18, 64], and the equality has been shown

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in [50, Theorem 5]. The case (t) = b follows from Theorem III.6, as when  $\rho^0 \leq \sigma^0$ ,

$$D_{\infty}^{\flat}(\varrho \| \sigma) = \sup_{\alpha > 1} \sup_{\tau \in S_{\varrho}(\mathcal{H})} \left\{ D(\tau \| \sigma) - \frac{\alpha}{\alpha - 1} D(\tau \| \varrho) \right\}$$
$$= \sup_{\tau \in S_{\varrho}(\mathcal{H})} \sup_{\alpha > 1} \left\{ D(\tau \| \sigma) - \frac{\alpha}{\alpha - 1} D(\tau \| \varrho) \right\}$$
$$= \sup_{\tau \in S_{\varrho}(\mathcal{H})} \left\{ D(\tau \| \sigma) - D(\tau \| \varrho) \right\}$$
$$= \sup_{\tau \in S_{\varrho}(\mathcal{H})} \left\{ \operatorname{Tr} \tau \left( \widehat{\log} \varrho - \widehat{\log} \sigma \right) \right\}$$
$$= \inf_{\tau \in S_{\varrho}(\mathcal{H})} \left\{ \operatorname{Tr} \tau \left( \widehat{\log} \varphi - \widehat{\log} \sigma \right) \right\}$$
$$= \inf_{\tau \in S_{\varrho}(\mathcal{H})} \left\{ \sum_{\alpha \in S_{\varrho}(\mathcal$$

Note that (64) is an extension of (39) to  $\alpha = +\infty$ .

Lemmas III.12 and III.5 with the definitions (59)-(60) yield the following

**Corollary III.15** Let  $\rho, \sigma \in \mathcal{L}(\mathcal{H})_+$ , and t be any of the three possible values. Then the function

 $\alpha \mapsto D_{\alpha}^{(t)}(\varrho \| \sigma) \in [0, +\infty]$  is continuous on  $[0, +\infty]$ .

# B. Convexity and monotonicity

It is easy to see that when  $\rho$  and  $\sigma$  commute, all the quantum Rényi divergences  $D_{\alpha}^{(t)}$  with  $(t) = \{ \}, (t) = *$  and  $(t) = \flat$  coincide, and are equal to the classical Rényi divergence of the eigenvalues of  $\rho$  and  $\sigma$ . The properties of the classical Rényi divergences (monotonicity under stochastic maps, joint convexity, etc.) are very well understood, and are fairly easy to prove. Corresponding properties of the various quantum generalizations are typically much harder to verify and need not hold for every value of t and  $\alpha$ . The cases  $(t) = \{ \}$  and (t) = \* are by now quite well understood, too; see, e.g., [30, 47, 61] for  $(t) = \{ \}$  and the recent papers [8, 21, 29, 48–50, 75] for (t) = \*. Hence, we will focus on the so far less studied  $D_{\alpha}^{\flat}$  below, and prove most of the claims only for this version, but state the various properties for all three values of t for completeness and for comparison.

We say that  $D_{\alpha}^{(t)}$  is monotone for a fixed  $\alpha$ , if for all finite-dimensional Hilbert spaces  $\mathcal{H}, \mathcal{K}$ , every  $\varrho, \sigma \in \mathcal{L}(\mathcal{H})_+$  and every linear completely positive trace-preserving map  $\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ , we have  $D_{\alpha}^{(t)}(\Phi(\varrho) \| \Phi(\sigma)) \leq D_{\alpha}^{(t)}(\varrho \| \sigma)$ . Monotonicity of  $s(\alpha)Q_{\alpha}^{(t)}$  is defined in the same way, and it is clear that for a fixed pair  $((t), \alpha), D_{\alpha}^{(t)}$  is monotone if and only if  $s(\alpha)Q_{\alpha}^{(t)}$  is monotone. Similarly, we say that  $s(\alpha)Q_{\alpha}^{(t)}$  is jointly convex, if for every finite-dimensional Hilbert space  $\mathcal{H}$ , the map  $(\varrho, \sigma) \mapsto s(\alpha)Q_{\alpha}^{(t)}(\varrho \| \sigma)$  is convex on  $\mathcal{L}(\mathcal{H})_+ \times \mathcal{L}(\mathcal{H})_+$ , where  $s(\alpha)$  is given by (36). By a standard argument, for any  $\alpha \in (0, +\infty) \setminus \{1\}$  and any value of (t), monotonicity of  $D_{\alpha}^{(t)}$  is equivalent to the joint convexity of  $s(\alpha)Q_{\alpha}^{(t)}$ . We have the following:

**Theorem III.16** The maximal interval of  $\alpha$  for which  $s(\alpha)Q_{\alpha}^{(t)}$  is jointly convex is

[0,2] for  $(t) = \{ \}$ ,  $[1/2, +\infty)$  for (t) = \*, and [0,1] for  $(t) = \flat$ .

These are also the maximal intervals in  $\mathbb{R}_+$  for which  $D_{\alpha}^{(t)}$  is monotone, except for  $D_{\alpha}^*$ , which is monotone also for  $\alpha = +\infty$ .

The case  $(t) = \{ \}$  was proved in [2, 44]; see also [61]. The case (t) = \* was proved in [21]; see also [50, 75]  $(\alpha \in (1, 2])$  and [8, 48]  $(\alpha > 1)$ . Either of these cases yield the monotonicity of the relative entropy under CPTP maps  $(\alpha = 1)$ , which is again equivalent to its joint convexity. Joint convexity for (t) = b and  $\alpha \in (0, 1)$  follows immediately from (37) and the joint convexity of the relative entropy. An alternative proof can be obtained from (i) of [32, Theorem 1.1] by taking  $A = \rho$ ,  $B = \sigma$ ,  $\Phi = \Psi = id$ ,  $p = \alpha/z$ ,  $q = (1 - \alpha)/z$ , taking the limit  $z \to +\infty$ , and using (27) from [7]. Failure of joint convexity

for  $(t) = \{ \}$  and  $\alpha > 2$  was pointed out in [50]; see also [48, Appendix A]. Failure of joint convexity for (t) = \* and  $\alpha < 1/2$ , was also pointed out in [50], based on numerical counterexamples; an analytic proof was given in [9]. We are only left to prove the failure of joint convexity of  $Q^{\flat}_{\alpha}$  (monotonicity of  $D^{\flat}_{\alpha}$ ) for  $\alpha > 1$ :

**Lemma III.17**  $Q^{\flat}_{\alpha}$  is not monotone under CPTP maps for any  $\alpha > 1$ . In fact, it is not even monotone under pinching by the reference operator; that is, for every  $\alpha > 1$ , there exist  $\varrho, \sigma \in \mathcal{L}(\mathcal{H})_+$  such that

$$Q^{\flat}_{\alpha}(\varrho \| \sigma) < Q^{\flat}_{\alpha}(\mathcal{E}_{\sigma} \varrho \| \sigma).$$

As a consequence,  $Q^{\flat}_{\alpha}$  is not jointly convex for  $\alpha > 1$ .

**Proof** Let  $\rho := \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$  and  $\sigma := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ , a, b > 0. A straightforward computation shows that

$$Q^{\flat}_{\alpha}(\varrho \| \sigma) = (ab)^{\frac{1-\alpha}{2}}.$$

If we take  $a \neq b$  then  $\mathcal{E}_{\sigma} \varrho = \frac{1}{2}I$ , and

$$Q^{\flat}_{\alpha}(\mathcal{E}_{\sigma}\varrho\|\sigma) = \frac{a^{1-\alpha} + b^{1-\alpha}}{2^{\alpha}}$$

for every  $\alpha > 1$ . Thus, our aim is to find a and b such that

$$\sqrt{a^{1-\alpha}b^{1-\alpha}} < \frac{a^{1-\alpha} + b^{1-\alpha}}{2^{\alpha}}, \quad \text{or equivalently}, \quad c \le \frac{1+c^2}{2^{\alpha}}, \quad \text{where} \quad c := \sqrt{b/a}^{1-\alpha}.$$

It is easy to see that this latter inequality has positive solutions, providing examples such that  $Q^{\flat}_{\alpha}(\varrho \| \sigma) < Q^{\flat}_{\alpha}(\mathcal{E}_{\sigma}\varrho \| \sigma)$ .

Since the log is concave and increasing,  $s(\alpha)\psi_{\alpha}^{(t)}$  is jointly convex for any  $\alpha \in [0, 1)$  for which  $s(\alpha)Q_{\alpha}^{(t)}$  is jointly convex. On the other hand, it is well-known and easy to verify that  $s(\alpha)\psi_{\alpha}^{(t)}$  is not jointly convex for  $\alpha > 1$  even for classical probability distributions. However, we have the following partial convexity properties:

**Proposition III.18** For every  $\rho \in \mathcal{L}(\mathcal{H})_+$ , the functions

$$\sigma \mapsto s(\alpha)Q_{\alpha}^{(t)}(\varrho \| \sigma), \qquad \sigma \mapsto s(\alpha)\psi_{\alpha}^{(t)}(\varrho \| \sigma), \qquad \sigma \mapsto D_{\alpha}^{(t)}(\varrho \| \sigma)$$

are convex on  $\mathcal{L}(\mathcal{H})_+$  for  $(t) = \flat$  and  $\alpha \in [0, +\infty)$ , for  $(t) = \ast$  and  $\alpha \in [1/2, +\infty)$ , and for  $(t) = \{ \}$  and  $\alpha \in [0, 2]$ . Moreover,  $\sigma \mapsto D^{\flat}_{\infty}(\varrho \| \sigma)$  and  $\sigma \mapsto D^{\ast}_{\infty}(\varrho \| \sigma)$  are also convex.

**Proof** When  $\alpha < 1$ , the assertions follow immediately from Theorem III.16. The assertions about  $Q_1^{(t)}$  and  $\psi_1^{(t)}$  are obvious from their definitions, and the assertion about  $D_1^{(t)}$  follows from the joint convexity of the relative entropy. Hence for the rest we assume that  $\alpha > 1$ . Note that convexity of  $\psi_{\alpha}^{(t)}(\varrho \| .)$  is equivalent to the log-convexity of  $Q_{\alpha}^{(t)}(\varrho \| .)$ , which is stronger than convexity. Moreover, since  $\psi_{\alpha}^{(t)}(\varrho \| .)$  and  $D_{\alpha}^{(t)}(\varrho \| .)$  only differ in a positive constant multiplier, it is enough to show the convexity of  $D_{\alpha}^{(t)}(\varrho \| .)$ . Thus, we have to show that for any  $\varrho, \sigma_1, \sigma_2 \in \mathcal{L}(\mathcal{H})_+$  and any  $\lambda \in (0, 1)$ ,

$$D_{\alpha}^{(t)}(\varrho \| (1-\lambda)\sigma_1 + \lambda\sigma_2) \le (1-\lambda)D_{\alpha}^{(t)}(\varrho \| \sigma_1) + \lambda D_{\alpha}^{(t)}(\varrho \| \sigma_2).$$
(65)

By (35), we may and will assume that  $\rho, \sigma_1, \sigma_2$  are all invertible.

We first consider the case (t) = b. By Theorem III.6,

$$D^{\flat}_{\alpha}(\varrho \| (1-\lambda)\sigma_{1}+\lambda\sigma_{2}) = \sup_{\tau \in \mathcal{S}(\mathcal{H})} \left\{ D(\tau \| (1-\lambda)\sigma_{1}+\lambda\sigma_{2}) - \frac{\alpha}{\alpha-1}D(\tau \| \varrho) \right\}$$

$$\leq \sup_{\tau \in \mathcal{S}(\mathcal{H})} \left\{ (1-\lambda)D(\tau \| \sigma_{1}) + \lambda D(\tau \| \sigma_{2}) - \frac{\alpha}{\alpha-1}D(\tau \| \varrho) \right\}$$

$$\leq \sup_{\tau_{1},\tau_{2} \in \mathcal{S}(\mathcal{H})} \left\{ (1-\lambda) \left( D(\tau_{1} \| \sigma_{1}) - \frac{\alpha}{\alpha-1}D(\tau_{1} \| \varrho) \right) + \lambda \left( D(\tau_{2} \| \sigma_{2}) - \frac{\alpha}{\alpha-1}D(\tau_{2} \| \varrho) \right) \right\}$$

$$= (1-\lambda)D^{\flat}_{\alpha}(\varrho \| \sigma_{1}) + \lambda D^{\flat}_{\alpha}(\varrho \| \sigma_{2}), \tag{66}$$

where the first inequality is due to the convexity of the relative entropy in its second argument, the second inequality is obvious, and the last line is again due to Theorem III.6. Convexity of  $\sigma \mapsto D_{\infty}^{\flat}(\varrho \| \sigma)$  follows by taking the limit  $\alpha \to +\infty$ .

Next, we consider (t) = \*. By [50, Lemma 12], we have

$$D^*_{\alpha}(\varrho \| \sigma) = \frac{\alpha}{\alpha - 1} \sup_{\tau \in \mathcal{L}(\mathcal{H})_+, \operatorname{Tr} \tau \leq 1} \log \omega_{\tau}(f_{\alpha}(\sigma)),$$

where  $f_{\alpha}(x) = x^{(1-\alpha)/\alpha}$ ,  $x \in (0, +\infty)$ , and  $\omega_{\tau}(X) := \text{Tr}(X \varrho^{1/2} \tau^{(\alpha-1)/\alpha} \varrho^{1/2})$ ,  $X \in \mathcal{L}(\mathcal{H})$ . Obviously,  $\omega_{\tau}$  is a positive linear functional, and for  $\alpha > 1$ ,  $f_{\alpha}$  is operator monotone decreasing [10]. Hence, by Lemma II.2,  $\sigma \mapsto D^*_{\alpha}(\varrho \| \sigma)$  is the supremum of convex functions, and hence is itself convex.

The case  $(t) = \{\}$  follows by a similar argument; see [47, Theorem II.1] for details.

### C. Further properties and relations

Here we establish some further properties of the Rényi divergences that are going to be useful later in the paper. The following Lemma is easy to verify:

**Lemma III.19** Let  $\varrho, \sigma \in \mathcal{L}(\mathcal{H})_+$ ,  $\alpha \in (0, +\infty] \setminus \{1\}$ , and t be any of the three possible values.

1. The Q quantities are multiplicative, and hence the corresponding Rényi divergences are additive in the sense that for every  $n \in \mathbb{N}$ ,

$$Q_{\alpha}^{(t)}(\varrho^{\otimes n} \| \sigma^{\otimes n}) = Q_{\alpha}^{(t)}(\varrho \| \sigma)^{n}, \qquad D_{\alpha}^{(t)}(\varrho^{\otimes n} \| \sigma^{\otimes n}) = n D_{\alpha}(\varrho \| \sigma).$$
(67)

2. For every  $\lambda > 0$ ,

$$\begin{aligned} Q_{\alpha}^{(t)}(\lambda \varrho \| \sigma) &= \lambda^{\alpha} Q_{\alpha}^{(t)}(\varrho \| \sigma), \\ Q_{\alpha}^{(t)}(\varrho \| \lambda \sigma) &= \lambda^{1-\alpha} Q_{\alpha}^{(t)}(\varrho \| \sigma), \end{aligned} \qquad \qquad D_{\alpha}^{(t)}(\lambda \varrho \| \sigma) = D_{\alpha}^{(t)}(\varrho \| \sigma) + \log \lambda, \\ D_{\alpha}^{(t)}(\varrho \| \lambda \sigma) &= D_{\alpha}^{(t)}(\varrho \| \sigma) - \log \lambda. \end{aligned}$$

We have the following ordering of the Rényi divergences:

**Proposition III.20** For any  $\rho, \sigma \in \mathcal{L}(\mathcal{H})_+$ , we have

$$D^*_{\alpha}(\varrho \| \sigma) \le D_{\alpha}(\varrho \| \sigma) \le D^{\flat}_{\alpha}(\varrho \| \sigma), \qquad \alpha \in [0, 1), \tag{68}$$

$$D^{\flat}_{\alpha}(\varrho \| \sigma) \le D^{*}_{\alpha}(\varrho \| \sigma) \le D_{\alpha}(\varrho \| \sigma), \qquad \alpha \in (1, +\infty].$$
(69)

**Proof** It is enough to prove the inequalities for positive definite  $\rho$  and  $\sigma$ , as the general case then follows by (18). The inequality  $D^*_{\alpha}(\rho \| \sigma) \leq D_{\alpha}(\rho \| \sigma)$  is equivalent to the Araki-Lieb-Thirring inequality [4, 43]. By the Golden-Thompson inequality [22, 71, 72],  $\operatorname{Tr} e^{A+B} \leq \operatorname{Tr} e^A e^B$  for any self-adjoint A, B. This yields that  $D_{\alpha}(\rho \| \sigma) \leq D^{\flat}_{\alpha}(\rho \| \sigma)$  for  $\alpha \in (0, 1)$ , and  $D_{\alpha}(\rho \| \sigma) \geq D^{\flat}_{\alpha}(\rho \| \sigma)$  for  $\alpha > 1$ . (Vice versa, the inequality  $s(\alpha)D_{\alpha}(\rho \| \sigma) \geq s(\alpha)D^{\flat}_{\alpha}(\rho \| \sigma)$  for a fixed  $\alpha \in (0, +\infty) \setminus \{1\}$  and every  $\rho, \sigma \in \mathcal{L}(\mathcal{H})_{++}$  implies the Golden-Thompson inequality.) Hence, we are left to prove the first inequality in (69).

Let us fix  $\rho, \sigma \in \mathcal{L}(\mathcal{H})_{++}$  and  $\alpha \in (1, +\infty)$ , and for every  $n \in \mathbb{N}$ , let  $\rho_n := \rho^{\otimes n}, \sigma_n := \sigma^{\otimes n}$ . Then

$$Q_{\alpha}^{\flat}(\varrho \| \sigma)^{n} = Q_{\alpha}^{\flat}(\varrho^{\otimes n} \| \sigma^{\otimes n}) = \operatorname{Tr} e^{\alpha \log \varrho_{n} + (1-\alpha) \log \sigma_{n}}$$
  

$$\leq \operatorname{Tr} e^{\alpha \log \mathcal{E}_{\sigma_{n}}(\varrho_{n}) + \alpha \log v(\sigma_{n}) + (1-\alpha) \log \sigma_{n}}$$
  

$$= v(\sigma_{n})^{\alpha} \operatorname{Tr} \left(\mathcal{E}_{\sigma_{n}}(\varrho_{n})\right)^{\alpha} \sigma_{n}^{1-\alpha} = v(\sigma_{n})^{\alpha} Q_{\alpha}^{*} \left(\mathcal{E}_{\sigma_{n}}(\varrho_{n}) \| \sigma_{n}\right)$$
  

$$\leq v(\sigma_{n})^{\alpha} Q_{\alpha}^{*} \left(\varrho_{n} \| \sigma_{n}\right) = v(\sigma_{n})^{\alpha} Q_{\alpha}^{*} \left(\varrho \| \sigma\right)^{n},$$

where the first and the last identities are due to (67), and the first inequality is due to the pinching inequality (6),  $\varrho_n \leq v(\sigma_n)\mathcal{E}_{\sigma_n}(\varrho_n)$ , the operator monotonicity of the logarithm, and Lemma II.1. The equalities in the third line are due to the fact that  $\sigma_n$  and  $\mathcal{E}_{\sigma_n}(\varrho_n)$  commute, and the last inequality is due to the monotonicity of  $D^*_{\alpha}$  under pinching [50, Proposition 14]. Taking now the *n*-th root and then the limit  $n \to +\infty$ , and using that  $v(\sigma_n) \leq (n+1)^{d-1}$ , we get the desired inequality.

**Remark III.21** It is known that equality in the inequality  $D^*_{\alpha}(\varrho \| \sigma) \leq D_{\alpha}(\varrho \| \sigma)$  holds if and only if  $\alpha = 1$  or  $\varrho$  and  $\sigma$  commute with each other [31]. It is also easy to see that the other inequalities don't hold with equality in general, either. Indeed, choosing  $\varrho := \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$  and  $\sigma := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ , a, b > 0, a straightforward computation shows that

$$Q_{\alpha}^{\flat}(\varrho\|\sigma) = (ab)^{\frac{1-\alpha}{2}}, \qquad Q_{\alpha}^{*}(\varrho\|\sigma) = \left(\frac{a^{\frac{1-\alpha}{\alpha}} + b^{\frac{1-\alpha}{\alpha}}}{2}\right)^{\alpha}, \qquad Q_{\alpha}(\varrho\|\sigma) = \frac{a^{1-\alpha} + b^{1-\alpha}}{2},$$

which are not equal to each other for general a and b.

All the Rényi divergences are strictly positive on pairs of states:

**Proposition III.22** For every  $\alpha > 0$ , and all three values of t,

$$D_{\alpha}^{(t)}(\varrho \| \sigma) \ge \log \operatorname{Tr} \varrho - \log \operatorname{Tr} \sigma, \qquad \varrho, \sigma \in \mathcal{L}(\mathcal{H})_{+},$$
(70)

with equality if and only if  $\rho$  is a constant multiple of  $\sigma$ , or equivalently,

$$D_{\alpha}^{(t)}(\varrho \| \sigma) \ge 0, \qquad \qquad \varrho, \sigma \in \mathcal{S}(\mathcal{H}), \tag{71}$$

with equality if and only if  $\rho = \sigma$ .

**Proof** The equivalence of (70) and (71) is immediate from the scaling properties in Lemma III.19, and hence we only prove (71). Moreover, by the ordering in (68) and the monotonicity in (57), it is enough to consider  $D^*_{\alpha}$  and  $\alpha \in (0, 1)$ . By the monotonicity under pinching [50, Proposition 14] and the classical Hölder inequality, we have  $Q^*_{\alpha}(\varrho \| \sigma) \leq Q^*_{\alpha}(\mathcal{E}_{\sigma}\varrho \| \sigma) \leq (\operatorname{Tr} \mathcal{E}_{\sigma}(\varrho))^{\alpha} (\operatorname{Tr} \sigma)^{1-\alpha} \leq 1$ , which yields (71). As a consequence,  $\psi^*_{\alpha}(\varrho \| \sigma) \leq 0$  for every  $\alpha \in (0, 1)$ . Now if  $D^*_{\alpha}(\varrho \| \sigma) = 0$  for some  $\alpha \in (0, 1)$  then  $\psi^*_{\alpha}(\varrho \| \sigma) = 0$ . By the convexity of  $\psi^*_{\alpha}(\varrho \| \sigma)$  in  $\alpha$ , (Lemma III.12), this is only possible if  $\psi^*_{\alpha}(\varrho \| \sigma) = 0$  for every  $\alpha \in (0, 1)$ . Hence,  $D^*_{\alpha}(\varrho \| \sigma) = 0$ ,  $\alpha \in (0, 1)$ , and taking the limit  $\alpha \nearrow 1$  yields  $D(\varrho \| \sigma) = 0$ . By (32) this implies  $\varrho = \sigma$ .

**Remark III.23** An alternative proof for the case (t) = \* has been given in [8, Theorem 5].

**Lemma III.24** For every  $\rho \in \mathcal{L}(\mathcal{H})_+$ , we have

$$\sigma' \ge \sigma \implies D_{\alpha}^{(t)}(\varrho \| \sigma') \le D_{\alpha}^{(t)}(\varrho \| \sigma)$$

for  $(t) = \{\}$  and  $\alpha \in [0,1]$ , for (t) = \* and  $\alpha \in [1/2, +\infty]$ , and for  $(t) = \flat$  and  $\alpha \in [0, +\infty]$ .

**Proof** By (35), we can assume without loss of generality that  $\rho, \sigma, \sigma'$  are invertible. The assertions then follow immeditely from the following: For  $(t) = \{ \}$  from the fact that  $x \mapsto x^{\alpha}, x \in (0, +\infty)$ , is operator monotone increasing for  $\alpha \in (0, 1)$ ; for (t) = \* from the fact that  $x \mapsto x^{\frac{1-\alpha}{\alpha}}$  is operator monotone increasing for  $\alpha \in [1/2, 1)$ , and operator monotone decreasing for  $\alpha \in (1, +\infty)$ , and from (7); and for  $(t) = \flat$  from the fact that the logarithm is operator monotone increasing. When necessary, the cases  $\alpha = 0, 1, +\infty$  can be obtained by taking the appropriate limit in  $\alpha$ .

**Remark III.25** The case (t) = \* has been shown in [50, Proposition 4], in a slightly different way.

The following technical lemma will play an important role in the minimax arguments used to establish the equivalence of the various definitions of the Rényi capacities in Proposition IV.2.

Recall the definition of  $s(\alpha)$  from (36).

Lemma III.26 For any of the three possible values of t, define

$$\mathcal{A} := \mathcal{A}^* := (0, +\infty) \setminus \{1\}, \qquad \mathcal{A}^\flat := (1, +\infty).$$

$$\tag{72}$$

Let  $\varrho, \sigma \in \mathcal{L}(\mathcal{H})_+$ , and let  $\alpha \in \mathcal{A}^{(t)}$ .

1. The function  $\varepsilon \mapsto s(\alpha)Q_{\alpha}^{(t)}(\varrho \| \sigma + \varepsilon I)$  is monotone decreasing in  $\varepsilon \in (0, +\infty)$ , and

$$s(\alpha)Q_{\alpha}^{(t)}(\varrho\|\sigma) = \lim_{\varepsilon \searrow 0} s(\alpha)Q_{\alpha}^{(t)}(\varrho\|\sigma + \varepsilon I) = \sup_{\varepsilon > 0} s(\alpha)Q_{\alpha}^{(t)}(\varrho\|\sigma + \varepsilon I).$$
(73)

The same hold for  $s(\alpha)\psi_{\alpha}^{(t)}$  and  $D_{\alpha}^{(t)}$  in place of  $s(\alpha)Q_{\alpha}^{(t)}$ , and also for  $D_{\infty}^{(t)}$ . 2. If  $\sigma$  is invertible then the function  $\varepsilon \mapsto Q_{\alpha}^{(t)}(\varrho + \varepsilon I \| \sigma)$  is monotone increasing in  $\varepsilon \in (0, +\infty)$ , and

$$Q_{\alpha}^{(t)}(\varrho \| \sigma) = \lim_{\varepsilon \searrow 0} Q_{\alpha}^{(t)}(\varrho + \varepsilon I \| \sigma) = \inf_{\varepsilon > 0} Q_{\alpha}^{(t)}(\varrho + \varepsilon I \| \sigma).$$
(74)

The same hold for  $\psi_{\alpha}^{(t)}$  and  $s(\alpha)D_{\alpha}^{(t)}$  in place of  $Q_{\alpha}^{(t)}$ . Moreover, these relations are valid also when  $(t) = \flat and \alpha \in (0, 1).$ 

3. We have

$$Q_{\alpha}^{(t)}(\varrho\|\sigma) = \lim_{\varepsilon \searrow 0} \lim_{\delta \searrow 0} Q_{\alpha}^{(t)}(\varrho + \delta I \|\sigma + \varepsilon I) = \begin{cases} \inf_{\varepsilon > 0} \inf_{\delta > 0} Q_{\alpha}^{(t)}(\varrho + \delta I \|\sigma + \varepsilon I), & \alpha \in (0, 1), \\ \sup_{\varepsilon > 0} \inf_{\delta > 0} Q_{\alpha}^{(t)}(\varrho + \delta I \|\sigma + \varepsilon I), & \alpha > 1. \end{cases}$$
(75)

and the same hold for  $\psi_{\alpha}^{(t)}$  and  $s(\alpha)D_{\alpha}^{(t)}$  in place of  $Q_{\alpha}^{(t)}$ .

**Proof** Note that (75) is immediate from (73) and (74), and the claims about the monotonicity are trivial to verify, and hence the second identities in (73) and (74) follow if we can prove the first identities. It is easy to see that for an invertible  $\sigma$ ,

$$\lim_{\varepsilon \searrow 0} Q_{\alpha}^{(t)}(\varrho + \varepsilon I \| \sigma) = \lim_{\varepsilon \searrow 0} Q_{\alpha}^{(t)}(\varrho + \varepsilon (I - \varrho^0) \| \sigma),$$

and thus (74) follows from (19) and Lemma III.1.

We only prove (73) for (t) = b, as the other cases follow by very similar, and slightly simpler arguments. Let  $P_s$  denote the spectral projection of  $\sigma$  corresponding to  $s \in \mathbb{R}$ ; if s is not an eigenvalue of  $\sigma$  then  $P_s = 0$ . Then

$$\varrho^{0}(\widehat{\log}(\sigma+\varepsilon I))\varrho^{0} = \sum_{s>0} \varrho^{0} P_{s} \varrho^{0} \log(s+\varepsilon) + \varrho^{0}(I-\sigma^{0})\varrho^{0} \log\varepsilon.$$

If  $\varrho^0 \leq \sigma^0$  then  $\varrho^0(I - \sigma^0)\varrho^0 = 0$ , and (73) follows trivially. Assume next that  $\varrho^0 \nleq \sigma^0$ . Then there exists some c > 0 such that  $\varrho^0(I - \sigma^0)\varrho^0 \geq cQ$ , where  $Q := (\varrho^0(I - \sigma^0)\varrho^0)^0 \neq 0$ , and  $Q \leq \varrho^0$ . Hence, for every  $\varepsilon \in (0, 1)$ ,

$$\varrho^0(\widehat{\log}(\sigma + \varepsilon I))\varrho^0 \le \kappa_{\varepsilon}\varrho^0 + (\log \varepsilon)cQ,$$

where  $\kappa_{\varepsilon} := \max\{0, \log(\|\sigma\| + \varepsilon)\}$ . Let  $\varrho_{\min}$  denote the smallest non-zero eigenvalue of  $\varrho$ . By the above,

$$\operatorname{Tr} \varrho^{0} e^{\alpha \operatorname{\widehat{\log}} \varrho + (1-\alpha) \varrho^{0} (\operatorname{\widehat{\log}} (\sigma + \varepsilon I)) \varrho^{0}} \geq \operatorname{Tr} \varrho^{0} e^{\alpha (\log \varrho_{\min}) \varrho^{0} + (1-\alpha) \kappa_{\varepsilon} \varrho^{0} + (1-\alpha) (\log \varepsilon) cQ}$$
$$= \varrho_{\min}^{\alpha} e^{\kappa_{\varepsilon} (1-\alpha)} \operatorname{Tr} \left[ \varepsilon^{c(1-\alpha)} Q + \varrho^{0} - Q \right],$$

and the last quantity goes to  $+\infty = Q^{\flat}_{\alpha}(\varrho \| \sigma)$  as  $\varepsilon \searrow 0$ .

For  $\alpha \in \mathcal{A}^{(t)}$ , the assertions about  $\psi_{\alpha}^{(t)}$  and  $D_{\alpha}^{(t)}$  follow trivially from (73) and (74). Finally,  $\varepsilon \mapsto$  $D_{\infty}^{(t)}(\rho \| \sigma + \varepsilon I)$  is the pointwise limit of monotone functions, and hence is itself monotone, and

$$\begin{split} \lim_{\varepsilon \searrow 0} D_{\infty}^{(t)}(\varrho \| \sigma + \varepsilon I) &= \sup_{\varepsilon > 0} D_{\infty}^{(t)}(\varrho \| \sigma + \varepsilon I) = \sup_{\varepsilon > 0} \sup_{\alpha \in (1, +\infty)} D_{\alpha}^{(t)}(\varrho \| \sigma + \varepsilon I) \\ &= \sup_{\alpha \in (1, +\infty)} \sup_{\varepsilon > 0} D_{\alpha}^{(t)}(\varrho \| \sigma + \varepsilon I) = \sup_{\alpha \in (1, +\infty)} D_{\alpha}^{(t)}(\varrho \| \sigma) = D_{\infty}^{(t)}(\varrho \| \sigma). \end{split}$$

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**Corollary III.27** Let t be any of the three possible values, and let  $\alpha \in \mathcal{A}^{(t)}$ , where  $\mathcal{A}^{(t)}$  is given in (72). For every  $\varrho \in \mathcal{L}(\mathcal{H})_+$ , the function

$$\sigma \mapsto s(\alpha)Q_{\alpha}^{(t)}(\varrho \| \sigma)$$
 is lower semicontinuous on  $\mathcal{L}(\mathcal{H})_+,$ 

and the same hold for  $s(\alpha)\psi_{\alpha}^{(t)}$  and  $D_{\alpha}^{(t)}$  in place of  $s(\alpha)Q_{\alpha}^{(t)}$ . For every  $\sigma \in \mathcal{L}(\mathcal{H})_{++}$ , the function

 $\varrho \mapsto Q_{\alpha}^{(t)}(\varrho \| \sigma)$  is upper semicontinuous on  $\mathcal{L}(\mathcal{H})_+$ ,

and the same hold for  $\psi_{\alpha}^{(t)}$  and  $s(\alpha)D_{\alpha}^{(t)}$  in place of  $Q_{\alpha}^{(t)}$ . Moreover,  $\sigma \mapsto D_{\infty}^{(t)}(\varrho \| \sigma)$  is also lower semicontinuous on  $\mathcal{L}(\mathcal{H})_+$ . The assertions about upper semicontinuity are also valid for  $(t) = \flat$  and  $\alpha \in (0, 1)$ .

**Proof** Let  $\alpha \in \mathcal{A}^{(t)}$  be fixed. For every  $\varepsilon > 0$ ,  $\sigma \mapsto s(\alpha)Q_{\alpha}^{(t)}(\varrho \| \sigma + \varepsilon)$  is continuous. Hence, by Lemma III.26, the function  $\sigma \mapsto s(\alpha)Q_{\alpha}^{(t)}(\varrho \| \sigma)$  is the supremum of continuous functions, and thus is itself lower semicontinuous. Similarly, if  $\sigma \in \mathcal{L}(\mathcal{H})_{++}$  then  $\varrho \mapsto Q_{\alpha}^{(t)}(\varrho + \varepsilon I \| \sigma)$  is continuous for every  $\varepsilon > 0$ , and hence, by Lemma III.26, the function  $\varrho \mapsto Q_{\alpha}^{(t)}(\varrho \| \sigma)$  is the infimum of continuous functions and thus upper semicontinuous. The assertions about  $\psi_{\alpha}^{(t)}$  and  $D_{\alpha}^{(t)}$  follow immediately. In particular,  $\sigma \mapsto D_{\infty}^{(t)}(\varrho \| \sigma)$  is the supremum of lower semicontinuous functions in  $\sigma$ , and hence is itself lower semicontinuous.

# IV. RÉNYI CAPACITIES

The celebrated Holevo-Schumacher-Westmoreland theorem [37, 65] states that the asymptotic classical information transmission capacity of a quantum channel under the constraint of asymptotically vanishing error probability coincides with its Holevo capacity; see Section VA for details. Based on results in classical information theory, it is natural to expect that a more refined description of the trade-off between the coding rate and the decoding error would involve Rényi generalizations of the Holevo capacity. In the main result of our paper, Theorem V.2, we will show that this is indeed the case for the strong converse exponent of classical-quantum channels. In this section we collect the necessary definitions and technicalities that we will need later in the proof of our main result.

#### A. Rényi mutual informations and equivalent definitions of the Rényi capacities

For a quantum channel  $W : \mathcal{X} \to \mathcal{S}(\mathcal{H})$  and a finitely supported probability distribution  $P \in \mathcal{P}_f(\mathcal{X})$ , the corresponding *Holevo quantity*  $\chi(W, P)$  is defined as the mutual information in the classical-quantum state W(P), expressable in the following equivalent ways:

$$\chi(W,P) := D(\mathbb{W}(P) \| P \otimes W(P)) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} D(\mathbb{W}(P) \| P \otimes \sigma)$$
(76)

$$= \sum_{x \in \mathcal{X}} P(x)D(W(x) \| W(P)) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sum_{x \in \mathcal{X}} P(x)D(W(x) \| \sigma).$$
(77)

(Recall that in  $P \otimes \sigma$  in (76), P stands for  $\sum_{x \in \mathcal{X}} P(x)|x\rangle\langle x|$ , the first marginal of  $\mathbb{W}(P)$  (14).) It is also customary to call  $\chi(W, P)$  the Holevo quantity of the ensemble of states  $\{\varrho, P(\varrho)\}_{\varrho \in \text{supp } P}$ .

The Holevo capacity  $\chi(W)$  of the channel W is then defined as the supremum of all such mutual informations over all finitely supported probability distributions at the input of the channel:

$$\chi(W) := \sup_{P \in \mathcal{P}_f(\mathcal{X})} \chi(W, P) = \sup_{P \in \mathcal{P}_f(\mathcal{X})} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} D(\mathbb{W}(P) \| P \otimes \sigma)$$
(78)

$$= \sup_{P \in \mathcal{P}_{f}(\mathcal{X})} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sum_{x \in \mathcal{X}} P(x) D(W(x) \| \sigma).$$
(79)

When the relative entropy is replaced with some Rényi divergence, the expressions in (76)-(77) may not coincide anymore, and therefore we have various options to formally define the Rényi analogues of the Holevo quantity. We will consider the following two options that will both turn out to be useful for later applications:

$$\chi_{\alpha,1}^{(t)}(W,P) := \inf_{\sigma \in \mathcal{S}(\mathcal{H})} D_{\alpha}^{(t)}(\mathbb{W}(P) \| P \otimes \sigma), \qquad \alpha \in [0, +\infty].$$
(80)

$$\chi_{\alpha,2}^{(t)}(W,P) := \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sum_{x \in \mathcal{X}} P(x) D_{\alpha}^{(t)}(W(x) \| \sigma), \qquad \alpha \in [0, +\infty].$$
(81)

These are exact analogues of the second formulas in (76) and (77), respectively. Note that (80) is a notion of Rényi mutual information in the classical-quantum state W(P); see, e.g. [29] for an operational interpretation of this quantity in the context of hypothesis testing (for  $(t) = \{ \}$  and (t) = \*). A straightforward computation verifies that for all  $\alpha$ ,

$$Q_{\alpha}^{(t)}(\mathbb{W}(P) \| P \otimes \sigma) = \sum_{x \in \mathcal{X}} P(x) Q_{\alpha}^{(t)}(W(x) \| \sigma),$$

and hence

$$\chi_{\alpha,1}^{(t)}(W,P) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \frac{1}{\alpha - 1} \log \sum_{x \in \mathcal{X}} P(x) Q_{\alpha}^{(t)}(W(x) \| \sigma).$$
(82)

We define the *Rényi capacities* of a channel W, corresponding to the Rényi  $\alpha$ -divergence  $D_{\alpha}^{(t)}$ , as

$$\chi_{\alpha,i}^{(t)}(W) := \sup_{P \in \mathcal{P}_f(\mathcal{X})} \chi_{\alpha,i}^{(t)}(W, P), \qquad i = 1, 2.$$
(83)

Although in general the identities in (76)–(77) do not extend to the Rényi quantities with  $\alpha \neq 1$ , we will show in Proposition IV.2 below that after the optimization over all finitely supported probability distributions, the possible differences disappear, and we have

$$\chi_{\alpha}^{(t)}(W) := \chi_{\alpha,1}^{(t)}(W) = \chi_{\alpha,2}^{(t)}(W), \tag{84}$$

at least for certain pairs of  $((t), \alpha)$ , including those important for us later. Thus, in these cases we can uniquely define the Rényi capacity of a channel W. We will show the equality in (84) by showing that both  $\chi_{\alpha,1}^{(t)}(W)$  and  $\chi_{\alpha,2}^{(t)}(W)$  are equal to the *Rényi divergence radius* of the image of the channel, defined as

$$R_{\alpha}^{(t)}(W) := \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{x \in \mathcal{X}} D_{\alpha}^{(t)}(W(x) \| \sigma).$$
(85)

The equality (84) will play an important role in proving our main theorem. Indeed, following the Dueck-Körner argument for classical channels, we naturally get a bound on the strong converse exponent in terms of the second type Rényi-Holevo quantities, more precisely, the  $\chi^{\flat}_{\alpha,2}$  quantities; see Theorems V.6, V.7, and V.12. To convert that to the correct form involving the  $\chi^{\ast}_{\alpha}$  capacities, we use additivity properties that are only known for the first type Rényi-Holevo quantities, more precisely for  $\chi_{\alpha,1}$  and  $\chi^{\ast}_{\alpha,1}$ ; see Lemma IV.8, Corollary IV.11, and the last part of the proof of Theorem V.14. The identity in (84) tells that we can freely switch between the two types of Rényi-Holevo quantities after optimization over the input probability distributions.

**Remark IV.1** Note that for  $\alpha \in \mathcal{A}^{(t)}$  (defined in (72)), the infima in (80), (81) and (85) can be replaced with minima, due to Corollary III.27.

**Proposition IV.2** We have

$$R_{\alpha}^{(t)}(W) = \sup_{\varepsilon > 0} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{x \in \mathcal{X}} D_{\alpha}^{(t)}(W(x) \| \sigma + \varepsilon I)$$
(86)

$$= \sup_{P \in \mathcal{P}_f(\mathcal{X})} \chi_{\alpha,2}^{(t)}(W,P), \tag{87}$$

$$= \sup_{P \in \mathcal{P}_f(\mathcal{X})} \chi_{\alpha,1}^{(t)}(W,P)$$
(88)

$$=\chi_{\alpha}^{(t)}(W).\tag{89}$$

for  $(t) = \{\}$  and  $\alpha \in (0,2]$ , for (t) = \* and  $\alpha \in [1/2, +\infty)$ , and for  $(t) = \flat$  and  $\alpha \in (1, +\infty)$ . Moreover, the expressions in (87) and (88) are also equal to each other for  $(t) = \flat$  and  $\alpha \in (0,1)$ .

**Proof** Let us fix a matching pair t and  $\alpha$  as in the statement of the Theorem. We assume that  $\alpha \neq 1$ , since that case is already known [59, 65]. By definition,

$$R_{\alpha}^{(t)}(W) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{x \in \mathcal{X}} D_{\alpha}^{(t)}(W(x) \| \sigma) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{x \in \mathcal{X}} \sup_{\varepsilon > 0} D_{\alpha}^{(t)}(W(x) \| \sigma + \varepsilon I)$$
(90)  
$$= \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{\varepsilon > 0} \sup_{x \in \mathcal{X}} D_{\alpha}^{(t)}(W(x) \| \sigma + \varepsilon I),$$

where the last expression in (90) follows from Lemma III.26. Note that  $\varepsilon \mapsto \sup_{x \in \mathcal{X}} D_{\alpha}^{(t)}(W(x) \| \sigma + \varepsilon I)$ is monotone decreasing for every  $\sigma \in \mathcal{S}(\mathcal{H})$ , due to Lemma III.26. On the other hand, for every  $\varepsilon > 0$ and  $x \in \mathcal{X}$ ,  $\sigma \mapsto D_{\alpha}^{(t)}(W(x) \| \sigma + \varepsilon I)$  is continuous, and hence  $\sigma \mapsto \sup_{x \in \mathcal{X}} D_{\alpha}^{(t)}(W(x) \| \sigma + \varepsilon I)$  is lower semi-continuous on the compact set  $\mathcal{S}(\mathcal{H})$ . Hence, by Lemma II.6,

$$R_{\alpha}^{(t)}(W) = \sup_{\varepsilon > 0} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{x \in \mathcal{X}} D_{\alpha}^{(t)}(W(x) \| \sigma + \varepsilon I),$$

proving (86).

By Lemma III.26,

$$\sup_{P \in \mathcal{P}_f(\mathcal{X})} \chi_{\alpha,1}^{(t)}(W,P) = \frac{1}{\alpha - 1} \log s(\alpha) \sup_{P \in \mathcal{P}_f(\mathcal{X})} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{\varepsilon > 0} s(\alpha) \sum_{x \in \mathcal{X}} P(x) Q_{\alpha}^{(t)}(W(x) \| \sigma + \varepsilon I),$$

where  $s(\alpha)$  is given in (36). Note that  $s(\alpha) \sum_{x \in \mathcal{X}} P(x) Q_{\alpha}^{(t)}(W(x) \| \sigma + \varepsilon I)$  is monotone decreasing in  $\varepsilon$  and continuous in  $\sigma$ , and hence, by Lemma II.6,

$$\sup_{P \in \mathcal{P}_{f}(\mathcal{X})} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{\varepsilon > 0} s(\alpha) \sum_{x \in \mathcal{X}} P(x) Q_{\alpha}^{(t)}(W(x) \| \sigma + \varepsilon I)$$
  
= 
$$\sup_{P \in \mathcal{P}_{f}(\mathcal{X})} \sup_{\varepsilon > 0} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} s(\alpha) \sum_{x \in \mathcal{X}} P(x) Q_{\alpha}^{(t)}(W(x) \| \sigma + \varepsilon I)$$
  
= 
$$\sup_{\varepsilon > 0} \sup_{P \in \mathcal{P}_{f}(\mathcal{X})} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} s(\alpha) \sum_{x \in \mathcal{X}} P(x) Q_{\alpha}^{(t)}(W(x) \| \sigma + \varepsilon I),$$

where the second equality is trivial. For every  $\varepsilon > 0$ ,  $s(\alpha) \sum_{x \in \mathcal{X}} P(x)Q_{\alpha}^{(t)}(W(x) \| \sigma + \varepsilon I)$  is convex and continuous in  $\sigma$  due to Proposition III.18, and it is affine (and thus concave) in P. Hence, by Lemma II.7,

$$\begin{split} \sup_{P \in \mathcal{P}_{f}(\mathcal{X})} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} s(\alpha) \sum_{x \in \mathcal{X}} P(x) Q_{\alpha}^{(t)}(W(x) \| \sigma + \varepsilon I) &= \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{P \in \mathcal{P}_{f}(\mathcal{X})} s(\alpha) \sum_{x \in \mathcal{X}} P(x) Q_{\alpha}^{(t)}(W(x) \| \sigma + \varepsilon I) \\ &= \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{x \in \mathcal{X}} s(\alpha) Q_{\alpha}^{(t)}(W(x) \| \sigma + \varepsilon I), \end{split}$$

where the second equality is trivial. This proves the equality of (86) and (88), and the equality of (88) and (89) is by definition.

Finally, the expression in (87) can be written as

$$\sup_{P \in \mathcal{P}_{f}(\mathcal{X})} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{\varepsilon > 0} \sum_{x \in \mathcal{X}} P(x) D_{\alpha}^{(t)}(W(x) \| \sigma + \varepsilon I) = \sup_{P \in \mathcal{P}_{f}(\mathcal{X})} \sup_{\varepsilon > 0} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sum_{x \in \mathcal{X}} P(x) D_{\alpha}^{(t)}(W(x) \| \sigma + \varepsilon I)$$
$$= \sup_{\varepsilon > 0} \sup_{P \in \mathcal{P}_{f}(\mathcal{X})} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sum_{x \in \mathcal{X}} P(x) D_{\alpha}^{(t)}(W(x) \| \sigma + \varepsilon I).$$

where the first expression is due to Lemma III.26. The second expression follows from the continuity of  $D_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I)$  in  $\sigma$  and its monotonicity in  $\varepsilon$ , due to Lemma II.6. The third expression follows trivially from the second. Using now the convexity of  $D_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I)$  in  $\sigma$ , due to Proposition III.18, and following the same argument as in the previous paragraph, we see that the last expression above is equal to (86).

**Remark IV.3** Proofs of some the above identities for various values of (t) and  $\alpha$  can be found scattered in the literature; see, e.g. [41, 47, 59, 65, 75]. The above Proposition unifies and extends all such previous results; the only exception is the equality  $R_{\alpha}(W) = \sup_{P \in \mathcal{P}_f(\mathcal{X})} \chi_{\alpha,1}(W, P)$  for  $\alpha > 2$ , that was shown in [41], but is not covered by the above Proposition. **Remark IV.4** As in the classical case (see, e.g. [16]), our proof of Proposition IV.2 is based on minimax arguments. However, unlike in the proof in [16], where  $\mathcal{X}$  is assumed to be finite, we cannot assume the compactness of  $\mathcal{P}_f(\mathcal{X})$ , and hence we need to use the compactness of  $\mathcal{S}(\mathcal{H})$  instead. This poses a technical difficulty, as the Rényi divergences can take infinite values when the reference state is not invertible, in which case the usual minimax theorems may not be applicable. We introduced the intermediate step in (86) to circumvent this difficulty. One might also extend the definitions of the Rényi-Holevo quantities using arbitrary (not necessarily finitely supported) probability measures on the image of the channel and work with the compact (w.r.t. the weak topology) set of such probability measures, as was done in [47]. However, the problem of infinite values persists in this case, and some step similar to the one in (86) is still necessary. Moreover, such a treatment requires mathematically more involved arguments, which we could avoid in the above proof.

**Corollary IV.5** For (t) = \* and  $(t) = \flat$ ,

$$\chi_{\infty}^{(t)}(W) = \sup_{P \in \mathcal{P}_{f}(\mathcal{X})} \chi_{\infty,1}^{(t)}(W,P) = \sup_{P \in \mathcal{P}_{f}(\mathcal{X})} \chi_{\infty,2}^{(t)}(W,P) = \sup_{\alpha \in (1,+\infty)} \chi_{\alpha}^{(t)}(W).$$
(91)

**Proof** The first equality in (91) is by definition, and

$$\sup_{P \in \mathcal{P}_{f}(\mathcal{X})} \chi_{\infty,1}^{(t)}(W,P) = \sup_{P \in \mathcal{P}_{f}(\mathcal{X})} \sup_{\alpha \in (1,+\infty)} \chi_{\alpha,1}^{(t)}(W,P) = \sup_{\alpha \in (1,+\infty)} \sup_{P \in \mathcal{P}_{f}(\mathcal{X})} \sup_{\alpha \in (1,+\infty)} \chi_{\alpha,1}^{(t)}(W,P)$$
$$= \sup_{\alpha \in (1,+\infty)} \sup_{P \in \mathcal{P}_{f}(\mathcal{X})} \chi_{\alpha,2}^{(t)}(W,P) = \sup_{P \in \mathcal{P}_{f}(\mathcal{X})} \sup_{\alpha \in (1,+\infty)} \chi_{\alpha,2}^{(t)}(W,P)$$
$$= \sup_{P \in \mathcal{P}_{f}(\mathcal{X})} \chi_{\infty,2}^{(t)}(W,P),$$

where we used Proposition IV.2.

**Lemma IV.6** For all three values of (t), any  $P \in \mathcal{P}_f(\mathcal{X})$ , and  $i = 1, 2, \alpha \mapsto \chi_{\alpha,i}^{(t)}(W, P)$  is monotone increasing on  $(0, +\infty]$ , and

$$\chi(W, P) = \lim_{\alpha \searrow 1} \chi_{\alpha, i}^{(t)}(W, P) = \inf_{\alpha > 1} \chi_{\alpha, i}^{(t)}(W, P),$$
(92)

$$\chi_{\infty,i}^{(t)}(W,P) = \lim_{\alpha \nearrow +\infty} \chi_{\alpha,i}^{(t)}(W,P) = \sup_{1 < \alpha < +\infty} \chi_{\alpha,i}^{(t)}(W,P).$$
(93)

Similarly,  $\alpha \mapsto \chi_{\alpha}^{(t)}(W)$  is monotone increasing on  $(0, +\infty]$ , and

$$\chi_{\infty}^{(t)}(W) = \lim_{\alpha \nearrow +\infty} \chi_{\alpha}^{(t)}(W) = \sup_{1 < \alpha < +\infty} \chi_{\alpha}^{(t)}(W), \tag{94}$$

$$\chi(W) = \lim_{\alpha \searrow 1} \chi_{\alpha}^{(t)}(W) = \inf_{1 < \alpha < +\infty} \chi_{\alpha}^{(t)}(W).$$
(95)

**Proof** The assertions about the monotonicity are obvious from Lemma III.12. Let  $f_{\alpha,1}^{(t)}(W, P, \sigma) := D_{\alpha}^{(t)}(W(P) \| P \otimes \sigma)$ , and  $f_{\alpha,2}^{(t)}(W, P, \sigma) := \sum_{x \in \mathcal{X}} P(x) D_{\alpha}^{(t)}(W(x) \| \sigma)$ . By Lemma III.5 and (76)–(77),

$$\lim_{\alpha \searrow 1} \chi_{\alpha,i}^{(t)}(W,P) = \inf_{\alpha > 1} \chi_{\alpha,i}^{(t)}(W,P) = \inf_{\alpha > 1} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} f_{\alpha,i}^{(t)}(W,P,\sigma) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \inf_{\alpha > 1} f_{\alpha,i}^{(t)}(W,P,\sigma)$$
$$= \inf_{\sigma \in \mathcal{S}(\mathcal{H})} D(\mathbb{W}(P) \| P \otimes \sigma) = \chi(W,P).$$

By Corollary III.27, both  $f_{\alpha,1}$  and  $f_{\alpha,2}$  are lower semicontinuous in  $\sigma$  on  $\mathcal{S}(\mathcal{H})$ , and hence, by Lemma II.6,

$$\lim_{\alpha \nearrow +\infty} \chi_{\alpha,i}^{(t)}(W,P) = \sup_{\alpha>1} \chi_{\alpha,i}^{(t)}(W,P) = \sup_{\alpha>1} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} f_{\alpha,i}^{(t)}(W,P,\sigma) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{\alpha>1} f_{\alpha,i}^{(t)}(W,P,\sigma)$$
$$= \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \left\{ \begin{array}{l} D_{\infty}^{(t)}(\mathbb{W}(P) \| P \otimes \sigma), & i = 1, \\ \sum_{x \in \mathcal{X}} P(x) D_{\infty}^{(t)}(W(x) \| \sigma), & i = 2, \end{array} \right\} = \chi_{\infty,i}^{(t)}(W,P).$$

The identities in (94) are immediate from Corollary IV.5. Finally, (95) has been proved for  $(t) = \{ \}$ in [47, Proposition B.5] (see also [56] for finite  $\mathcal{X}$ ). Thus, by Proposition III.20 and the monotonicity of  $\alpha \mapsto \chi_{\alpha}^{(t)}(W)$ , we get

$$\chi(W) \le \liminf_{\alpha \searrow 1} \chi_{\alpha}^{(t)}(W) \le \limsup_{\alpha \searrow 1} \chi_{\alpha}^{(t)}(W) \le \limsup_{\alpha \searrow 1} \chi_{\alpha}(W) = \chi(W).$$

proving (95) for (t) = \* and  $(t) = \flat$ .

**Remark IV.7** The limit relation in (95) for (t) = \* has been proved in [75, Section 8] by a different method.

#### в. Rényi capacities of pinched channels

For the rest of the section, we fix a channel  $W: \mathcal{X} \to \mathcal{S}(\mathcal{H})$ . For every  $n \in \mathbb{N}$ , let  $\sigma_{u,n}$  be a universal symmetric state on  $\mathcal{H}^{\otimes n}$  as in Lemma II.8. We denote by  $\mathcal{E}_n$  the pinching by  $\sigma_{u,n}$ . If we use the construction from Appendix A then  $\mathcal{E}_n$  can be explicitly written as

$$\mathcal{E}_n(X) = \sum_{\lambda \in Y_{n,d}} (I_{U_\lambda} \otimes I_{V_\lambda}) X (I_{U_\lambda} \otimes I_{V_\lambda}), \qquad X \in \mathcal{L}(\mathcal{H}^{\otimes n}),$$

where  $d := \dim \mathcal{H}$ . By the pinching inequality (6) and Lemma II.8,

$$X \le v(\sigma_{u,n}) \mathcal{E}_n(X) \le v_{n,d} \mathcal{E}_n(X), \qquad X \in \mathcal{L}(\mathcal{H}^{\otimes n})_+,$$

where  $v_{n,d} \leq (n+1)^{\frac{(d+2)(d-1)}{2}}$ . For every  $n \in \mathbb{N}$ , we define the pinched channel  $\mathcal{E}_n W^{\otimes n} : \mathcal{X}^n \to \mathcal{S}(\mathcal{H}^{\otimes n})$  as

$$(\mathcal{E}_n W^{\otimes n})(\underline{x}) := \mathcal{E}_n(W^{\otimes n}(\underline{x})), \qquad \underline{x} \in \mathcal{X}^n.$$

We use the shorthand notation  $\mathcal{E}_n \mathbb{W}^{\otimes n}$  for its lifted channel, i.e.,

$$(\mathcal{E}_n \mathbb{W}^{\otimes n})(\underline{x}) := ((\mathrm{id} \otimes \mathcal{E}_n) \mathbb{W}^{\otimes n})(\underline{x}) = |\underline{x}\rangle \langle \underline{x}| \otimes \mathcal{E}_n(W^{\otimes n}(\underline{x})), \qquad \underline{x} \in \mathcal{X}^n.$$

Our aim in the rest of the section is to relate the  $\chi^{\flat}_{\alpha,1}$ -quantity for the pinched channel  $\mathcal{E}_n W^{\otimes n}$  to the  $\chi^*_{\alpha,1}$ -quantity of the original channel  $W^{\otimes n}$ . We obtain such a relation in Corollary IV.11, which will be a key technical tool to determine the strong converse exponent of W in Section V.

We will benefit from the following additivity properties:

**Lemma IV.8** For every  $P \in \mathcal{P}_f(\mathcal{X})$  and every  $\alpha > 1$ ,

$$\chi_{\alpha,1}(W^{\otimes n}, P^{\otimes n}) = n\chi_{\alpha,1}(W, P), \qquad \qquad \chi_{\alpha,1}^*(W^{\otimes n}, P^{\otimes n}) = n\chi_{\alpha,1}^*(W, P), \qquad n \in \mathbb{N}.$$

**Proof** In the case of  $\chi_{\alpha,1}$ , the unique minimizer state in (80) can be determined explicitly due to the quantum Sibson's identity, and one can observe that the minimizer for a general n is the n-th tensor power of the minimizer for n = 1; see, e.g., [46, Section 4.4] for details. The additivity of  $\chi_{\alpha,1}^*$  is a special case of [8, Theorem 11]. 

For every  $\pi \in \mathfrak{S}_n$ , we denote its natural action on  $\mathcal{X}^n$  by the same symbol  $\pi$ , i.e.,

$$\pi(x_1, \dots, x_n) := (x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)}), \qquad x_1, \dots, x_n \in \mathcal{X}.$$

We say that a probability density  $P_n \in \mathcal{P}(\mathcal{X}^n)$  is symmetric if  $P_n \circ \pi = P_n$  for every  $\pi \in \mathfrak{S}_n$ .

**Lemma IV.9** Let  $P_n \in \mathcal{P}(\mathcal{X}^n)$  be a symmetric probability density on  $\mathcal{X}^n$ . Then for any  $\alpha > 1$  and  $(t) = \flat \ or \ (t) = *,$ 

$$\chi_{\alpha,1}^{(t)}(\mathcal{E}_n W^{\otimes n}, P_n) = \min_{\sigma_n \in \mathcal{S}_{\text{sym}}(\mathcal{H}^{\otimes n})} D_{\alpha}^{(t)} \left( \mathcal{E}_n \mathbb{W}^{\otimes n}(P_n) \| P_n \otimes \sigma_n \right),$$
(96)

$$\chi_{\alpha,2}^{(t)}(\mathcal{E}_n W^{\otimes n}, P_n) = \min_{\sigma_n \in \mathcal{S}_{\text{sym}}(\mathcal{H}^{\otimes n})} \sum_{\underline{x} \in \mathcal{X}^n} P_n(\underline{x}) D_{\alpha}^{(t)} \left( \mathcal{E}_n W^{\otimes n}(\underline{x}) \| \sigma_n \right),$$
(97)

*i.e.*, the minimizations in (80) and (81) can be restricted to symmetric states. Moreover, the minima in (96)–(97) can be replaced with infima over  $S_{sym}(\mathcal{H}^{\otimes n})_{++}$ , the set of invertible symmetric states.

The same hold for  $(t) = \{ \}$  and  $\alpha \in (1, 2]$ .

**Proof** Let us fix  $\alpha > 1$ . We only prove (96) and for (t) = b, as the proofs for the other cases follow completely similar lines. Thus, with the shorthand notation

$$f(\sigma_n) := Q^{\flat}_{\alpha}(\mathcal{E}_n \mathbb{W}^{\otimes n}(P_n) \| P_n \otimes \sigma_n) = \sum_{\underline{x} \in \mathcal{X}^n} P_n(\underline{x}) Q^{\flat}_{\alpha}(\mathcal{E}_n(W^{\otimes n}(\underline{x})) \| \sigma_n),$$

our aim is to show that

$$\min_{\sigma_n \in \mathcal{S}(\mathcal{H}^{\otimes n})} f(\sigma_n) = \min_{\sigma_n \in \mathcal{S}_{\text{sym}}(\mathcal{H}^{\otimes n})} f(\sigma_n),$$
(98)

which follows immediately if we can show that for any  $\sigma_n \in \mathcal{S}(\mathcal{H}_n^{\otimes n})$ ,

$$f\left(\frac{1}{n!}\sum_{\pi\in\mathfrak{S}_n}\pi_{\mathcal{H}}^*\sigma_n\pi_{\mathcal{H}}\right) \le \max_{\pi\in\mathfrak{S}_n}f(\pi_{\mathcal{H}}^*\sigma_n\pi_{\mathcal{H}}) = f(\sigma_n).$$
(99)

Note that the minima in (98) exist because of the lower semicontinuity established in Corollary III.27, and the inequality in (99) follows from the convexity of  $Q^{\flat}_{\alpha}$  in its second argument, Proposition III.18. Hence, the assertion follows if we can prove the permutation invariance of f, i.e., that  $f(\pi_{\mathcal{H}}^*\sigma_n\pi_{\mathcal{H}}) =$  $f(\sigma_n)$  for any  $\sigma_n \in \mathcal{S}(\mathcal{H}^{\otimes n})$  and any  $\pi \in \mathfrak{S}_n$ . Let us introduce the shorthand notation  $\varrho_{\underline{x}} := \mathcal{E}_n(W^{\otimes n}(\underline{x})), \, \underline{x} \in \mathcal{X}^n$ . Then

$$f(\pi_{\mathcal{H}}^* \sigma_n \pi_{\mathcal{H}} + \varepsilon I) = \sum_{\underline{x} \in \mathcal{X}^n} P_n(\underline{x}) \operatorname{Tr} \varrho_{\underline{x}}^0 \exp\left(\alpha \operatorname{\widehat{\log}} \varrho_{\underline{x}} + (1 - \alpha)\varrho_{\underline{x}}^0 (\log(\pi_{\mathcal{H}}^* (\sigma_n + \varepsilon I)\pi_{\mathcal{H}})) \varrho_{\underline{x}}^0\right).$$

Note that the spectral projections of  $\sigma_{u,n}$  commute with all  $\pi_{\mathcal{H}}, \pi \in \mathfrak{S}_n$ , thus

$$\pi_{\mathcal{H}} \varrho_{\underline{x}} \pi_{\mathcal{H}}^* = \pi_{\mathcal{H}} \mathcal{E}_n(W^{\otimes n}(\underline{x})) \pi_{\mathcal{H}}^* = \mathcal{E}_n\left(\pi_{\mathcal{H}} W^{\otimes n}(\underline{x}) \pi_{\mathcal{H}}^*\right) = \mathcal{E}_n\left(W^{\otimes n}(\pi(\underline{x}))\right) = \varrho_{\pi(\underline{x})}, \quad \pi \in \mathfrak{S}_n.$$

As a consequence,

$$\varrho_{\underline{x}}^{0}(\log(\pi_{\mathcal{H}}^{*}(\sigma_{n}+\varepsilon I)\pi_{\mathcal{H}}))\varrho_{\underline{x}}^{0} = \varrho_{\underline{x}}^{0}\pi_{\mathcal{H}}^{*}(\log(\sigma_{n}+\varepsilon I))\pi_{\mathcal{H}}\varrho_{\underline{x}}^{0} = \pi_{\mathcal{H}}^{*}\varrho_{\pi(\underline{x})}^{0}(\log(\sigma_{n}+\varepsilon I))\varrho_{\pi(\underline{x})}^{0}\pi_{\mathcal{H}},$$

and

$$\widehat{\log} \, \varrho_{\underline{x}} = \widehat{\log}(\pi_{\mathcal{H}}^* \varrho_{\pi(\underline{x})} \pi_{\mathcal{H}}) = \pi_{\mathcal{H}}^* (\widehat{\log} \, \varrho_{\pi(\underline{x})}) \pi_{\mathcal{H}}.$$

Putting it together, we get

$$f(\pi_{\mathcal{H}}^* \sigma_n \pi_{\mathcal{H}} + \varepsilon I) = \sum_{\underline{x} \in \mathcal{X}^n} P_n(\underline{x}) \operatorname{Tr} \varrho_{\underline{x}}^0 \exp\left(\alpha \pi_{\mathcal{H}}^* (\widehat{\log} \varrho_{\pi(\underline{x})}) \pi_{\mathcal{H}} + (1 - \alpha) \pi_{\mathcal{H}}^* \varrho_{\pi(\underline{x})}^0 (\log(\sigma_n + \varepsilon I)) \varrho_{\pi(\underline{x})}^0 \pi_{\mathcal{H}}\right)$$
$$= \sum_{\underline{x} \in \mathcal{X}^n} P_n(\underline{x}) \operatorname{Tr} \pi_{\mathcal{H}} \varrho_{\underline{x}}^0 \pi_{\mathcal{H}}^* \exp\left(\alpha \widehat{\log} \varrho_{\pi(\underline{x})} + (1 - \alpha) \varrho_{\pi(\underline{x})}^0 (\log(\sigma_n + \varepsilon I)) \varrho_{\pi(\underline{x})}^0\right)$$
$$= \sum_{\underline{x} \in \mathcal{X}^n} P_n(\pi(\underline{x})) \operatorname{Tr} \varrho_{\pi(\underline{x})}^0 \exp\left(\alpha \widehat{\log} \varrho_{\pi(\underline{x})} + (1 - \alpha) \varrho_{\pi(\underline{x})}^0 (\log(\sigma_n + \varepsilon I)) \varrho_{\pi(\underline{x})}^0\right)$$
$$= f(\sigma_n + \varepsilon I),$$

where in the third line we used that  $P_n$  is symmetric. Taking the limit  $\varepsilon \to 0$  proves the desired permutation invariance, due to Lemma III.26.

To show the assertion that the minimization can be restricted to invertible states, let  $\sigma_n$  be a state where the minimum in (96) is attained, and for every  $\varepsilon \in (0, 1)$ , let  $\sigma_{n,\varepsilon} := (1 - \varepsilon)\sigma_n + \varepsilon(I - \sigma_n^0) / \operatorname{Tr}(I - \sigma_n^0)$ . Note that  $\sigma_{n,\varepsilon}$  is symmetric and invertible for every  $\varepsilon \in (0, 1)$ . Since  $\sigma_n$  is a minimizer, we have  $\varrho_{\underline{x}}^0 = (\mathcal{E}_n W^{\otimes n}(\underline{x}))^0 \leq \sigma_n^0$  for every  $\underline{x} \in \mathcal{X}^n$  such that  $P_n(\underline{x}) > 0$ , and thus

$$\varrho_{\underline{x}}^{0}\left(\widehat{\log}\,\sigma_{n,\varepsilon}\right)\varrho_{\underline{x}}^{0} = \varrho_{\underline{x}}^{0}\left(\widehat{\log}(1-\varepsilon)\sigma_{n}\right)\varrho_{\underline{x}}^{0} = \varrho_{\underline{x}}^{0}\left(\widehat{\log}\,\sigma_{n}\right)\varrho_{\underline{x}}^{0} + \varrho_{\underline{x}}^{0}\log(1-\varepsilon),$$

which in turn yields

$$f(\sigma_{n,\varepsilon}) = (1-\varepsilon)^{1-\alpha} f(\sigma_n) \xrightarrow[\varepsilon \searrow 0]{} f(\sigma_n).$$

Hence,

$$f(\sigma_n) = \inf_{\varepsilon \in (0,1)} f(\sigma_{n,\varepsilon}) \ge \inf_{\sigma_n \in \mathcal{S}_{sym}(\mathcal{H}^{\otimes n})_{++}} f(\sigma_n)$$

and the converse inequality is obvious.

**Lemma IV.10** For every  $P \in \mathcal{P}_f(\mathcal{X})$ , every  $\alpha > 1$ , and i = 1, 2,

$$\chi_{\alpha,i}^{\flat}(\mathcal{E}_{n}W^{\otimes n}, P^{\otimes n}) \geq \begin{cases} \chi_{\alpha,i}(\mathcal{E}_{n}W^{\otimes n}, P^{\otimes n}) - \log v_{n,d}, \\ \\ \chi_{\alpha,i}^{*}(W^{\otimes n}, P^{\otimes n}) - 3\log v_{n,d}. \end{cases}$$
(100)

**Proof** We prove the assertion only for i = 2, since the other case is completely similar. Let  $\sigma_n \in S_{\text{sym}}(\mathcal{H}^{\otimes n})_{++}$  be an invertible symmetric state. Since  $\sigma_{u,n}$  is a universal symmetric state, we have  $\sigma_n \leq v_{n,d}\sigma_{u,n}$ , and thus for every  $\underline{x} \in \mathcal{X}^n$ ,

$$D^{\flat}_{\alpha}(\mathcal{E}_{n}W^{\otimes n}(\underline{x})\|\sigma_{n}) \ge D^{\flat}_{\alpha}(\mathcal{E}_{n}W^{\otimes n}(\underline{x})\|\sigma_{u,n}) - \log v_{n,d} = D_{\alpha}(\mathcal{E}_{n}W^{\otimes n}(\underline{x})\|\sigma_{u,n}) - \log v_{n,d},$$
(101)

where the inequality is due to Lemma III.24 and Lemma III.19, and the equality is due to the fact that  $\mathcal{E}_n W^{\otimes n}(\underline{x})$  and  $\sigma_{u,n}$  commute with each other. Hence,

$$\chi_{\alpha,2}^{\flat}(\mathcal{E}_{n}W^{\otimes n},P^{\otimes n}) = \inf_{\sigma_{n}\in\mathcal{S}_{\mathrm{sym}}(\mathcal{H}^{\otimes n})_{++}} \sum_{\underline{x}\in\mathcal{X}^{n}} P^{\otimes n}(\underline{x}) D_{\alpha}^{\flat}(\mathcal{E}_{n}W^{\otimes n}(\underline{x}) \| \sigma_{n})$$

$$\geq \sum_{\underline{x}\in\mathcal{X}^{n}} P^{\otimes n}(\underline{x}) D_{\alpha}(\mathcal{E}_{n}W^{\otimes n}(\underline{x}) \| \sigma_{u,n}) - \log v_{n,d}$$

$$\geq \chi_{\alpha,2}(\mathcal{E}_{n}W^{\otimes n},P^{\otimes n}) - \log v_{n,d},$$
(102)

where the first equality is due to Lemma IV.9, the first inequality is due to (101), and the last inequality is due to the definition (81). This proves the first bound in (100).

By [29, Lemma 2], we have

$$D_{\alpha}(\mathcal{E}_{n}W^{\otimes n}(\underline{x})\|\sigma_{u,n}) \ge D_{\alpha}^{*}(W^{\otimes n}(\underline{x})\|\sigma_{u,n}) - 2\log v_{n,d}$$

Plugging it into (102), we get

$$\chi_{\alpha,2}^{\flat}(\mathcal{E}_{n}W^{\otimes n},P^{\otimes n}) \geq \sum_{\underline{x}\in\mathcal{X}^{n}} P^{\otimes n}(\underline{x})D_{\alpha}^{*}(W^{\otimes n}(\underline{x})\|\sigma_{u,n}) - 3\log v_{n,d} \geq \chi_{\alpha,2}^{*}(W^{\otimes n},P^{\otimes n}) - 3\log v_{n,d},$$

proving the second bound in (100).

**Corollary IV.11** For every  $P \in \mathcal{P}_f(\mathcal{X})$  and every  $\alpha > 1$ ,

$$\lim_{n \to +\infty} \frac{1}{n} \chi_{\alpha,1}^{\flat}(\mathcal{E}_n W^{\otimes n}, P^{\otimes n}) = \lim_{n \to +\infty} \frac{1}{n} \chi_{\alpha,1}(\mathcal{E}_n W^{\otimes n}, P^{\otimes n}) = \chi_{\alpha,1}^*(W, P).$$
(103)

**Proof** We have

$$\chi_{\alpha,1}^*(W^{\otimes n}, P^{\otimes n}) - 3\log v_{n,d} \le \chi_{\alpha,1}^{\flat}(\mathcal{E}_n W^{\otimes n}, P^{\otimes n}) \le \chi_{\alpha,1}^*(\mathcal{E}_n W^{\otimes n}, P^{\otimes n}) \le \chi_{\alpha,1}^*(W^{\otimes n}, P^{\otimes n}),$$

where the first inequality is due to Lemma IV.10, the second one is due to Proposition III.20, and the last one follows from the monotonicity of  $D^*_{\alpha}$  under pinching [50, Proposition 14]. By Lemma IV.8,  $\chi^*_{\alpha,1}(W^{\otimes n}, P^{\otimes n}) = n\chi^*_{\alpha,1}(W, P)$ . Thus, dividing the above chain of inequalities by n, taking the limit  $n \to +\infty$ , and using (13), we obtain

$$\lim_{n \to +\infty} \frac{1}{n} \chi_{\alpha,1}^{\flat}(\mathcal{E}_n W^{\otimes n}, P^{\otimes n}) = \chi_{\alpha,1}^*(W, P).$$
(104)

Next, we use

$$\chi_{\alpha,1}^{\flat}(\mathcal{E}_n W^{\otimes n}, P^{\otimes n}) \le \chi_{\alpha,1}(\mathcal{E}_n W^{\otimes n}, P^{\otimes n}) \le \chi_{\alpha,1}^{\flat}(\mathcal{E}_n W^{\otimes n}, P^{\otimes n}) + \log v_{n,d},$$

where the first inequality is due to Proposition III.20, and the second one is due to Lemma IV.10. Combining with (104), we get

$$\lim_{n \to +\infty} \frac{1}{n} \chi_{\alpha,1}(\mathcal{E}_n W^{\otimes n}, P^{\otimes n}) = \chi_{\alpha,1}^*(W, P).$$

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### V. THE STRONG CONVERSE EXPONENT FOR CLASSICAL-QUANTUM CHANNELS

#### A. Classical-quantum channel coding and the strong converse exponent

Let  $W : \mathcal{X} \to \mathcal{S}(\mathcal{H})$  be a classical-quantum channel, as described in Section II E. The encoding and decoding process of message transmission over the *n*-fold extension of the channel is described as follows. Each message  $k \in \{1, 2, \ldots, M_n\}$  is encoded to a codeword by an encoder  $\phi_n$ :

$$\phi_n: k \in \{1, 2, \dots, M_n\} \longmapsto \phi_n(k) = x_{k,1}, x_{k,2}, \dots, x_{k,n} \in \mathcal{X}^n$$

and is mapped by n uses of the channel to

$$W^{\otimes n}(\phi_n(k)) = W(x_{k,1}) \otimes W(x_{k,2}) \otimes \cdots \otimes W(x_{k,n}) \in \mathcal{S}(\mathcal{H}^{\otimes n}).$$

The set  $\{\phi_n(k)\}_{k=1}^{M_n} \subset \mathcal{X}^n$  is called a codebook, which is agreed upon by the sender and the receiver in advance. The decoding process, called the decoder, is described by a POVM  $D_n = \{D_n(k)\}_{k=1}^{M_n}$  on  $\mathcal{H}^{\otimes n}$ , where the outcomes  $1, 2, \ldots, M_n$  indicate decoded messages. The pair  $\mathcal{C}_n = (\phi_n, D_n)$  is called a code with cardinality  $|\mathcal{C}_n| := M_n$ .

When the message k was sent, the probability of obtaining the outcome l is given by

$$P(l|k) = \operatorname{Tr} W^{\otimes n}(\phi_n(k))D_n(l).$$

The average error probability of the code  $\mathcal{C}_n$  is then given by

$$P_e(W^{\otimes n}, \mathcal{C}_n) = 1 - \frac{1}{M_n} \sum_{k=1}^{M_n} \operatorname{Tr} W^{\otimes n}(\phi_n(k)) D_n(k),$$

which is required to vanish asymptotically for reliable communication. At the same time, the aim of classical-quantum channel coding is to make the transmission rate  $\liminf_{n\to+\infty} \frac{1}{n} \log |\mathcal{C}_n|$  as large as possible. The channel capacity C(W) is defined as the supremum of achievabile rates with asymptotically vanishing error probabilities, i.e.,

$$C(W) = \sup \Big\{ R \ \Big| \ \exists \{ \mathcal{C}_n \}_{n=1}^{\infty} \text{ such that } \liminf_{n \to \infty} \frac{1}{n} \log |\mathcal{C}_n| \ge R \text{ and } \lim_{n \to \infty} P_e(\mathcal{C}_n, W^{\otimes n}) = 0 \Big\}.$$

According to the Holevo-Schumacher-Westmoreland theorem [37, 65],

$$C(W) = \chi(W), \tag{105}$$

where  $\chi(W)$  is the Holevo capacity from (78).

By the definition of C(W), (105) means that for any rate R below the Holevo capacity, there exists a sequence of codes with rate R and asymptotically vanishing error probability. Moreover, it is known that the error probability can be made to vanish with an exponential speed [26]. On the other hand, the strong converse theorem of classical-quantum channel coding [56, 76] tells that for any sequence of codes with a rate above the Holevo capacity, the error probability inevitably goes to 1, with an exponential speed, or equivalently, the success probability

$$P_s(W^{\otimes n}, \mathcal{C}_n) := 1 - P_e(W^{\otimes n}, \mathcal{C}_n)$$

decays to zero exponentially fast. The optimal achievable exponent of this decay for a given rate R is called the strong converse exponent sc(R, W):

**Definition V.1 (strong converse exponent)** The success rate r is said to be R-achievable, if there exists a sequence of codes  $\{C_n\}_{n=1}^{\infty}$  such that

$$\liminf_{n \to \infty} \frac{1}{n} \log |\mathcal{C}_n| \ge R \quad and \quad \liminf_{n \to \infty} \frac{1}{n} \log P_s(\mathcal{C}_n, W^{\otimes n}) \ge -r.$$
(106)

The strong converse exponent corresponding to the rate R is the infimum of all R-achievable rates:

 $sc(R, W) = \inf \{ r \mid r \text{ is } R\text{-achievable} \}.$ 

Alternatively, the strong converse exponent can be expressed as

$$sc(R,W) = \inf\left\{-\liminf_{n \to \infty} \frac{1}{n} \log P_s(\mathcal{C}_n, W^{\otimes n}) \mid \liminf_{n \to \infty} \frac{1}{n} \log |\mathcal{C}_n| \ge R\right\},\$$

where the infimum is taken over all sequences of codes  $\{C_n\}_{n\in\mathbb{N}}$ . Note that we take the infimum here, as our aim is to make the success probability vanish as slow as possible.

Our main result is the following expression for the strong converse exponent, which is an exact analogue of the Arimoto-Dueck-Körner exponent for classical channels.

**Theorem V.2** Let  $W: \mathcal{X} \to \mathcal{S}(\mathcal{H})$  be a classical-quantum channel. For any rate  $R \geq 0$ ,

$$sc(R,W) = \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \chi_{\alpha}^{*}(W) \right\} =: H_{R,c}^{*}(W).$$
(107)

The proof follows from Lemma V.5 and Theorem V.14 below.

**Remark V.3** We remark that the inequality  $sc(R, W) \ge H^*_{R,c}(W)$  (the optimality part of the theorem) follows by standard techniques; we explain it in the next section for readers' convenience. Hence, the novelty of Theorem V.2 is the converse inequality  $sc(R, W) \le H^*_{R,c}(W)$  (the so-called achievability part).

**Remark V.4** In binary state discrimination, the strong converse exponent is given by a similar transform of the Rényi divergences, known as the Hoeffding anti-divergence [25, 48]; for two states  $\rho$  and  $\sigma$ and a rate R it is defined as

$$H_{R,c}^*(\varrho \| \sigma) := \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - D_{\alpha}^*(\varrho \| \sigma) \right\}.$$

The expression in (107) is a direct analogue of this for capacities, which we can also extend to other Rényi capacities as

$$H_{R,c}^{(t)}(W) := \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \chi_{\alpha}^{(t)}(W) \right\}.$$
 (108)

We call these quantities converse Hoeffding capacities. Theorem V.2 shows that it is the converse Hoeffding capacity corresponding to  $D^*_{\alpha}$  that is operationally relevant for the strong converse of c-q channel coding, (just as in state discrimination), but in our proof in Section V, the quantity corresponding to  $D^{\flat}_{\alpha}$  also plays an important role.

### B. Lower bound for the strong converse exponent

Applying the method developed in [51, 63, 66] to the new Rényi relative entropies, we have the following lemma.

**Lemma V.5** For any classical-quantum channel  $W \in C(\mathcal{H}|\mathcal{X})$  and  $R \geq 0$ , we have

$$sc(R,W) \ge \sup_{\alpha>1} \frac{\alpha-1}{\alpha} \left\{ R - \chi_{\alpha}^{*}(W) \right\}.$$
(109)

**Proof** Suppose that r is R-achievable. Then there exists a sequence of codes  $C_n = (\phi_n, D_n), n \in \mathbb{N}$ , such that (106) holds. Let  $\sigma \in \mathcal{S}(\mathcal{H})$ , and define density operators on  $\mathcal{K}_n := \bigoplus_{k=1}^{M_n} \mathcal{H}^{\otimes n}$  by

$$R_n := \frac{1}{M_n} \bigoplus_{k=1}^{M_n} W^{\otimes n}(\phi_n(k)), \qquad S_n := \frac{1}{M_n} \bigoplus_{k=1}^{M_n} \sigma^{\otimes n},$$

where  $M_n := |\mathcal{C}_n|$ . Note that  $T_n := \bigoplus_{k=1}^{M_n} D_n(k)$  and  $T_n^{\perp} := I_{\mathcal{K}_n} - T_n$  form a two-valued POVM  $\{T_n, T_n^{\perp}\}$  on  $\bigoplus_{k=1}^{M_n} \mathcal{H}^{\otimes n}$ , and we have  $\operatorname{Tr} R_n T_n = P_s(W^{\otimes n}, \mathcal{C}_n)$  and  $\operatorname{Tr} S_n T_n = \frac{1}{M_n}$ . With the notation  $\phi_n(k) = x_{k,1}, x_{k,2}, \ldots, x_{k,n}$ , the monotonicity of  $Q_{\alpha}^*$  yields that for  $\alpha \geq 1$ ,

$$P_{s}(W^{\otimes n}, \mathcal{C}_{n})^{\alpha} \frac{1}{M_{n}^{1-\alpha}} = (\operatorname{Tr} R_{n}T_{n})^{\alpha} (\operatorname{Tr} S_{n}T_{n})^{1-\alpha}$$

$$\leq (\operatorname{Tr} R_{n}T_{n})^{\alpha} (\operatorname{Tr} S_{n}T_{n})^{1-\alpha} + (\operatorname{Tr} R_{n}T_{n}^{\perp})^{\alpha} (\operatorname{Tr} S_{n}T_{n}^{\perp})^{1-\alpha}$$

$$\leq Q_{\alpha}^{*}(R_{n}||S_{n})$$

$$= \frac{1}{M_{n}} \sum_{k=1}^{M_{n}} \prod_{i=1}^{n} Q_{\alpha}^{*}(W(x_{k,i})||\sigma)$$

$$\leq \left\{ \sup_{x \in \mathcal{X}} Q_{\alpha}^{*}(W(x)||\sigma) \right\}^{n}, \qquad (110)$$

where the last equality follows from the multiplicativity of  $Q^*_{\alpha}$ . Since this holds for every  $\sigma \in \mathcal{S}(\mathcal{H})$ , we get

$$\frac{\alpha}{n}\log P_s(W^{\otimes n}, \mathcal{C}_n) + \frac{\alpha - 1}{n}\log M_n \le \inf_{\sigma\in\mathcal{S}(\mathcal{H})} \sup_{x\in\mathcal{X}}\log Q^*_{\alpha}(W(x)\|\sigma),$$

or equivalently,

$$\frac{1}{n}\log P_s(W^{\otimes n}, \mathcal{C}_n) \le -\frac{\alpha - 1}{\alpha} \left\{ \frac{1}{n}\log M_n - \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{x \in \mathcal{X}} D^*_{\alpha}(W(x) \| \sigma) \right\}.$$

Using Proposition IV.2, and taking the limsup in n, we get

$$-\frac{\alpha-1}{\alpha}\left\{R-\chi_{\alpha}^{*}(W)\right\} \geq \limsup_{n \to +\infty} \frac{1}{n} \log P_{s}(W^{\otimes n}, \mathcal{C}_{n}) \geq \liminf_{n \to +\infty} \frac{1}{n} \log P_{s}(W^{\otimes n}, \mathcal{C}_{n}) \geq -r.$$

Since this is true for every  $\alpha > 1$ , the assertion follows.

# C. Dueck-Körner exponent

In this section we show the following weak converse to Lemma V.5:

**Theorem V.6** For every R > 0,

$$sc(R,W) \le \sup_{\alpha>1} \frac{\alpha-1}{\alpha} \left\{ R - \chi_{\alpha}^{\flat}(W) \right\}.$$
 (111)

This will follow immediately from Theorems V.7 and V.12 and Lemma V.13. We give the proof at the end of the section.

Note that for classical channels, the right-hand sides of the bounds in (109) and (111) coincide, and the two bounds together give Theorem V.2. For quantum channels, however, they need not be the same. In Section V D we will combine the bound in (111) with block pinching to obtain an upper bound on sc(R, W) that matches (109).

Given classical-quantum channels  $V, W \in C(\mathcal{H}|\mathcal{X})$  and a finitely supported probability distribution  $P \in \mathcal{P}_f(\mathcal{X})$ , the quantum relative entropy between the outputs of the extended channels,  $\mathbb{V}(P)$  and  $\mathbb{W}(P)$ , is written as

$$D(\mathbb{V}(P)\|\mathbb{W}(P)) = \sum_{x \in \mathcal{X}} P(x)D(V(x)\|W(x)) =: D(V\|W|P),$$

which is called the conditional quantum relative entropy.

For every  $P \in \mathcal{P}_f(\mathcal{X})$  and  $R \ge 0$ , let

$$F(P, R, W) := \inf_{V \in C(\mathcal{H}|\mathcal{X})} \{ D(V ||W|P) + |R - \chi(V, P)|^+ \},$$
(112)

where

$$|x|^+ = \max\{0, x\}, \qquad x \in \mathbb{R}.$$

Note that for D(V||W|P) and  $\chi(V, P)$ , only the values of V and W on the support of P are relevant, and therefore we can replace  $\mathcal{X}$  with supp P without loss of generality. Moreover,  $D(V||W|P) = +\infty$  if there exists an  $x \in \text{supp } P$  such that  $V(x)^0 \not\leq W(x)^0$ . Hence, we can restrict the infimum to

 $C_{W,P} := \left\{ V \in C(\mathcal{H}|\operatorname{supp} P) : V(x)^0 \le W(x)^0, \, x \in \operatorname{supp} P \right\}.$ 

Let us introduce the norm  $||F|| := \sum_{x \in \text{supp } P} ||F(x)||_1$  on  $\{F : \text{supp } P \to \mathcal{L}(\mathcal{H})\}$ . Then  $C_{W,P}$  is compact w.r.t. this norm, and it is easy to see that  $V \mapsto D(V||W|P) + |R - \chi(V,P)|^+$  is continuous on  $C_{W,P}(\mathcal{H}|\mathcal{X})$ . Hence, we have

$$F(P, R, W) = \min_{V \in C_{W, P}} \{D(V || W | P) + |R - \chi(V, P)|^+\}$$
$$= \min_{V \in C(\mathcal{H}|\mathcal{X})} \{D(V || W | P) + |R - \chi(V, P)|^+\}.$$

The following is a direct analogue of the Dueck-Körner upper bound [19]:

**Theorem V.7** For any rate R > 0, and any  $P \in \mathcal{P}_f(\mathcal{X})$ ,

$$sc(R,W) \le F(P,R,W).$$

**Proof** Let

$$F_1(P, R, W) := \inf_{V: \, \chi(V, P) > R} D(V || W | P),$$
(113)

$$F_2(P, R, W) := \inf_{V: \chi(V, P) \le R} \{ D(V || W | P) + R - \chi(V, P) \}.$$
(114)

Then it is easy to see that

$$F(P, R, W) = \min\{F_1(P, R, W), F_2(P, R, W)\},\$$

and hence the assertion follows from Lemmas V.10 and V.11 below.

We will need the following two lemmas to prove Lemma V.10. The first one is a key tool in the information spectrum method [12, 53], and is a consequence of the quantum Stein's lemma [33, 57].

**Lemma V.8** For any states  $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ , we have

$$\lim_{n \to \infty} \operatorname{Tr}(\varrho^{\otimes n} - e^{na} \sigma^{\otimes n})_{+} = \begin{cases} 1, & a < D(\varrho \| \sigma), \\ 0, & a > D(\varrho \| \sigma). \end{cases}$$

The second one is the key observation behind the dummy channel technique:

**Lemma V.9** Let  $W \in C(\mathcal{H}|\mathcal{X})$  be a classical-quantum channel. For any code  $C_n = (\phi_n, D_n)$ , any  $a \in \mathbb{R}$ , and any classical-quantum channel  $V \in C(\mathcal{H}|\mathcal{X})$  (the dummy channel), we have

$$P_s(W^{\otimes n}, \mathcal{C}_n) \ge e^{-na} \left\{ P_s(V^{\otimes n}, \mathcal{C}_n) - \frac{1}{M_n} \sum_{k=1}^{M_n} \operatorname{Tr} \left( V^{\otimes n}(\phi_n(k)) - e^{na} W^{\otimes n}(\phi_n(k)) \right)_+ \right\}.$$
(115)

**Proof** Let  $C_n = (\phi_n, D_n)$  be a code with  $|C_n| = M_n$ . According to (2), we have

$$\operatorname{Tr}\left(V^{\otimes n}(\phi_n(k)) - e^{na} W^{\otimes n}(\phi_n(k))\right)_+ \ge \operatorname{Tr}\left(V^{\otimes n}(\phi_n(k)) - e^{na} W^{\otimes n}(\phi_n(k))\right) D_n(k),$$

and hence

$$\operatorname{Tr} W^{\otimes n}(\phi_n(k))D_n(k) \ge e^{-na}\operatorname{Tr} V^{\otimes n}(\phi_n(k))D_n(k) - e^{-na}\operatorname{Tr} \left(V^{\otimes n}(\phi_n(k)) - e^{na}W^{\otimes n}(\phi_n(k))\right)_+$$

for every  $k \in \{1, \ldots, M_n\}$ . Summing over k and dividing by  $M_n$  yields (115).

**Lemma V.10** In the setting of Theorem V.7, we have  $sc(R, W) \leq F_1(P, R, W)$ .

**Proof** We show that any rate r satisfying  $r > F_1(P, R, W)$  is R-achievable. By the definition (113), there exists a classical-quantum channel V such that

$$r > D(V||W|P), \tag{116}$$

$$R < \chi(V, P). \tag{117}$$

By Lemma V.9, we have

$$P_s(W^{\otimes n}, \mathcal{C}_n) \ge e^{-nr} \left\{ P_s(V^{\otimes n}, \mathcal{C}_n) - \frac{1}{M_n} \sum_{k=1}^{M_n} \operatorname{Tr} \left( V^{\otimes n}(\phi_n(k)) - e^{nr} W^{\otimes n}(\phi_n(k)) \right)_+ \right\}$$
(118)

for any code  $C_n = (\phi_n, D_n)$ . Now we apply the random coding argument and choose codewords  $\phi_n(k) \in \mathcal{X}^n$ ,  $k = 1, 2, \ldots, M_n = \lceil e^{nR} \rceil$ , independently and identically according to  $P^{\otimes n}$ . For the decoder, we choose the Hayashi-Nagaoka decoder [28].

Let  $E[\cdot]$  denote the expectation w.r.t. the random coding ensemble. Taking the expectation of both sides of (118) w.r.t. E, we get

$$E\left[P_s(W^{\otimes n}, \mathcal{C}_n)\right] \ge e^{-nr} \left\{ E\left[P_s(V^{\otimes n}, \mathcal{C}_n)\right] - \sum_{\underline{x} \in \mathcal{X}^n} P^n(\underline{x}) \operatorname{Tr} \left(V^{\otimes n}(\underline{x}) - e^{nr} W^{\otimes n}(\underline{x})\right)_+ \right\}$$
$$= e^{-nr} \left\{ E\left[P_s(V^{\otimes n}, \mathcal{C}_n)\right] - \operatorname{Tr} \left(\mathbb{V}(P)^{\otimes n} - e^{nr} \mathbb{W}(P)^{\otimes n}\right)_+ \right\}.$$

Since  $R < \chi(V, P)$ , the results of [28] yield

$$\lim_{n \to \infty} E\left[P_s(V^{\otimes n}, \mathcal{C}_n)\right] = 1,\tag{119}$$

and, since  $r > D(V || W | P) = D(\mathbb{V}(P) || \mathbb{W}(P)$ , Lemma V.8 yields

$$\lim_{n \to +\infty} \operatorname{Tr} \left( \mathbb{V}(P)^{\otimes n} - e^{nr} \mathbb{W}(P)^{\otimes n} \right)_+ = 0.$$

Hence,

$$\begin{split} \liminf_{n \to \infty} \frac{1}{n} \log \max_{\mathcal{C}_n} P_s(W^{\otimes n}, \mathcal{C}_n) \\ &\geq \liminf_{n \to \infty} \frac{1}{n} \log E \left[ P_s(W^{\otimes n}, \mathcal{C}_n) \right] \\ &\geq -r + \lim_{n \to \infty} \frac{1}{n} \log \left\{ E \left[ P_s(V^{\otimes n}, \mathcal{C}_n) \right] - \sum_{\underline{x} \in \mathcal{X}^n} P^n(\underline{x}) \operatorname{Tr} \left( V^{\otimes n}(\underline{x}) - e^{nr} W^{\otimes n}(\underline{x}) \right)_+ \right\} \\ &= -r, \end{split}$$

where the maximum in the first line is taken over all codes with cardinality  $|\mathcal{C}_n| = \lceil e^{nR} \rceil$ . Thus, r is R-achievable.

# **Lemma V.11** In the setting of Theorem V.7, we have $sc(R, W) \leq F_2(P, R, W)$ .

**Proof** First note that if P is supported on one single point  $x_0 \in \mathcal{X}$  then  $\chi(V, P) = 0$  for every channel V, and hence  $F_2(P, R, W) = R$ . It is easy to see that the trivial encoding  $\phi_n$  that assigns  $\phi_n(k)_i := x_0, i \in \{1, \ldots, n\}$ , to every message  $k \in \{1, \ldots, \lceil e^{nR} \rceil\}$ , together with any decoding yields a code  $C_n$  with transmission rate R and success rate R, showing that  $sc(R, W) \leq R = F_2(P, R, W)$ . Hence for the rest we will assume that  $|\operatorname{supp} P| > 2$ .

To prove the assertion, it is enough to show that any rate r satisfying  $r > F_2(P, R, W)$  is R-achievable. By the definition (114), there exists a classical-quantum channel V such that

$$r > D(V||W|P) + R - \chi(V, P), \tag{120}$$

$$R \ge \chi(V, P). \tag{121}$$

$$r_1 := r - R + \chi(V, P) - \delta > D(V ||W|P),$$
  

$$R_1 < \chi(V, P).$$

Since (116) and (117) are satisfied for  $r_1$  and  $R_1$ , we can see that  $r_1$  is  $R_1$ -achievable from Lemma V.10, i.e., there exists a sequence of codes  $\Psi_n = (\psi_n, Y_n)$  such that

$$\liminf_{n \to \infty} \frac{1}{n} \log |\Psi_n| \ge -R_1, \quad \liminf_{n \to \infty} \frac{1}{n} \log P_s(W^{\otimes n}, \Psi_n) \ge -r_1.$$

Let  $N_n = \lceil e^{nR_1} \rceil$  and  $M_n = \lceil e^{nR} \rceil$ . Since  $N_n \leq M_n$  holds, we can expand the code  $\Psi_n = (\psi_n, Y_n)$  to construct a code  $C_n = (\phi_n, D_n)$  with the rate R by

$$\phi_n(k) := \begin{cases} \psi_n(k) & (1 \le k \le N_n) \\ \psi_n(1) & (N_n < k \le M_n) \end{cases}, \qquad D_n(k:) = \begin{cases} Y_n(k) & (1 \le k \le N_n) \\ 0 & (N_n < k \le M_n). \end{cases}$$

Then we have

$$P_s(W^{\otimes n}, \mathcal{C}_n) = \frac{1}{M_n} \sum_{k=1}^{M_n} \operatorname{Tr} W^{\otimes n}(\phi_n(k)) D_n(k) = \frac{N_n}{M_n} \frac{1}{N_n} \sum_{k=1}^{N_n} \operatorname{Tr} W^{\otimes n}(\psi_n(k)) Y_n(k) = \frac{N_n}{M_n} P_s(W^{\otimes n}, \Psi_n),$$

and hence,

$$\liminf_{n \to \infty} \frac{1}{n} \log P_s(W^{\otimes n}, \mathcal{C}_n) \ge \liminf_{n \to \infty} \frac{1}{n} \log P_s(W^{\otimes n}, \Psi_n) + \lim_{n \to \infty} \frac{1}{n} \log \frac{N_n}{M_n}$$
$$\ge -r_1 + R_1 - R$$
$$= -(r - R + \chi(V, P) - \delta) + \chi(V, P) - \delta - R$$
$$= -r,$$

proving that r is R-achievable.

Our next step is deriving another representation for F(P, R, W) defined in (112).

**Theorem V.12** Given  $W \in C(\mathcal{H}|\mathcal{X})$  and  $R \geq 0$ , for any  $P \in \mathcal{P}_f(\mathcal{X})$ , we have

$$F(P, R, W) = \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \chi_{\alpha, 2}^{\flat}(W, P) \right\}.$$
(122)

**Proof** Since  $\sup_{0 < \delta < 1} \delta(a - b) = |a - b|^+$  holds for any  $a, b \in \mathbb{R}$ , we have

 $x \in \mathcal{X}$ 

$$F(P, R, W) = \min_{V \in C_{W,P}} \{ D(V ||W|P) + |R - \chi(V, P)|^+ \}$$
  
= 
$$\min_{V \in C_{W,P}} \sup_{0 < \delta < 1} \{ D(V ||W|P) + \delta \{ R - \chi(V, P) \} \}.$$
 (123)

Note that  $\chi(V, P) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})_{++}} D(V || \sigma | P)$ , where  $D(V || \sigma | P)$  denotes, with a slight abuse of notation, the conditional relative entropy of V with respect to the constant classical-quantum channel  $x \in \mathcal{X} \mapsto \sigma \in \mathcal{S}(\mathcal{H})$ . Thus, we can rewrite (123) as

$$F(P, R, W) = \min_{V \in C_{W,P}} \sup_{0 < \delta < 1} \sup_{\sigma \in \mathcal{S}(\mathcal{H})_{++}} G(P, \delta, \sigma, V),$$
(124)

 $x \in \mathcal{X}$ 

where for every  $\delta \in [0, 1]$ ,  $V \in C_{W,P}$  and  $\sigma \in \mathcal{S}(\mathcal{H})_{++}$ ,

$$G(P, \delta, \sigma, V) := D(V ||W|P) + \delta \{R - D(V ||\sigma|P)\}$$

$$= \delta R - (1 - \delta) \sum_{x \in \mathcal{X}} P(x) H(V(x))$$

$$- \sum P(x) \operatorname{Tr} V(x) \widehat{\log} W(x) + \delta \sum P(x) \operatorname{Tr} V(x) \log \sigma,$$
(126)

In the above,  $H(\varrho) := -\operatorname{Tr} \varrho \log \varrho$  stands for the von Neumann entropy of a state  $\varrho \in \mathcal{S}(\mathcal{H})$ . Moreover,

 $\sup_{\sigma \in \mathcal{S}(\mathcal{H})_{++}} G(P, \delta, \sigma, V) = D(V ||W|P) + \delta\{R - \chi(V, P)\}$ (127)

$$= \sum_{x \in \mathcal{X}} P(x)D(V(x)||W(x)) + \delta R - \delta H\left(\sum_{x \in \mathcal{X}} P(x)V(x)\right) + \delta \sum_{x \in \mathcal{X}} P(x)H(V(x)).$$
(128)

The following properties of G are easy to verify:

- (i)  $G(P, \delta, \sigma, V)$  is concave and continuous with respect to  $\sigma \in \mathcal{S}(\mathcal{H})_{++}$ ,
- (ii)  $G(P, \delta, \sigma, V)$  is convex and continuous with respect to  $V \in C_{W,P}$ ,
- (iii)  $\sup_{\sigma \in \mathcal{S}(\mathcal{H})_{++}} G(P, \delta, \sigma, V)$  is convex and continuous with respect to  $V \in C_{W,P}$ ,
- (iv)  $\sup_{\sigma \in \mathcal{S}(\mathcal{H})_{++}} G(P, \delta, \sigma, V)$  is affine with respect to  $0 < \delta < 1$ .

Indeed, the claim (iv) is obvious from (127), and (i) is immediate from (126) due to the operator concavity of the logarithm. Also by (126) and the concavity of the von Neumann entropy, we get (ii). The convexity property in (iii) is obvious from (ii), and the continuity is clear from (128).

Applying now the minimax theorem in Lemma II.7 first to  $\sup_{\sigma \in \mathcal{S}(\mathcal{H})_{++}} G(P, \delta, \sigma, V)$  and then to  $G(P, \delta, \sigma, V)$ , and benefiting both times from the compactness of  $C_{W,P}$ , (124) can be rewritten as

$$F(P, R, W) = \min_{V \in C_{W,P}} \sup_{0 < \delta < 1} \sup_{\sigma \in \mathcal{S}(\mathcal{H})_{++}} G(P, \delta, \sigma, V)$$
  
= 
$$\sup_{0 < \delta < 1} \min_{V \in C_{W,P}} \sup_{\sigma \in \mathcal{S}(\mathcal{H})_{++}} G(P, \delta, \sigma, V)$$
(129)

$$= \sup_{0<\delta<1} \sup_{\sigma\in\mathcal{S}(\mathcal{H})_{++}} \min_{V\in C_{W,P}} G(P,\delta,\sigma,V).$$
(130)

Moreover, in each line the supremum over  $\delta \in (0, 1)$  can be replaced with supremum over  $\delta \in [0, 1]$ . Note that by (39) and (64) we have

$$\min_{\tau \in S_{\varrho}(\mathcal{H})} \left\{ D(\tau \| \varrho) - \delta D(\tau \| \sigma) \right\} = \begin{cases} 0, & \delta = 0, \\ -\delta D^{\flat}_{\frac{1}{1-\delta}}(\varrho \| \sigma), & \delta \in (0,1), \\ -D^{\flat}_{\infty}(\varrho \| \sigma), & \delta = 1 \end{cases}$$

for any  $\rho, \sigma$  such that  $\rho^0 \wedge \sigma^0 \neq 0$ . Hence,

$$\min_{V \in C_{W,P}} G(P, \delta, \sigma, V) = \delta R + \sum_{x \in \mathcal{X}} P(x) \min_{V(x) \in S_{W(x)}(\mathcal{H})} \left\{ D(V(x) \| W(x)) - \delta D(V(x) \| \sigma) \right\}$$

$$= \begin{cases} \delta R, & \delta = 0, \\ \delta R - \delta \sum_{x \in \mathcal{X}} P(x) D_{\frac{1}{1-\delta}}^{\flat}(W(x) \| \sigma), & \delta \in (0, 1), \\ \delta R - \sum_{x \in \mathcal{X}} P(x) D_{\infty}^{\flat}(W(x) \| \sigma), & \delta = 1. \end{cases}$$
(131)

Now let us introduce a new parameter  $\alpha := \frac{1}{1-\delta}$ , for which the interval  $0 < \delta < 1$  corresponds to  $\alpha > 1$ . Then (130) can be rewritten as

$$F(P, R, W) = \sup_{\alpha > 1} \sup_{\sigma \in \mathcal{S}(\mathcal{H})_{++}} \min_{V \in C_{W,P}} G(P, \frac{\alpha - 1}{\alpha}, \sigma, V)$$

$$= \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \inf_{\sigma \in \mathcal{S}(\mathcal{H})_{++}} \sum_{x \in \mathcal{X}} P(x) D_{\alpha}^{\flat}(W(x) \| \sigma) \right\}$$

$$= \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \chi_{\alpha,2}^{\flat}(W, P) \right\},$$
(132)

as required.

By Theorems V.7 and V.12, we have

$$sc(R,W) \le \inf_{P \in \mathcal{P}_f(\mathcal{X})} F(P,R,W) = \inf_{P \in \mathcal{P}_f(\mathcal{X})} \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \chi_{\alpha,2}^{\flat}(W,P) \right\}.$$
 (133)

To arrive at the expression in Theorem V.2, we need the following minimax-type lemma (for i = 2):

**Lemma V.13** For every  $R \ge 0$ ,

$$H_{R,c}^{\flat}(W) = \sup_{\alpha > 1} \inf_{P \in \mathcal{P}_{f}(\mathcal{X})} \frac{\alpha - 1}{\alpha} \left\{ R - \chi_{\alpha,i}^{\flat}(W, P) \right\} = \inf_{P \in \mathcal{P}_{f}(\mathcal{X})} \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \chi_{\alpha,i}^{\flat}(W, P) \right\}, \quad (134)$$

where the equalities hold for both i = 1 and i = 2.

**Proof** The first identity is due to the definition (108), so our aim is to show that the infimum and the supremum in (134) can be interchanged. Let us introduce the notation  $\delta = (\alpha - 1)/\alpha$ ,  $\alpha \in (1, +\infty)$ , and for every  $P \in \mathcal{P}_f(\mathcal{X}), \sigma \in \mathcal{S}(\mathcal{H})_{++}$  and  $\delta \in [0, 1)$ , let

$$G_1(P,\delta,\sigma) := \delta R - \delta D^{\flat}_{\frac{1}{1-\delta}}(\mathbb{W}(P) \| P \otimes \sigma),$$
(135)

$$G_2(P,\delta,\sigma) := \delta R - \delta \sum_{x \in \mathcal{X}} P(x) D_{\frac{1}{1-\delta}}^{\flat}(W(x) \| \sigma).$$
(136)

We define

$$G_i(P,1,\sigma) := \lim_{\delta \nearrow 1} G_i(P,\delta,\sigma) = \begin{cases} R - D^{\flat}_{\infty}(\mathbb{W}(P) \| P \otimes \sigma), & i = 1\\ R - \sum_{x \in \mathcal{X}} P(x) D^{\flat}_{\infty}(W(x) \| \sigma), & i = 2 \end{cases}$$

Let

$$G_i(P,\delta) := \sup_{\sigma \in \mathcal{S}(\mathcal{H})_{++}} G_i(P,\delta,\sigma) = \begin{cases} \delta R - \delta \chi^{\flat}_{\frac{1-\delta}{1-\delta},i}(W,P), & \delta \in [0,1), \\ R - \chi^{\flat}_{\infty,i}(W,P), & \delta = 1. \end{cases}$$
(137)

Then (134) is equivalent to

$$\sup_{\delta \in (0,1)} \inf_{P \in \mathcal{P}_f(\mathcal{X})} G_i(P, \delta) = \inf_{P \in \mathcal{P}_f(\mathcal{X})} \sup_{\delta \in (0,1)} G_i(P, \delta).$$
(138)

Note that  $\delta \mapsto G_i(P, \delta)$  is continuous on [0, 1], and

$$\lim_{\delta \searrow 0} \inf_{P \in \mathcal{P}_f(\mathcal{X})} G_i(P, \delta) = \lim_{\delta \searrow 0} \left\{ \delta R - \delta \chi^{\flat}_{\frac{1}{1-\delta}, i}(W) \right\} = 0 = \inf_{P \in \mathcal{P}_f(\mathcal{X})} G_i(P, 0),$$
$$\lim_{\delta \nearrow 1} \inf_{P \in \mathcal{P}_f(\mathcal{X})} G_i(P, \delta) = \lim_{\delta \nearrow 1} \left\{ \delta R - \delta \chi^{\flat}_{\frac{1}{1-\delta}, i}(W) \right\} = R - \chi^{\flat}_{\infty}(W) = \inf_{P \in \mathcal{P}_f(\mathcal{X})} G_i(P, 1).$$

Hence, (138) can be rewritten as

$$\sup_{\delta \in [0,1]} \inf_{P \in \mathcal{P}_f(\mathcal{X})} G_i(P,\delta) = \inf_{P \in \mathcal{P}_f(\mathcal{X})} \sup_{\delta \in [0,1]} G_i(P,\delta).$$
(139)

This will follow from Lemma II.7 if we can show that  $G_i(P, \delta)$  is convex in  $P \in \mathcal{P}_f(\mathcal{X})$  and concave and upper semi-continuous in  $\delta \in [0, 1]$ . We will only give a proof for the case i = 2, since this is what we need for our main result, and because the proof for i = 1 follows very similar lines.

By (136) and (137),  $G_2(P, \delta)$  is the supremum of convex and continuous functions in P, and hence is itself convex and lower semi-continuous for every  $\delta \in [0, 1]$ .

By (131) and (136), we have

$$G_2(P,\delta,\sigma) = \min_{V \in C_{W,P}} G(P,\delta,\sigma,V).$$

Using properties (i) and (ii) in the proof of Theorem V.12, and Lemma II.7, we get

$$G_2(P,\delta) = \sup_{\sigma \in \mathcal{S}(\mathcal{H})_{++}} \min_{V \in C_{W,P}} G(P,\delta,\sigma,V) = \min_{V \in C_{W,P}} \sup_{\sigma \in \mathcal{S}(\mathcal{H})_{++}} G(P,\delta,\sigma,V).$$

Using (127),

$$G_2(P,\delta) = \min_{V \in C_{W,P}} \left\{ D(V ||W|P) + \delta \{ R - \chi(V,P) \} \right\},\$$

and hence  $G_2(P, \delta)$  is the infimum of affine functions in  $\delta$ , and therefore is itself concave and upper semi-continuous. This finishes the proof.

# Proof of Theorem V.6: We have

$$sc(R,W) \le \inf_{P \in \mathcal{P}_f(\mathcal{X})} F(P,R,W) = \inf_{P \in \mathcal{P}_f(\mathcal{X})} \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \chi_{\alpha,2}^{\flat}(W,P) \right\}$$
(140)

$$= \sup_{\alpha>1} \frac{\alpha - 1}{\alpha} \left\{ R - \chi_{\alpha}^{\flat}(W) \right\}$$
(141)

where the first line is due to (133), and the second line follows by Lemma V.13.

# D. Strong converse exponent: achievability

Here we prove the following converse to Lemma V.5:

**Theorem V.14** Let  $W: \mathcal{X} \to \mathcal{S}(\mathcal{H})$  be a classical-quantum channel. For any R > 0, we have

$$sc(R,W) \le \sup_{\alpha>1} \frac{\alpha-1}{\alpha} \left\{ R - \chi_{\alpha}^{*}(W) \right\}.$$
(142)

**Proof** For every  $m \in \mathbb{N}$ , define the pinched channel

$$W_m(\underline{x}) := (\mathcal{E}_m W^{\otimes m})(\underline{x}) = \mathcal{E}_m(W(x_1) \otimes \ldots \otimes W(x_m)), \qquad \underline{x} = (x_1, \ldots, x_m) \in \mathcal{X}^m,$$

where  $\mathcal{E}_m = \mathcal{E}_{\sigma_{u,m}}$  is the pinching by a universal symmetric state  $\sigma_{u,m}$ . By Theorem V.6 and (134), for every R > 0, there exists a sequence of codes  $\mathcal{C}_k^{(m)} = (\phi_k^{(m)}, \mathcal{D}_k^{(m)}), k \in \mathbb{N}$ , such that

$$\liminf_{k \to +\infty} \frac{1}{k} \log |\mathcal{C}_k^{(m)}| \ge Rm,\tag{143}$$

$$\liminf_{k \to +\infty} \frac{1}{k} \log P_s\left(W_m^{\otimes k}, \mathcal{C}_k^{(m)}\right) \ge -\sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ Rm - \chi_\alpha^\flat(\mathcal{E}_m W^{\otimes m}) \right\}.$$
 (144)

Note that we can assume that the elements of the decoding POVM  $\mathcal{D}_k^{(m)} = \{D_k^{(m)}(i) : i = 1, \ldots, |\mathcal{C}_k^{(m)}|\}$  are invariant under  $\mathcal{E}_m^{\otimes k}$ , i.e.,

$$\mathcal{E}_m^{\otimes k}(D_k^{(m)}(i)) = D_k^{(m)}(i) \qquad \forall i \ \forall k.$$
(145)

Indeed,

$$\begin{split} P_{s}(W_{m}^{\otimes k},\mathcal{C}_{k}^{(m)}) &= \frac{1}{|\mathcal{C}_{k}^{(m)}|} \sum_{i=1}^{|\mathcal{C}_{k}^{(m)}|} \operatorname{Tr} W_{m}^{\otimes k}(\phi_{k}^{(m)}(i)) D_{k}^{(m)}(i) \\ &= \frac{1}{|\mathcal{C}_{k}^{(m)}|} \sum_{i=1}^{|\mathcal{C}_{k}^{(m)}|} \operatorname{Tr} \mathcal{E}_{m}^{\otimes k} \left( W^{\otimes mk}(\phi_{k}^{(m)}(i)) \right) D_{k}^{(m)}(i) \\ &= \frac{1}{|\mathcal{C}_{k}^{(m)}|} \sum_{i=1}^{|\mathcal{C}_{k}^{(m)}|} \operatorname{Tr} \mathcal{E}_{m}^{\otimes k} \left( W^{\otimes mk}(\phi_{k}^{(m)}(i)) \right) \mathcal{E}_{m}^{\otimes k}(D_{k}^{(m)}(i)) \\ &= \frac{1}{|\mathcal{C}_{k}^{(m)}|} \sum_{i=1}^{|\mathcal{C}_{k}^{(m)}|} \operatorname{Tr} W_{m}^{\otimes k}(\phi_{k}^{(m)}(i)) \mathcal{E}_{m}^{\otimes k}(D_{k}^{(m)}(i)), \end{split}$$

i.e., the success probability does not change if we replace  $\{D_k^{(m)}(i) : i = 1, \dots, |\mathcal{C}_k^{(m)}|\}$  with  $\{\mathcal{E}_m^{\otimes k}(D_k^{(m)}(i)) : i = 1, \dots, |\mathcal{C}_k^{(m)}|\}$ .

Now, from the above code  $C_k^{(m)}$ , we construct a code  $C_n$  for  $W^{\otimes n}$  for every  $n \in \mathbb{N}$ . For a given  $n \in \mathbb{N}$ , let  $k \in \mathbb{N}$  be such that  $km \leq n < (k+1)m$  (we assume that m > 1). For every  $i \in \{1, \ldots, |\mathcal{C}_k^{(m)}|\}$ , define  $\phi_n(i)$  as any continuation of  $\phi_k^{(m)}(i)$  in  $\mathcal{X}^n$ , and define the decoding POVM elements as  $D_n(i) := D_k^{(m)}(i) \otimes I^{\otimes (n-mk)}$ . Then  $|\mathcal{C}_n| = |\mathcal{C}_k^{(m)}|$ , and hence, by (143)

$$\liminf_{n \to +\infty} \frac{1}{n} \log |\mathcal{C}_n| = \frac{1}{m} \liminf_{k \to +\infty} \frac{1}{k} \log |\mathcal{C}_k^{(m)}| \ge R.$$
(146)

Moreover,

$$P_{s}(W^{\otimes n}, \mathcal{C}_{n}) = \frac{1}{|\mathcal{C}_{n}|} \sum_{i=1}^{|\mathcal{C}_{n}|} \operatorname{Tr} W^{\otimes n}(\phi_{n}(i)) \left( D_{k}^{(m)}(i) \otimes I^{\otimes (n-mk)} \right)$$
$$= \frac{1}{|\mathcal{C}_{k}^{(m)}|} \sum_{i=1}^{|\mathcal{C}_{k}^{(m)}|} \operatorname{Tr} W^{\otimes km}(\phi_{k}^{(m)}(i)) D_{k}^{(m)}(i)$$
$$= \frac{1}{|\mathcal{C}_{k}^{(m)}|} \sum_{i=1}^{|\mathcal{C}_{k}^{(m)}|} \operatorname{Tr} W^{\otimes km}(\phi_{k}^{(m)}(i)) \mathcal{E}_{m}^{\otimes k}(D_{k}^{(m)}(i))$$
$$= \frac{1}{|\mathcal{C}_{k}^{(m)}|} \sum_{i=1}^{|\mathcal{C}_{k}^{(m)}|} \operatorname{Tr} \mathcal{E}_{m}^{\otimes k}(W^{\otimes km}(\phi_{k}^{(m)}(i))) D_{k}^{(m)}(i)$$
$$= P_{s}(W_{m}^{\otimes k}, \mathcal{C}_{k}^{(m)}),$$

where in the third line we used (145). Hence, by (144),

$$\liminf_{n \to +\infty} \frac{1}{n} \log P_s(W^{\otimes n}, \mathcal{C}_n) = \frac{1}{m} \liminf_{k \to +\infty} \frac{1}{k} \log P_s\left(W_m^{\otimes k}, \mathcal{C}_k^{(m)}\right)$$
$$\geq -\sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \frac{1}{m} \chi_\alpha^\flat(W_m) \right\}.$$
(147)

Now we have

$$\chi_{\alpha}^{\flat}(W_m) = \sup_{P_m \in \mathcal{P}_f(\mathcal{X}^m)} \chi_{\alpha,1}^{\flat}(\mathcal{E}_m W^{\otimes n}, P_m) \ge \sup_{P \in \mathcal{P}_f(\mathcal{X})} \chi_{\alpha,1}^{\flat}(\mathcal{E}_m W^{\otimes n}, P^{\otimes m})$$
$$\ge \sup_{P \in \mathcal{P}_f(\mathcal{X})} \chi_{\alpha,1}^*(W^{\otimes n}, P^{\otimes m}) - 3\log v_{m,d} = m \sup_{P \in \mathcal{P}_f(\mathcal{X})} \chi_{\alpha,1}^*(W, P) - 3\log v_{m,d}$$
$$= m \chi_{\alpha}^*(W) - 3\log v_{m,d},$$

where the first equality is by definition, the first inequality is trivial, the second inequality is due to Lemma IV.10, the following identity is due to Lemma IV.8, and the last identity is again by definition. Thus, by (147),

$$\liminf_{n \to +\infty} \frac{1}{n} \log P_s(W^{\otimes n}, \mathcal{C}_n) \ge -\sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \chi^*_{\alpha}(W) \right\} - 3\frac{1}{m} \log v_{m,d}.$$

Taking the limit  $m \to +\infty$  and using (13), we obtain

$$\liminf_{n \to +\infty} \frac{1}{n} \log P_s(W^{\otimes n}, \mathcal{C}_n) \ge -\sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \chi^*_{\alpha}(W) \right\}.$$
(148)

Combining this with (146), the assertion follows.

# VI. APPLICATION TO QUANTUM CHANNELS

By a quantum channel we mean a completely positive trace-preserving (CPTP) linear map from  $\mathcal{L}(\mathcal{H}_{in})$  to  $\mathcal{L}(\mathcal{H}_{out})$ , where  $\mathcal{H}_{in}, \mathcal{H}_{out}$  are finite-dimensional Hilbert spaces. The *n*-fold product extension of a quantum channel  $\Phi : \mathcal{L}(\mathcal{H}_{in}) \to \mathcal{L}(\mathcal{H}_{out})$  is the unique linear extension of the map  $\Phi^{\otimes n} : X_1 \otimes \ldots X_n \mapsto \Phi(X_1) \otimes \ldots \otimes \Phi(X_n)$  to  $\mathcal{L}(\mathcal{H}_{in}^{\otimes n})$ . Obviously, any quantum channel  $\Phi : \mathcal{L}(\mathcal{H}_{in}) \to \mathcal{L}(\mathcal{H}_{out})$  can be viewed as a classical-quantum channel with  $\mathcal{X} = \mathcal{H}_{in}, \mathcal{H} = \mathcal{H}_{out}$ , and  $W(x) := \Phi(x), x \in \mathcal{L}(\mathcal{H}_{in})$ . However, the *n*-fold product extension of a quantum channel is different from its *n*-fold product extension as a classical-quantum channel, as the latter can only take product inputs, while the former can take entangled inputs. The coding process and the definition of the strong converse exponent  $sc(R, \Phi)$  remains the same as described in Section V A, with the exception that the codewords  $\phi_n(k)$  can be any density operators in  $\mathcal{H}_{in}^{\otimes n}$ . By Theorem V.14, for any rate *R*, there exists a sequence of codes  $\mathcal{C}_n, n \in \mathbb{N}$ , with product codewords such that

$$\liminf_{n \to \infty} \frac{1}{n} \log |\mathcal{C}_n| \ge R \quad \text{and} \quad \liminf_{n \to \infty} \frac{1}{n} \log P_s(\mathcal{C}_n, \Phi^{\otimes n}) \ge -\sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \chi^*_{\alpha}(\Phi) \right\}.$$

Hence,

$$sc(R,\Phi) \leq \sup_{\alpha>1} \frac{\alpha-1}{\alpha} \{R - \chi^*_{\alpha}(\Phi)\}.$$

On the other hand, the same argument as in Lemma V.5 yields that for any sequence of codes with rate R,

$$\limsup_{n \to +\infty} \frac{1}{n} \log P_s(\Phi^{\otimes n}, \mathcal{C}_n) \le -\sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \limsup_{n \to +\infty} \frac{1}{n} \chi_{\alpha}^*(\Phi^{\otimes n}) \right\}.$$

Hence, we arrive at the following

**Theorem VI.1** For any quantum channel  $\Phi$  and any rate R > 0,

$$\sup_{\alpha>1} \frac{\alpha-1}{\alpha} \left\{ R - \limsup_{n \to +\infty} \frac{1}{n} \chi_{\alpha}^*(\Phi^{\otimes n}) \right\} \le sc(R, \Phi) \le \sup_{\alpha>1} \frac{\alpha-1}{\alpha} \left\{ R - \chi_{\alpha}^*(\Phi) \right\}.$$

In particular, if  $\limsup_{n \to +\infty} \frac{1}{n} \chi_{\alpha}^*(\Phi^{\otimes n}) \leq \chi_{\alpha}^*(\Phi)$  for every  $\alpha \in (1, +\infty)$  then

$$sc(R,\Phi) = \sup_{\alpha>1} \frac{\alpha-1}{\alpha} \left\{ R - \chi_{\alpha}^{*}(\Phi) \right\}.$$
(149)

# A. Entanglement breaking channels

It has been shown in [75, Theorem 18] that if  $\Phi$  is entanglement breaking then  $\chi^*_{\alpha}(\Phi^{\otimes n}) \leq n\chi^*_{\alpha}(\Phi)$ for every  $n \in \mathbb{N}$  and  $\alpha \in (1, 2]$ . The range of  $\alpha$  for which this subadditivity property holds was limited to (1, 2] in [75] because the equality  $\chi^*_{\alpha}(\Phi) = R_{\alpha}(\Phi)$  was only showed for this parameter range (in [75, Lemma 14]), which in turn was due to the fact that convexity of  $\sigma \mapsto Q^*_{\alpha}(\varrho || \sigma)$  for a fixed  $\varrho$  was only known for this parameter range. Having Propositions III.18 and IV.2 at our disposal, we can conclude that  $\chi^*_{\alpha}(\Phi^{\otimes n}) \leq n\chi^*_{\alpha}(\Phi)$  holds for every  $n \in \mathbb{N}$  and  $\alpha \in (1, +\infty)$ , and thus the strong converse exponent of an entanglement breaking channel  $\Phi$  is given by (149).

# B. Covariant channels

We say that a quantum channel  $\Phi : \mathcal{L}(\mathcal{H}_{in}) \to \mathcal{L}(\mathcal{H}_{out})$  is group covariant if there exists a compact group G and continuous unitary representations  $U_{in}$  and  $U_{out}$  on  $\mathcal{H}_{in}$  and  $\mathcal{H}_{out}$ , respectively, such that

- (i)  $U_{\text{out}}$  is irreducible, and
- (ii)  $\Phi(U_{\rm in}(g)XU_{\rm in}(g)^*) = U_{\rm out}(g)\Phi(X)U_{\rm out}(g)^*$  for all  $g \in G$  and all  $X \in \mathcal{L}(\mathcal{H}_{\rm in})$ .

For a density operator  $\sigma$ , let

$$H_{\alpha}(\sigma) := \frac{1}{1-\alpha} \log \operatorname{Tr} \sigma^{\alpha}$$

denote its  $\alpha$ -entropy. The minimum output  $\alpha$ -entropy  $H^{\min}_{\alpha}(\Phi)$  of a channel  $\Phi : \mathcal{L}(\mathcal{H}_{in}) \to \mathcal{L}(\mathcal{H}_{out})$  is defined as

$$H^{\min}_{\alpha}(\Phi) := \min_{\varrho \in \mathcal{S}(\mathcal{H}_{\mathrm{in}})} H_{\alpha}(\Phi(\varrho))$$

We say that a channel  $\Phi$  belongs to the KW-class (after [41]) if it is group covariant, and has additive minimum output  $\alpha$ -entropy for all  $\alpha \in (1, +\infty)$ , i.e.,

(iii) 
$$H^{\min}_{\alpha}(\Phi^{\otimes n}) = nH^{\min}_{\alpha}(\Phi), \qquad n \in \mathbb{N}, \ \alpha \in (1, +\infty).$$

Typical examples for channels in the KW-class are the depolarizing channels [39] and unital qubit channels [38].

It has been shown in [41] that all channels in the KW-class have the strong converse property for classical information transmission, i.e., for all sequences of codes with rate above the Holevo capacity, the success probability goes to zero with the number of channel uses. Note that the properties (i)–(iii) imply that the classical information transmission capacity of a channel in the KW-class is equal to its single-shot Holevo capacity, according to the Holevo-Schumacher-Westmoreland theorem [37, 65]. Moreover, the results of [41] yield the following bound on the strong converse exponent of any channel  $\Phi$  in the KW-class:

$$sc(R, \Phi) \ge \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \{R - \chi_{\alpha}(\Phi)\}.$$

Note that in the above expression we have  $\chi_{\alpha}(\Phi)$ , which can be strictly larger than  $\chi_{\alpha}^{*}(\Phi)$ .

It has been shown in [75, Section 8.1] that for any quantum channel  $\Phi : \mathcal{L}(\mathcal{H}_{in}) \to \mathcal{L}(\mathcal{H}_{out})$ ,

$$\chi_{\alpha}^{*}(\Phi^{\otimes n}) \leq n \log \dim \mathcal{H}_{\text{out}} - H_{\alpha}^{\min}(\Phi^{\otimes n})$$

In particular, for channels satisfying (iii) above, we have

$$\chi_{\alpha}^{*}(\Phi^{\otimes n}) \leq n \left[ \log \dim \mathcal{H}_{\text{out}} - H_{\alpha}^{\min}(\Phi) \right].$$
(150)

The following is an analogue of [41, Lemma 1.3]. While we follow the main idea of the proof of [41, Lemma 1.3], some technical details are different; in particular, Proposition III.18 plays a crucial role.

**Lemma VI.2** For any group covariant channel  $\Phi : \mathcal{L}(\mathcal{H}_{in}) \to \mathcal{L}(\mathcal{H}_{out})$ , and any  $\alpha \in [1/2, +\infty)$ , we have

$$\chi^*_{\alpha}(\Phi) = \log \dim \mathcal{H}_{out} - H^{\min}_{\alpha}(\Phi).$$

**Proof** Let  $\Phi : \mathcal{L}(\mathcal{H}_{in}) \to \mathcal{L}(\mathcal{H}_{out})$  be a group covariant channel (with covariace group G), and  $\alpha \in [1/2, +\infty)$  be fixed. By Proposition IV.2

$$\chi_{\alpha}^{*}(\Phi) = \inf_{\sigma \in \mathcal{S}(\mathcal{H}_{out})} \sup_{\varrho \in S(\mathcal{H}_{in})} D_{\alpha}^{*}(\Phi(\varrho) \| \sigma) \leq \sup_{\varrho \in S(\mathcal{H}_{in})} D_{\alpha}^{*}\left(\Phi(\varrho) \left\| \frac{I_{\mathcal{H}_{out}}}{\dim \mathcal{H}_{out}} \right) = \log \dim \mathcal{H}_{out} - H_{\alpha}^{\min}(\Phi).$$

On the other hand, for any  $\rho \in \mathcal{S}(\mathcal{H}_{in}), \sigma \in \mathcal{S}(\mathcal{H}_{out})$ , and any  $g \in G$ , we have

$$\sup_{\varrho \in \mathcal{S}(\mathcal{H}_{\mathrm{in}})} D^*_{\alpha}(\Phi(\varrho) \| \sigma) \ge D^*_{\alpha}(\Phi(U_{\mathrm{in}}(g) \varrho U_{\mathrm{in}}(g)^*) \| \sigma) = D^*_{\alpha}(U_{\mathrm{out}}(g) \Phi(\varrho) U_{\mathrm{out}}(g)^* \| \sigma)$$
$$= D^*_{\alpha}(\Phi(\varrho) \| U_{\mathrm{out}}(g)^* \sigma U_{\mathrm{out}}(g)).$$

Integrating both sides with respect to the normalized Haar measure  $\mu$  on G, we get

$$\sup_{\varrho' \in \mathcal{S}(\mathcal{H}_{\mathrm{in}})} D^*_{\alpha}(\Phi(\varrho') \| \sigma) \ge \int_G D^*_{\alpha}(\Phi(\varrho) \| U_{\mathrm{out}}(g)^* \sigma U_{\mathrm{out}}(g)) \, d\mu \ge D^*_{\alpha} \left( \Phi(\varrho) \left\| \int_G U_{\mathrm{out}}(g)^* \sigma U_{\mathrm{out}}(g) \, d\mu \right) \right.$$
$$= D^*_{\alpha} \left( \Phi(\varrho) \left\| \frac{I_{\mathcal{H}_{\mathrm{out}}}}{\dim \mathcal{H}_{\mathrm{out}}} \right) = \log \dim \mathcal{H}_{\mathrm{out}} - H_{\alpha}(\Phi(\varrho)),$$

where the second inequality is due to Proposition III.18, and in the second line we used the irreducibility of  $U_{\text{out}}$ . Taking first the infimum over  $\sigma \in S(\mathcal{H}_{\text{out}})$  and then the supremum over  $\varrho \in S(\mathcal{H}_{\text{in}})$ , we get

$$\chi^*_{\alpha}(\Phi) \ge \log \dim \mathcal{H}_{out} - H^{\min}_{\alpha}(\Phi).$$

Lemma VI.2 and (150) yield immediately the following

**Corollary VI.3** For any channel  $\Phi$  in the KW-class, we have

$$\chi_{\alpha}^{*}(\Phi^{\otimes n}) \leq n\chi_{\alpha}^{*}(\Phi), \qquad n \in \mathbb{N}, \quad \alpha \in [1/2, +\infty).$$

Finally, Corollary VI.3 and Theorem VI.1 yield

**Theorem VI.4** For any quantum channel  $\Phi$  in the KW-class, and any rate R > 0,

$$sc(R, \Phi) = \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \{ R - \chi_{\alpha}^{*}(\Phi) \}$$
$$= \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \{ R - \log \dim \mathcal{H}_{out} + H_{\alpha}^{min}(\Phi) \}.$$

# Appendix A: Universal symmetric states

In this section we review the construction of a universal symmetric state that was given in [27] (see Lemma II.8).

For every  $n \in \mathbb{N}$ , let  $\mu_{\mathcal{H},n}$  be the *n*-th tensor power representation of the identical representation of  $\mathrm{SU}(\mathcal{H})$  on  $\mathcal{H}$ , i.e.,  $\mu_{\mathcal{H},n} : \mathrm{SU}(\mathcal{H}) \ni A \longmapsto A^{\otimes n}$ , and let  $\mathcal{L}_{\mu_{\mathcal{H},n}}(\mathcal{H}^{\otimes n}) := \{A \in \mathcal{L}(\mathcal{H}^{\otimes n}) : \mu_{\mathcal{H},n}(U)A = A\mu_{\mathcal{H},n}(U) \quad \forall U \in \mathrm{SU}(\mathcal{H})\}$  be the commutant algebra of the representation. According to the Schur-Weyl duality (see, e.g., [23, Chapter 9]),  $\mathcal{L}_{\mathrm{sym}}(\mathcal{H}^{\otimes n})$  and  $\mathcal{L}_{\mu_{\mathcal{H},n}}(\mathcal{H}^{\otimes n})$  are each other's commutants, i.e.,

$$\mathcal{L}_{\text{sym}}(\mathcal{H}^{\otimes n}) = \mathcal{L}_{\mu_{\mathcal{H},n}}(\mathcal{H}^{\otimes n})' = \mu_{\mathcal{H},n}(\text{SU}(\mathcal{H}))'', \qquad \mathcal{L}_{\mu_{\mathcal{H},n}}(\mathcal{H}^{\otimes n}) = \mathcal{L}_{\text{sym}}(\mathcal{H}^{\otimes n})' = \{\pi_{\mathcal{H}} | \pi \in \mathfrak{S}_n\}''.$$

(Note that the double commutant is equal to the algebra generated by the given set.) Moreover,  $\mathcal{H}^{\otimes n}$  decomposes as

$$\mathcal{H}^{\otimes n} \simeq \bigoplus_{\lambda \in Y_{n,d}} U_{\lambda} \otimes V_{\lambda},$$

where  $U_{\lambda}$  and  $V_{\lambda}$  carry irreducible representations of  $\mathfrak{S}_n$  and  $\mathrm{SU}(\mathcal{H})$ , respectively, and we have

$$\mathcal{L}_{\text{sym}}(\mathcal{H}^{\otimes n}) = \bigoplus_{\lambda \in Y_{n,d}} \mathcal{L}(U_{\lambda}) \otimes I_{V_{\lambda}}, \qquad \mathcal{L}_{\mu_{\mathcal{H},n}}(\mathcal{H}^{\otimes n}) = \bigoplus_{\lambda \in Y_{n,d}} I_{U_{\lambda}} \otimes \mathcal{L}(V_{\lambda}).$$

Here,  $Y_{n,d}$  is the set of the Young diagrams up to the depth  $d := \dim \mathcal{H}$ , defined as

$$Y_{n,d} = \left\{ \lambda = (n_1, n_2, \dots, n_d) \ \middle| \ n_1 \ge n_2 \ge \dots \ge n_d \ge 0, \ \sum_{i=1}^n n_i = n \right\}.$$

In particular, every permutation invariant state  $\rho_n \in \mathcal{L}_{sym}(\mathcal{H}^{\otimes n})$  can be written according to the above decomposition as

$$\varrho_n = \bigoplus_{\lambda \in Y_{n,d}} p_\lambda \cdot \varrho_\lambda \otimes \frac{I_{V_\lambda}}{\dim V_\lambda},$$

where  $\rho_{\lambda}$  is a density operator acting on  $U_{\lambda}$  and  $\{p_{\lambda}\}$  is a probability function on  $Y_{n,d}$ . Using inequalities  $\rho_{\lambda} \leq I_{U_{\lambda}}$  and  $p_{\lambda} \leq 1$ , we have

$$\varrho_n \leq \bigoplus_{\lambda \in Y_{n,d}} \frac{1}{\dim V_{\lambda}} \cdot I_{U_{\lambda}} \otimes I_{V_{\lambda}} \leq \max_{\lambda} \{\dim U_{\lambda}\} \cdot |Y_{n,d}| \cdot \sigma_{u,n},$$

where

$$\sigma_{u,n} := \bigoplus_{\lambda \in Y_{n,d}} \frac{1}{|Y_{n,d}|} \cdot \frac{I_{U_{\lambda}}}{\dim U_{\lambda}} \otimes \frac{I_{V_{\lambda}}}{\dim V_{\lambda}}$$

It is known that

$$\max_{\lambda} \{\dim U_{\lambda}\} \le (n+1)^{\frac{d(d-1)}{2}}, \quad |Y_{n,d}| \le (n+1)^{d-1},$$

and hence  $\sigma_{u,n}$  satisfies the criteria in Lemma II.8.

# Appendix B: The auxiliary function

In this section we prove that various versions of the so-called auxiliary function are concave. This was not needed in the main body of the paper, but since the proof follows very naturally from some simple considerations in the paper, and the problem itself is interesting for information theory, we decided to include a brief discussion here. For more on the background, we refer to the recent paper [14].

Recall the definitions of the  $Q_{\alpha}$  quantities for classical probability distributions, and the  $Q_{\alpha}, Q_{\alpha}^{*}, Q_{\alpha}^{\flat}$ quantities for positive semidefinite operators, given in Section III A. For the rest, let  $Q_{\alpha}^{(t)}$  be any function on pairs of positive semidefinite operators, such that it reduces to the classical  $Q_{\alpha}$  for commuting operators, i.e., for any non-zero non-negative functions  $p, q \in \mathbb{R}^{\mathcal{X}}_{+} \setminus \{0\}$  on some finite set  $\mathcal{X}$ , any orthonormal system  $\{|x\rangle\}_{x\in\mathcal{X}}$  in some Hilbert space, and any  $\alpha \in (0, +\infty) \setminus \{1\}$ ,

$$Q_{\alpha}^{(t)}\left(\sum_{x\in\mathcal{X}}p(x)|x\rangle\langle x|\Big\|\sum_{x\in\mathcal{X}}q(x)|x\rangle\langle x|\right)=Q_{\alpha}(p||q),$$

where the latter is the classical  $Q_{\alpha}$  quantity of p and q. This is satisfied by  $Q_{\alpha}^{(t)}$  for  $(t) = \{ \}, (t) = *$  and  $(t) = \flat$ . For any  $\varrho, \sigma \in \mathcal{L}(\mathcal{H})_+$ , and any  $\alpha \in (0, +\infty) \setminus \{1\}$ , define

$$\psi_{\alpha}^{(t)}(\varrho \| \sigma) := \log Q_{\alpha}^{(t)}(\varrho \| \sigma), \qquad D_{\alpha}^{(t)}(\varrho \| \sigma) := \frac{1}{\alpha - 1} \log Q_{\alpha}^{(t)}(\varrho \| \sigma) - \frac{1}{\alpha - 1} \log \operatorname{Tr} \varrho \tag{B1}$$

as in (27) and (26), and let  $\psi_1^{(t)}(\varrho \| \sigma) := \log \operatorname{Tr} \varrho$ . For a map  $W : \mathcal{X} \to \mathcal{L}(\mathcal{H})_+$ , where  $\mathcal{H}$  is a finitedimensional Hilbert space, and for any finitely supported probability distribution  $P \in \mathcal{P}_f(\mathcal{X})$ , let

$$\chi_{1,\alpha}^{(t)}(W,P) := \inf_{\sigma \in \mathcal{S}(\mathcal{H})} D_{\alpha}^{(t)} \left( \sum_{x} P(x) |x\rangle \langle x| \otimes W_{x} \right\| \sum_{x} P(x) |x\rangle \langle x| \otimes \sigma \right)$$
$$\chi_{2,\alpha}^{(t)}(W,P) := \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sum_{x} P(x) D_{\alpha}^{(t)} (W_{x} \| \sigma).$$

When W is a classical-quantum channel (i.e., all W(x) have unit trace), the above quantities are exactly the generalized Holevo quantities given in (80) and (81), at least for the three (t) values considered there. We define the corresponding *auxiliary functions* as

$$E_{0,i}^{(t)}(s,W,P) := -\frac{\alpha - 1}{\alpha} \chi_{i,\alpha}^{(t)}(W,p) \bigg|_{\alpha = \frac{1}{1+s}},$$

i = 1, 2. Note that with  $\alpha = \frac{1}{1+s}$ , we have

$$E_{0,1}^{(t)}(s,W,P) = -\sup_{\sigma\in\mathcal{S}(\mathcal{H})}\frac{1}{\alpha}\psi_{\alpha}^{(t)}(\mathbb{W}(P)\|P\otimes\sigma) = -\sup_{\sigma\in\mathcal{S}(\mathcal{H})}(1+s)\psi_{\frac{1}{1+s}}^{(t)}(\mathbb{W}(P)\|P\otimes\sigma)$$
(B2)

$$E_{0,2}^{(t)}(s,W,p) = -\sup_{\sigma \in \mathcal{S}(\mathcal{H})} \sum_{x \in \mathcal{X}} P(x) \frac{1}{\alpha} \psi_{\alpha}^{(t)}(W(x) \| \sigma) = -\sup_{\sigma \in \mathcal{S}(\mathcal{H})} \sum_{x \in \mathcal{X}} P(x)(1+s) \psi_{\frac{1}{1+s}}^{(t)}(W(x) \| \sigma)$$
(B3)

for  $\alpha \in (0, 1)$ , or equivalently, s > 0, and the same formulas hold with minima instead of the maxima for  $\alpha > 1$ , or equivalently,  $s \in (-1, 0)$ .

**Proposition B.1** Assume that  $Q^{(t)}$  is such that for any  $\rho, \sigma \in \mathcal{L}(\mathcal{H})_+$ , the map  $\alpha \mapsto \psi_{\alpha}^{(t)}(\rho \| \sigma)$  is convex on (0, 1). Then for any map  $W : \mathcal{X} \to \mathcal{L}(\mathcal{H})$ , and any  $P \in \mathcal{P}_f(\mathcal{X})$ , the maps  $s \mapsto E_{0,i}^{(t)}(s, W, P)$  are concave on  $(0, +\infty)$  for i = 1, 2.

**Proof** Applying Lemma II.5 to  $f(\alpha) := \psi_{\alpha}^{(t)}(.,.)$  with the affine function  $\varphi(s) := 1 + s$ , we get that  $s \mapsto (1+s)\psi_{\frac{1}{1+s}}^{(t)}(.,.)$  is convex, and by Lemma II.3, taking the supremum over  $\sigma$  in (B2), (B3), preserves this convexity.

**Corollary B.2** By Lemma III.12,  $Q_{\alpha}^{(t)}$  with  $(t) = \{ \}, (t) = *, (t) = \flat$  satisfy the convexity assumption of Proposition B.1, and hence the corresponding auxiliary functions are concave.

The case  $(t) = \{ \}$  is special in the sense that there is an explicit expression for  $E_{0,1}^{(t)}(s, W, P)$ , due to the Sibson identity [41, 68]. Indeed, one can easily see that

$$D_{\alpha}\left(\sum_{x} P(x)|x\rangle\langle x|\otimes W_{x} \|\sum_{x} P(x)|x\rangle\langle x|\otimes\sigma\right)$$
  
=  $\frac{1}{\alpha-1}\log\sum_{x} P(x)\operatorname{Tr} W_{x}^{\alpha}\sigma^{1-\alpha} = \frac{1}{\alpha-1}\log\operatorname{Tr}\omega(\alpha,P)^{\alpha}\sigma^{1-\alpha} + \frac{\alpha}{\alpha-1}\log\operatorname{Tr}\left(\sum_{x} P(x)W_{x}^{\alpha}\right)^{1/\alpha}$ 

where  $\omega(\alpha, P) := (\sum_x P(x)W_x^{\alpha})^{1/\alpha} / \operatorname{Tr} (\sum_x P(x)W_x^{\alpha})^{1/\alpha}$ . By the strict positivity of  $D_{\alpha}$  on states, we get

$$\min_{\sigma \in \mathcal{S}(\mathcal{H})} D_{\alpha} \left( \sum_{x} P(x) |x\rangle \langle x| \otimes W_{x} \right\| \sum_{x} P(x) |x\rangle \langle x| \otimes \sigma \right) = \frac{\alpha}{\alpha - 1} \log \operatorname{Tr} \left( \sum_{x} P(x) W_{x}^{\alpha} \right)^{1/\alpha}, \quad (B4)$$

or equivalently, with  $\alpha = \frac{1}{1+s}$ ,

$$E_{0,1}(s, W, P) = -\log \operatorname{Tr}\left(\sum_{x} P(x) W_x^{\alpha}\right)^{1/\alpha} = -\log \operatorname{Tr}\left(\sum_{x} P(x) W_x^{\frac{1}{1+s}}\right)^{1+s}$$

By Corollary B.2, we have the following:

**Corollary B.3** For any map  $W : \mathcal{X} \to \mathcal{L}(\mathcal{H})_+$ , and any finitely supported probability distribution  $P \in \mathcal{P}_f(\mathcal{X})$ , the map  $s \mapsto -\log \operatorname{Tr}\left(\sum_x P(x) W_x^{\frac{1}{1+s}}\right)^{1+s}$  is concave.

**Remark B.4** The same result was proved very recently in [14] by completely different methods, using the properties of certain operator means. Our proof is considerably simpler, and Proposition B.1 and Corollary B.2 also give extensions to auxiliary functions defined from other Rényi divergences, and for the value i = 2.

Unfortunately, the proof method of Proposition B.1 does not work for  $\alpha > 1$ , equivalently, for  $s \in (-1,0)$ ; this is due to the fact that in this case we have infima in (B2) and (B3) instead of suprema, and in general, the infimum of convex functions need not be convex. Note, however, that the proof of Lemma V.13 yields the following:

**Proposition B.5** For any map  $W : \mathcal{X} \to \mathcal{L}(\mathcal{H})_+$ , and any finitely supported probability distribution  $P \in \mathcal{P}_f(\mathcal{X})$ , the maps  $s \mapsto E_{0,i}^{\flat}(s, W, P)$  are concave on (-1, 0) for i = 1, 2.

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