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ON GENERALIZATIONS RELATED TO THE LEFT SIDE OF FEJER'S INEQUALITY VIA FRACTIONAL INTEGRAL OPERATOR

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Abstract. In the paper, firstly, a new fractional integral identity is obtained. Then, some new results related to the left side of Fejér's inequality for differentiable mappings whose derivatives in absolute value are convex via fractional integral operator, using this identity with fundamental inequalities such as Hölder's integral inequality, power-mean inequality and triangle inequality for integral, are presented. The results presented here would provide extensions of those proved in [11].

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1. INTRODUCTION AND PRELIMINARIES

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality (*see e.g.* [5]). The most well-known inequalities related to the integral mean of a convex function f are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities. In [6], Fejér established the following inequality:

Let $f : [a,b] \to \mathbb{R}$ be convex function. Then the inequality

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{a+b}{2}\int_a^b g(x)dx \tag{1.2}$$

holds, where $g : [a,b] \to \mathbb{R}$ is nonnegative,integrable and symmetric to $\frac{a+b}{2}$.

For some results which generalize, improve, and extend the inequalities (1.1) and (1.2), (*see e.g.* [7, 10–13]).

In the following, we will give some necessary definitions and preliminary results which are used and referred to throughout this paper.

Definition 1. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} u^{\alpha-1} du$. Here is $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

Some recent result and properties concerning the this integral operators can be found in [2–4, 8].

In [9], Raina introduced a class of functions defined formally by

$$\mathcal{F}_{\rho,\lambda}^{\sigma}(x) = \mathcal{F}_{\rho,\lambda}^{\sigma(0),\sigma(1),\ldots}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \quad (\rho, \lambda > 0; |x| < \mathbf{R}), \tag{1.3}$$

where the coefficients $\sigma(k)\,(k \in \mathbb{N} = \mathbb{N} \cup \{0\})$ is a bounded sequence of positive real numbers and $\mathbf{R}$ is the set of real numbers. With the help of (1.3), Raina [9] and Agarwal *et al.* [1] defined the following left-sided and right-sided fractional integral operators respectively, as follows:

$$\left(\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} \varphi\right)(x) = \int_a^x (x-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}[w(x-t)^{\rho}]\varphi(t)dt \quad (x > a), \tag{1.4}$$

$$\left(\mathcal{J}_{\rho,\lambda,b-;w}^{\sigma} \varphi\right)(x) = \int_x^b (t-x)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}[w(t-x)^{\rho}]\varphi(t)dt \quad (x < b), \tag{1.5}$$

where $\lambda, \rho > 0$, $w \in \mathbb{R}$ and $\varphi(t)$ is such that the integral on the right side exits.

It is easy to verify that $\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} \varphi(x)$ and $\mathcal{J}_{\rho,\lambda,b-;w}^{\sigma} \varphi(x)$ are bounded integral operators on $L(a,b)$, if

$$\mathfrak{M} := \mathcal{F}_{\rho,\lambda+1}^{\sigma}[w(b-a)^{\rho}] < \infty. \tag{1.6}$$

In fact, for $\varphi \in L(a,b)$, we have

$$||\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} \varphi(x)||_1 \leq \mathfrak{M}(b-a)^{\lambda} ||\varphi||_1 \tag{1.7}$$

and

$$||\mathcal{J}_{\rho,\lambda,b-;w}^{\sigma} \varphi(x)||_1 \leq \mathfrak{M}(b-a)^{\lambda} ||\varphi||_1 \tag{1.8}$$

where

$$||\varphi||_p := \left(\int_a^b |\varphi(t)|^p dt \right)^{\frac{1}{p}}.$$

Here, many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. For instance the classical Riemann-Liouville fractional integrals J_{a+}^{α} and J_{b-}^{α} of order α follow easily by setting $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in (1.4) and (1.5).

Set *et al.* established a new identity and some Hermite-Hadamard-Fejér type inequalities for differentiable mappings whose derivatives in absolute value are convex via Riemann-Liouville fractional integrals as the following:

Lemma 1 ([11]). *Let $f : [a,b] \to \mathbb{R}$ be a differentiable on mapping (a,b) with $a < b$ and let $g : [a,b] \to \mathbb{R}$. If $f', g \in L[a,b]$, then the following identity for fractional integrals holds:*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\left[J_{\left(\frac{a+b}{2}\right)-}^{\alpha} g(a) + J_{\left(\frac{a+b}{2}\right)+}^{\alpha} g(b)\right] \\ & - \left[J_{\left(\frac{a+b}{2}\right)-}^{\alpha} (fg)(a) + J_{\left(\frac{a+b}{2}\right)+}^{\alpha} (fg)(b)\right] \\ & = \frac{1}{\Gamma(\alpha)} \int_a^b k(t) f'(t)\,dt, \end{aligned} \tag{1.9}$$

where

$$k(t) = \begin{cases} \int_a^t (s-a)^{\alpha-1} g(s)\,ds, & t \in \left[a, \frac{a+b}{2}\right], \\ \int_b^t (b-s)^{\alpha-1} g(s)\,ds, & t \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

Theorem 1 ([11]). *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$ and $g : [a,b] \to \mathbb{R}$ is continuous. If $|f'|$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right)\left[J_{\left(\frac{a+b}{2}\right)-}^{\alpha} g(a) + J_{\left(\frac{a+b}{2}\right)+}^{\alpha} g(b)\right] \right. \\ & \left. - \left[J_{\left(\frac{a+b}{2}\right)-}^{\alpha} (fg)(a) + J_{\left(\frac{a+b}{2}\right)+}^{\alpha} (fg)(b)\right] \right| \\ & \le \frac{(b-a)^{\alpha+1} \|g\|_{[a,b],\infty}}{2^{\alpha+1}(\alpha+1)\Gamma(\alpha+1)} \left(|f'(a)| + |f'(b)|\right) \end{aligned} \tag{1.10}$$

with $\alpha > 0$.

Theorem 2. ([11]) *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$ and let $g : [a,b] \to \mathbb{R}$ is continuous. If $|f'|^q$ is convex on $[a,b]$,*

$q \geq 1$, then the following inequality for fractional integrals holds:

$$\begin{aligned}&\left| f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right]\right.\\&\left.-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right|\\&\leq \frac{(b-a)^{\alpha+1}||g||_{[a,b],\infty}}{2^{\alpha+1+\frac{1}{q}}(\alpha+1)(\alpha+2)^{\frac{1}{q}}\Gamma(\alpha+1)}\\&\times\left\{\left((\alpha+3)|f'(a)|^q+(\alpha+1)|f'(b)|^q\right)^{\frac{1}{q}}+\left((\alpha+1)|f'(a)|^q+(\alpha+3)|f'(b)|^q\right)^{\frac{1}{q}}\right\}\end{aligned} \tag{1.11}$$

with $\alpha > 0$.

Theorem 3 ([11]). *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$ and let $g : [a,b] \to \mathbb{R}$ is continuous. If $|f'|^q$ is convex on $[a,b]$, $q > 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left| f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right]\right.\\&\left.-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right|\\&\leq \frac{(b-a)^{\alpha+1}||g||_{[a,b],\infty}}{2^{\alpha+1+\frac{2}{q}}(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\\&\times\left[\left(3|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}+\left(|f'(a)|^q+3|f'(b)|^q\right)^{\frac{1}{q}}\right],\end{aligned} \tag{1.12}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Motivated by results works done in [1,9,11], in this paper we show that Fejér type inequalities containing the Reimann-Liouville fractional integral operator and given in [11] can be extended to fractional integral operator introduced in [1,9].

2. Main results

Lemma 2. *Let $f : [a,b] \to \mathbb{R}$ be differentiable mapping on (a,b) with $a < b$ and let $g : [a,b] \to \mathbb{R}$. If $f', g \in L[a,b]$, then the following identity for fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right)\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}g(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}g(a)\right] \tag{2.1}$$

$$
\begin{aligned}
&-\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}(fg)(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}(fg)(a)\right]\\
&=\int_a^{\frac{a+b}{2}}\left(\int_a^t (s-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}]g(s)ds\right)f'(t)dt\\
&+\int_{\frac{a+b}{2}}^{b}\left(\int_b^t (b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)ds\right)f'(t)dt.
\end{aligned}
$$

Proof. It suffices to note that

$$
\begin{aligned}
I&=\int_a^{\frac{a+b}{2}}\left(\int_a^t (s-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}]g(s)ds\right)f'(t)dt\\
&+\int_{\frac{a+b}{2}}^{b}\left(\int_b^t (b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)ds\right)f'(t)dt\\
&=I_1+I_2.
\end{aligned}
\tag{2.2}
$$

By integration by parts, we get

$$
\begin{aligned}
I_1&=\left(\int_a^t (s-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}]g(s)ds\right)f(t)dt\Big|_a^{\frac{a+b}{2}}\\
&-\int_a^{\frac{a+b}{2}}(t-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(t-a)^{\rho}]g(t)f(t)dt\\
&=\left(\int_a^{\frac{a+b}{2}}(s-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}]g(s)ds\right)f\left(\frac{a+b}{2}\right)\\
&-\int_a^{\frac{a+b}{2}}(t-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(t-a)^{\rho}](fg)(t)dt\\
&=f\left(\frac{a+b}{2}\right)\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}g(a)-\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}(fg)(a)
\end{aligned}
\tag{2.3}
$$

and similarly

$$
\begin{aligned}
I_2&=\left(\int_b^t (b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)ds\right)f(t)dt\Big|_{\frac{a+b}{2}}^{b}\\
&-\int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-t)^{\rho}]g(t)f(t)dt\\
&=\left(\int_{\frac{a+b}{2}}^{b}(b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)ds\right)f\left(\frac{a+b}{2}\right)
\end{aligned}
\tag{2.4}
$$

$$-\int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha-1}\mathcal{F}_{\rho,\alpha}^{\sigma}[w(b-t)^{\rho}](fg)(t)dt$$
$$=f\left(\frac{a+b}{2}\right)\mathcal{J}_{\rho,\alpha,\frac{a+b}{2}+;w}^{\sigma}g(b)-\mathcal{J}_{\rho,\alpha,\frac{a+b}{2}+;w}^{\sigma}(fg)(b).$$

Putting (2.3) and (2.4) in (2.2), we obtain (2.1) which completes the proof.

□

Theorem 4. *Let $f: I\to\mathbb{R}$ be a differentiable mapping on I° and $f'\in L[a,b]$ with $a<b$ and $g:[a,b]\to\mathbb{R}$ is continuous. If $|f'|$ is convex on $[a,b]$, then the following for fractional integrals holds:*

$$\left|f\left(\frac{a+b}{2}\right)\left[\mathcal{J}_{\rho,\alpha,\frac{a+b}{2}+;w}^{\sigma}g(b)+\mathcal{J}_{\rho,\alpha,\frac{a+b}{2}-;w}^{\sigma}g(a)\right]\right. \tag{2.5}$$
$$\left.-\left[\mathcal{J}_{\rho,\alpha,\frac{a+b}{2}+;w}^{\sigma}(fg)(b)+\mathcal{J}_{\rho,\alpha,\frac{a+b}{2}-;w}^{\sigma}(fg)(a)\right]\right|$$
$$\leq(b-a)^{\alpha+1}||g||_{[a,b],\infty}\mathcal{F}_{\rho,\alpha+1}^{\sigma_1}[|w|(b-a)^{\rho}]\left(|f'(a)|+|f'(b)|\right)$$

where

$$\sigma_1(k):=\sigma(k)\frac{1}{2^{\alpha+\rho k+1}(\alpha+\rho k+1)}$$

with $\alpha>0$.

Proof. Since $|f'|$ is convex on $[a,b]$, we know that for $t\in[a,b]$

$$|f'(t)|=\left|f'\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|\leq\frac{b-t}{b-a}|f'(a)|+\frac{t-a}{b-a}|f'(b)|.$$

From Lemma 2 we have

$$\left|f\left(\frac{a+b}{2}\right)\left[\mathcal{J}_{\rho,\alpha,\frac{a+b}{2}+;w}^{\sigma}g(b)+\mathcal{J}_{\rho,\alpha,\frac{a+b}{2}-;w}^{\sigma}g(a)\right]\right.$$
$$\left.-\left[\mathcal{J}_{\rho,\alpha,\frac{a+b}{2}+;w}^{\sigma}(fg)(b)+\mathcal{J}_{\rho,\alpha,\frac{a+b}{2}-;w}^{\sigma}(fg)(a)\right]\right|$$
$$\leq\int_{a}^{\frac{a+b}{2}}\left|\int_{a}^{t}(s-a)^{\alpha-1}\mathcal{F}_{\rho,\alpha}^{\sigma}[w(s-a)^{\rho}]g(s)ds\right||f'(t)|dt$$
$$+\int_{\frac{a+b}{2}}^{b}\left|\int_{b}^{t}(b-s)^{\alpha-1}\mathcal{F}_{\rho,\alpha}^{\sigma}[w(b-s)^{\rho}]g(s)ds\right||f'(t)|dt$$
$$\leq\frac{||g||_{[a,\frac{a+b}{2}],\infty}}{b-a}\int_{a}^{\frac{a+b}{2}}\left(\int_{a}^{t}(s-a)^{\alpha-1}\left(\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^{k}}{\Gamma(\alpha+\rho k)}(s-a)^{\rho k}\right)ds\right)$$
$$\times\left[(b-t)|f'(a)|+(t-a)|f'(b)|\right]dt$$

$$+\frac{||g||_{[\frac{a+b}{2},b],\infty}}{b-a}\int_{\frac{a+b}{2}}^{b}\left(\int_{b}^{t}(b-s)^{\alpha-1}\left(\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}(b-s)^{\rho k}\right)ds\right)$$
$$\times\left[(b-t)|f'(a)|+(t-a)|f'(b)|\right]dt$$
$$=\frac{||g||_{[a,\frac{a+b}{2}],\infty}}{b-a}\int_{a}^{\frac{a+b}{2}}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\left(\int_{a}^{t}(s-a)^{\alpha+\rho k-1}ds\right)$$
$$\times\left[(b-t)|f'(a)|+(t-a)|f'(b)|\right]dt$$
$$+\frac{||g||_{[\frac{a+b}{2},b],\infty}}{b-a}\int_{\frac{a+b}{2}}^{b}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\left(\int_{b}^{t}(b-s)^{\alpha+\rho k-1}ds\right)$$
$$\times\left[(b-t)|f'(a)|+(t-a)|f'(b)|\right]dt$$
$$=\frac{||g||_{[a,\frac{a+b}{2}],\infty}}{b-a}$$
$$\times\sum_{k=0}^{\infty}\left(\int_{a}^{\frac{a+b}{2}}(t-a)^{\alpha+\rho k}\left[(b-t)|f'(a)|+(t-a)|f'(b)|\right]dt\right)\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k+1)}$$
$$+\frac{||g||_{[\frac{a+b}{2},b],\infty}}{b-a}$$
$$\times\sum_{k=0}^{\infty}\left(\int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha+\rho k}\left[(b-t)|f'(a)|+(t-a)|f'(b)|\right]dt\right)\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k+1)}$$

$$=\frac{||g||_{[a,\frac{a+b}{2}],\infty}}{b-a}$$
$$\times\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k+1)}\left[\left(\frac{(\alpha+\rho k+3)(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+1)(\alpha+\rho k+2)}\right)|f'(a)|\right.\right.$$
$$\left.\left.+\left(\frac{(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+2)}\right)|f'(b)|\right]\right\}$$
$$+\frac{||g||_{[\frac{a+b}{2},b],\infty}}{b-a}$$
$$\times\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k+1)}\left[\left(\frac{(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+2)}\right)|f'(a)|\right.\right.$$
$$\left.\left.+\left(\frac{(\alpha+\rho k+3)(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+1)(\alpha+\rho k+2)}\right)|f'(b)|\right]\right\}$$

$$\leq \frac{||g||_{[a,b],\infty}}{b-a}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k+1)}\frac{(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+1}(\alpha+\rho k+1)}\left(|f'(a)|+|f'(b)|\right)\right\}$$
$$= ||g||_{[a,b],\infty}(b-a)^{\alpha+1}\mathcal{F}^{\sigma_1}_{\rho,\alpha+1}[w(b-a)^{\rho}](|f'(a)|+|f'(b)|),$$

where

$$\int_a^{\frac{a+b}{2}}(t-a)^{\alpha+\rho k+1}dt = \int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha+\rho k+1}dt = \frac{(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+2)},$$

$$\int_a^{\frac{a+b}{2}}(t-a)^{\alpha+\rho k}(b-t)dt = \int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha+\rho k}(t-a)dt$$
$$= \frac{(\alpha+\rho k+3)(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+1)(\alpha+\rho k+2)}.$$

This completes the proof. □

Remark 1. If we choose $\sigma(0)=1$ and $w=0$ in Theorem 4, then the inequality (2.5) reduces to (1.10).

Theorem 5. *Let $f: I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a<b$ and let $g:[a,b]\to\mathbb{R}$ is continuous. If $|f'|^q$ is convex on $[a,b]$, $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\Bigg|f\left(\frac{a+b}{2}\right)\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}g(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}g(a)\right] \tag{2.6}$$
$$-\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}(fg)(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}(fg)(a)\right]\Bigg|$$
$$\leq ||g||_{[a,b],\infty}((b-a)^{\alpha+1})^{\frac{1}{q}}\left((b-a)^{\alpha+1}\mathcal{F}^{\sigma_1}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left[\mathcal{F}^{\sigma_2}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]|f'(a)|^q+\mathcal{F}^{\sigma_3}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]|f'(b)|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\mathcal{F}^{\sigma_3}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]|f'(a)|^q+\mathcal{F}^{\sigma_2}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]|f'(b)|^q\right]^{\frac{1}{q}}\right\}$$

where

$$\sigma_1(k) := \sigma(k)\frac{1}{(\alpha+\rho k+1)2^{\alpha+\rho k+1}},$$
$$\sigma_2(k) := \sigma(k)\frac{\alpha+\rho k+3}{(\alpha+\rho k+1)(\alpha+\rho k+2)2^{\alpha+\rho k+2}}$$

and

$$\sigma_3(k) := \sigma(k)\frac{1}{(\alpha+\rho k+2)2^{\alpha+\rho k+2}}$$

with $\alpha > 0$.

Proof. Since $|f'|^q$ is convex on $[a,b]$, we know that for $t \in [a,b]$

$$|f'(t)|^q = \left|f'\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b\right)\right|^q \leq \frac{b-t}{b-a}|f'(a)|^q + \frac{t-a}{b-a}|f'(b)|^q.$$

Using Lemma 2, power-mean inequality and the convexity of $|f'|^q$, it follows that

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}g(b) + \mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}g(a)\right]\right.\\
&\left.-\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}(fg)(b) + \mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}(fg)(a)\right]\right|\\
&\leq \left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}]g(s)ds\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}]g(s)ds\right||f'(t)|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{a+b}{2}}^b\left|\int_b^t (b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)ds\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_{\frac{a+b}{2}}^b\left|\int_b^t (b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)ds\right||f'(t)|^q dt\right)^{\frac{1}{q}}\\
&\leq ||g||_{[a,\frac{a+b}{2}],\infty}\left(\int_a^{\frac{a+b}{2}}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\left|\int_a^t (s-a)^{\alpha+\rho k-1}ds\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_a^{\frac{a+b}{2}}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\left|\int_a^t (s-a)^{\alpha+\rho k-1}ds\right||f'(t)|^q dt\right)^{\frac{1}{q}}\\
&+||g||_{[\frac{a+b}{2},b],\infty}\left(\int_{\frac{a+b}{2}}^b\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\left|\int_b^t (b-s)^{\alpha+\rho k-1}ds\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_{\frac{a+b}{2}}^b\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\left|\int_b^t (b-s)^{\alpha+\rho k-1}ds\right||f'(t)|^q dt\right)^{\frac{1}{q}}\\
&\leq \left((b-a)^{\alpha+1}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\frac{(b-a)^{\rho k}}{(\alpha+\rho k)(\alpha+\rho k+1)2^{\alpha+\rho k+1}}\right)^{1-\frac{1}{q}}
\end{aligned}$$

$$\times\left\{\frac{||g||_{[a,\frac{a+b}{2}],\infty}}{(b-a)^{\frac{1}{q}}}\left(\int_a^{\frac{a+b}{2}}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)(\alpha+\rho k)}\right.\right.$$

$$\left.\times\left[(t-a)^{\alpha+\rho k}(b-t)|f'(a)|^q+(t-a)^{\alpha+\rho k+1}|f'(b)|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\frac{||g||_{[\frac{a+b}{2},b],\infty}}{(b-a)^{\frac{1}{q}}}\left(\int_{\frac{a+b}{2}}^{b}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)(\alpha+\rho k)}\right.$$

$$\left.\left.\times\left[(b-t)^{\alpha+\rho k+1}|f'(a)|^q+(b-t)^{\alpha+\rho k}(t-a)|f'(b)|^q\right]dt\right)^{\frac{1}{q}}\right\}$$

$$=\left((b-a)^{\alpha+1}\mathcal{F}^{\sigma_1}_{\rho,\alpha+1}[w(b-a)^{\rho}]\right)^{1-\frac{1}{q}}$$

$$\times\left\{\frac{||g||_{[a,\frac{a+b}{2}],\infty}}{(b-a)^{\frac{1}{q}}}\left(\sum_{k=0}^{\infty}\left(\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)(\alpha+\rho k)}\right.\right.\right.$$

$$\times\left[\frac{(\alpha+\rho k+3)(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+1)(\alpha+\rho k+2)}|f'(a)|^q\right.$$

$$\left.\left.\left.+\frac{(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+2)}|f'(b)|^q\right]\right)\right)^{\frac{1}{q}}$$

$$+\frac{||g||_{[\frac{a+b}{2},b],\infty}}{(b-a)^{\frac{1}{q}}}\left(\sum_{k=0}^{\infty}\left(\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)(\alpha+\rho k)}\right.\right.$$

$$\times\left[\frac{(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+2)}|f'(a)|^q\right.$$

$$\left.\left.\left.\left.+\frac{(\alpha+\rho k+3)(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+1)(\alpha+\rho k+2)}|f'(b)|^q\right]\right)\right)^{\frac{1}{q}}\right\}$$

$$\leq ||g||_{[a,b],\infty}\left((b-a)^{\alpha+1}\right)^{\frac{1}{q}}\left((b-a)^{\alpha+1}\mathcal{F}^{\sigma_1}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left[\mathcal{F}^{\sigma_2}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]|f'(a)|^q+\mathcal{F}^{\sigma_3}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]|f'(b)|^q\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\mathcal{F}^{\sigma_3}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]|f'(a)|^q+\mathcal{F}^{\sigma_2}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]|f'(b)|^q\right]^{\frac{1}{q}}\right\},$$

where it is easily seen that

$$\int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha+\rho k-1} ds \right| dt = \int_{\frac{a+b}{2}}^b \left| \int_b^t (b-s)^{\alpha+\rho k-1} ds \right| dt$$
$$= \frac{(b-a)^{\alpha+\rho k+1}}{2^{\alpha+\rho k+1}(\alpha+\rho k)(\alpha+\rho k+1)},$$

$$\int_a^{\frac{a+b}{2}} (t-a)^{\alpha+\rho k}(b-t)dt = \int_{\frac{a+b}{2}}^b (b-t)^{\alpha+\rho k}(t-a)dt$$
$$= \frac{(\alpha+\rho k+3)(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+1)(\alpha+\rho k+2)}.$$

Hence the proof is completed. □

Remark 2. If we choose $\sigma(0) = 1$ and $w = 0$ in Theorem 5, then the inequality (2.6) reduces to (1.11).

Theorem 6. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$ and let $g : [a,b] \to \mathbb{R}$ is continuous. If $|f'|^q$ is convex on $[a,b]$, $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\left| f\left(\frac{a+b}{2}\right)\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}g(b) + \mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}g(a)\right] \right. \tag{2.7}$$
$$\left. -\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}(fg)(b) + \mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}(fg)(a)\right]\right|$$
$$\leq \frac{(b-a)^{\alpha+1}||g||_{[a,b]\infty}}{2^{\frac{3}{q}}}\left(\mathcal{F}^{\sigma_1}_{\rho,\alpha+1}[|w|(b-a)^\rho]\right)^{\frac{1}{p}}$$
$$\times\left\{[3|f'(a)|^q + |f'(b)|^q]^{\frac{1}{q}} + [|f'(a)|^q + 3|f'(b)|^q]^{\frac{1}{q}}\right\}$$

where

$$\sigma_1(k) := \sigma(k)\frac{1}{2^{\alpha+\rho k+\frac{1}{p}}(\alpha p+\rho k p+1)^{\frac{1}{p}}}$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Lemma 2, Hölder's inequality and the convexity of $|f'|^q$, it follows that

$$\left| f\left(\frac{a+b}{2}\right)\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}g(b) + \mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}g(a)\right] \right.$$
$$\left. -\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}(fg)(b) + \mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}(fg)(a)\right]\right|$$

$$
\begin{aligned}
&\leq \left| \int_a^{\frac{a+b}{2}} \left(\int_a^t (s-a)^{\alpha-1} \mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}] g(s) ds \right) f'(t) dt \right. \\
&\left. + \int_{\frac{a+b}{2}}^b \left(\int_b^t (b-s)^{\alpha-1} \mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}] g(s) ds \right) f'(t) dt \right| \\
&\leq \int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1} \mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}] g(s) ds \right| |f'(t)| dt \\
&+ \int_{\frac{a+b}{2}}^b \left| \int_b^t (b-s)^{\alpha-1} \mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}] g(s) ds \right| |f'(t)| dt \\
&= \int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1} \sum_{k=0}^{\infty} \frac{\sigma(k) w^k (s-a)^{\rho k}}{\Gamma(\alpha+\rho k)} g(s) ds \right| |f'(t)| dt \\
&+ \int_{\frac{a+b}{2}}^b \left| \int_b^t (b-s)^{\alpha-1} \sum_{k=0}^{\infty} \frac{\sigma(k) w^k (b-s)^{\rho k}}{\Gamma(\alpha+\rho k)} g(s) ds \right| |f'(t)| dt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_a^{\frac{a+b}{2}} \left| \sum_{k=0}^{\infty} \frac{\sigma(k) w^k}{\Gamma(\alpha+\rho k)} \int_a^t (s-a)^{\alpha+\rho k-1} g(s) ds \right| |f'(t)| dt \\
&+ \int_{\frac{a+b}{2}}^b \left| \sum_{k=0}^{\infty} \frac{\sigma(k) w^k}{\Gamma(\alpha+\rho k)} \int_b^t (b-s)^{\alpha+\rho k-1} g(s) ds \right| |f'(t)| dt \\
&\leq \int_a^{\frac{a+b}{2}} \left(\sum_{k=0}^{\infty} \frac{\sigma(k) |w|^k}{\Gamma(\alpha+\rho k)} \left| \int_a^t (s-a)^{\alpha+\rho k-1} g(s) ds \right| \right) |f'(t)| dt \\
&+ \int_{\frac{a+b}{2}}^b \left(\sum_{k=0}^{\infty} \frac{\sigma(k) |w|^k}{\Gamma(\alpha+\rho k)} \left| \int_b^t (b-s)^{\alpha+\rho k-1} g(s) ds \right| \right) |f'(t)| dt \\
&\leq ||g||_{[a,\frac{a+b}{2}],\infty} \sum_{k=0}^{\infty} \frac{\sigma(k) |w|^k}{\Gamma(\alpha+\rho k)} \left(\int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha+\rho k-1} ds \right| |f'(t)| dt \right) \\
&+ ||g||_{[\frac{a+b}{2},b],\infty} \sum_{k=0}^{\infty} \frac{\sigma(k) |w|^k}{\Gamma(\alpha+\rho k)} \left(\int_{\frac{a+b}{2}}^b \left| \int_b^t (b-s)^{\alpha+\rho k-1} ds \right| |f'(t)| dt \right) \\
&\leq ||g||_{[a,b],\infty} \left\{ \sum_{k=0}^{\infty} \frac{\sigma(k) |w|^k}{\Gamma(\alpha+\rho k)} \left(\int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha+\rho k-1} ds \right|^p dt \right)^{\frac{1}{p}} \left(\int_a^{\frac{a+b}{2}} |f'(t)|^q dt \right)^{\frac{1}{q}} \right. \\
&\left. + \sum_{k=0}^{\infty} \frac{\sigma(k) |w|^k}{\Gamma(\alpha+\rho k)} \left(\int_{\frac{a+b}{2}}^b \left| \int_b^t (b-s)^{\alpha+\rho k-1} ds \right|^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{a+b}{2}}^b |f'(t)|^q dt \right)^{\frac{1}{q}} \right\}
\end{aligned}
$$

$$= ||g||_{[a,b],\infty}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\left(\int_a^{\frac{a+b}{2}}\left|\frac{(s-a)^{\alpha+\rho k}}{(\alpha+\rho k)}\Big|_a^t\right|^p dt\right)^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}|f'(t)|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\left(\int_{\frac{a+b}{2}}^{b}\left|\frac{(b-s)^{\alpha+\rho k}}{(\alpha+\rho k)}\Big|_b^t\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^{b}|f'(t)|^q dt\right)^{\frac{1}{q}}\right\}$$

$$= ||g||_{[a,b],\infty}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\left(\int_a^{\frac{a+b}{2}}\frac{(t-a)^{(\alpha+\rho k)p}}{(\alpha+\rho k)^p}dt\right)^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}|f'(t)|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\left(\int_{\frac{a+b}{2}}^{b}\frac{(b-t)^{(\alpha+\rho k)p}}{(\alpha+\rho k)^p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^{b}|f'(t)|^q dt\right)^{\frac{1}{q}}\right\}$$

$$= ||g||_{[a,b],\infty}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k+1)}\left(\frac{(t-a)^{\alpha p+\rho kp+1}}{\alpha p+\rho kp+1}\Big|_a^{\frac{a+b}{2}}\right)^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}|f'(t)|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k+1)}\left(-\frac{(b-t)^{\alpha p+\rho kp+1}}{\alpha p+\rho kp+1}\Big|_{\frac{a+b}{2}}^{b}\right)^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^{b}|f'(t)|^q dt\right)^{\frac{1}{q}}\right\}$$

$$= ||g||_{[a,b],\infty}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k+1)}\left(\frac{(b-a)^{\alpha p+\rho kp+1}}{2^{\alpha p+\rho kp+1}(\alpha p+\rho kp+1)}\right)^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}|f'(t)|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k+1)}\left(\frac{(b-a)^{\alpha p+\rho kp+1}}{2^{\alpha p+\rho kp+1}(\alpha p+\rho kp+1)}\right)^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^{b}|f'(t)|^q dt\right)^{\frac{1}{q}}\right\}$$

$$= ||g||_{[a,b],\infty}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k+1)}\left(\frac{(b-a)^{\alpha+\rho k+\frac{1}{p}}}{2^{\alpha+\rho k+\frac{1}{p}}(\alpha p+\rho kp+1)^{\frac{1}{p}}}\right)\left(\int_a^{\frac{a+b}{2}}|f'(t)|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k+1)}\left(\frac{(b-a)^{\alpha+\rho k+\frac{1}{p}}}{2^{\alpha+\rho k+\frac{1}{p}}(\alpha p+\rho kp+1)^{\frac{1}{p}}}\right)\left(\int_{\frac{a+b}{2}}^{b}|f'(t)|^q dt\right)^{\frac{1}{q}}\right\}$$

$$\leq ||g||_{[a,b],\infty}(b-a)^{\alpha+1}\mathscr{F}^{\sigma_1}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]$$

$$\times\left\{\left(\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{\frac{1}{q}}\right\}.$$

Here we use

$$\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha+\rho k-1}ds\right|^p dt=\int_{\frac{a+b}{2}}^{b}\left|\int_b^t (b-s)^{\alpha+\rho k-1}ds\right|^p dt$$

$$=\frac{(b-a)^{\alpha p+\rho kp+1}}{2^{\alpha p+\rho kp+1}(\alpha+\rho k)^p(\alpha p+\rho kp+1)},$$

$$\int_a^{\frac{a+b}{2}} |f'(t)|^q dt \le \frac{1}{b-a}\int_a^{\frac{a+b}{2}} \left[(b-t)|f'(a)|^q + (t-a)|f'(b)|^q\right] dt$$
$$= (b-a)\frac{3|f'(a)|^q + |f'(b)|^q}{8}$$

and

$$\int_{\frac{a+b}{2}}^{b} |f'(t)|^q dt \le \frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} \left[(b-t)|f'(a)|^q + (t-a)|f'(b)|^q\right] dt$$
$$= (b-a)\frac{|f'(a)|^q + 3|f'(b)|^q}{8}.$$

Hence the inequality (2.7) is proved. □

Remark 3. If we choose $\sigma(0) = 1$ and $w = 0$ in Theorem 6, then the inequality (2.7) reduces to (1.12).

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