Miskolc Mathematical Notes
Vol. 19 (2018), No. 1, pp. 385–396
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2018.1611

MORE ON LIE DERIVATIONS OF A GENERALIZED MATRIX ALGEBRA

A.H. MOKHTARI AND H.R. EBRAHIMI VISHKI

Received 07 March, 2017

Abstract. Motivated by the elaborate work of Cheung [Linear Multilinear Algebra, **51** (2003), 299-310], we apply the construction of a Lie derivation on a generalized matrix algebra to give a characterization for a Lie derivation on a generalized matrix algebra to be proper. Our approach not only provides a direct proof for some known results in the theory, but also it presents several sufficient conditions assuring the properness of Lie derivations on certain generalized matrix algebras.

2010 *Mathematics Subject Classification:* 16W25; 15A78; 47B47

Keywords: Derivation, Lie derivation, generalized matrix algebra, triangular algebra

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ be a unital algebra (over a commutative unital ring $\mathbf{R}$) and $\mathcal{M}$ be an $\mathcal{A}-$module. A linear mapping $\mathcal{D} : \mathcal{A} \to \mathcal{M}$ is said to be a derivation if

$$\mathcal{D}(ab) = \mathcal{D}(a)b + a\mathcal{D}(b) \qquad (a, b \in \mathcal{A}).$$

A linear mapping $\mathcal{L} : \mathcal{A} \to \mathcal{M}$ is called a Lie derivation if

$$\mathcal{L}([a,b]) = [\mathcal{L}(a), b] + [a, \mathcal{L}(b)] \qquad (a, b \in \mathcal{A}),$$

where $[a,b] = ab - ba$. Every derivation is trivially a Lie derivation. If $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ is a derivation and $\tau : \mathcal{A} \to Z(\mathcal{A})(:=$ the center of $\mathcal{A})$ is a linear map then $\mathcal{D} + \tau$ is a Lie derivation if and only if τ vanishes at commutators (i.e. $\tau([a,b]) = 0$, for all $a, b \in \mathcal{A}$). Lie derivations of this form are called proper Lie derivations. Therefore a Lie derivation $\mathcal{L}$ is proper if and only if $\mathcal{L} = \mathcal{D} + \tau$ for some derivation $\mathcal{D}$ and a linear center valued map τ on $\mathcal{A}$ (i.e. $\tau(\mathcal{A}) \subseteq Z(\mathcal{A})$). The fundamental question is that under what conditions a Lie derivation on an algebra is proper. We say that an algebra $\mathcal{A}$ has Lie derivation property if every Lie derivation from $\mathcal{A}$ into itself is proper.

In this paper, we study the Lie derivation property for the general matrix algebras. First we briefly introduce a general matrix algebra. A Morita context $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \Phi_{\mathcal{M}\mathcal{N}}, \Psi_{\mathcal{N}\mathcal{M}})$ consists of two unital algebras $\mathcal{A}$, $\mathcal{B}$, an $(\mathcal{A}, \mathcal{B})-$module $\mathcal{M}$, a $(\mathcal{B},\mathcal{A})-$module $\mathcal{N}$, and two module homomorphisms $\Phi_{\mathcal{M}\mathcal{N}}:\mathcal{M}\otimes_{\mathcal{B}}\mathcal{N}\longrightarrow\mathcal{A}$ and $\Psi_{\mathcal{N}\mathcal{M}}:\mathcal{N}\otimes_{\mathcal{A}}\mathcal{M}\longrightarrow\mathcal{B}$ satisfying the following commutative diagrams:

$$\begin{array}{ccc}\mathcal{M}\otimes_{\mathcal{B}}\mathcal{N}\otimes_{\mathcal{A}}\mathcal{M} & \xrightarrow{\Phi_{\mathcal{M}\mathcal{N}}\otimes I_{\mathcal{M}}} & \mathcal{A}\otimes_{\mathcal{A}}\mathcal{M}\\ \Big\downarrow I_{\mathcal{M}}\otimes\Psi_{\mathcal{N}\mathcal{M}} & & \Big\downarrow\cong\\ \mathcal{M}\otimes_{\mathcal{B}}\mathcal{B} & \xrightarrow{\cong} & \mathcal{M}\end{array}$$

and

$$\begin{array}{ccc}\mathcal{N}\otimes_{\mathcal{A}}\mathcal{M}\otimes_{\mathcal{B}}\mathcal{N} & \xrightarrow{\Psi_{\mathcal{N}\mathcal{M}}\otimes I_{\mathcal{N}}} & \mathcal{B}\otimes_{\mathcal{B}}\mathcal{N}\\ \Big\downarrow I_{\mathcal{N}}\otimes\Phi_{\mathcal{M}\mathcal{N}} & & \Big\downarrow\cong\\ \mathcal{N}\otimes_{\mathcal{A}}\mathcal{A} & \xrightarrow{\cong} & \mathcal{N}.\end{array}$$

For a Morita context $(\mathcal{A},\mathcal{B},\mathcal{M},\mathcal{N},\Phi_{\mathcal{M}\mathcal{N}},\Psi_{\mathcal{N}\mathcal{M}})$, the set

$$\mathcal{G}=\begin{pmatrix}\mathcal{A} & \mathcal{M}\\ \mathcal{N} & \mathcal{B}\end{pmatrix}=\left\{\begin{pmatrix}a & m\\ n & b\end{pmatrix}\Big| a\in\mathcal{A}, m\in\mathcal{M}, n\in\mathcal{N}, b\in\mathcal{B}\right\}$$

forms an algebra under the usual matrix operations, where at least one of two modules $\mathcal{M}$ and $\mathcal{N}$ is distinct from zero. The algebra $\mathcal{G}$ is called a generalized matrix algebra. In the case where $\mathcal{N}=0$, $\mathcal{G}$ becomes the so-called triangular algebra $\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$ whose Lie derivations are extensively investigated by Cheung [4]. Generalized matrix algebras were first introduced by Sands [12], where he studied various radicals of algebras occurring in Morita contexts.

A direct verification reveals that the center $Z(\mathcal{G})$ of $\mathcal{G}$ is

$$Z(\mathcal{G})=\{a\oplus b|a\in Z(\mathcal{A}), b\in Z(\mathcal{B}),\ am=mb, na=bn \quad \text{for all} \quad m\in\mathcal{M}, n\in\mathcal{N}\},$$

where $a\oplus b=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\in\mathcal{G}$. We also define two natural projections $\pi_{\mathcal{A}}:\mathcal{G}\longrightarrow\mathcal{A}$ and $\pi_{\mathcal{B}}:\mathcal{G}\longrightarrow\mathcal{B}$ by

$$\pi_{\mathcal{A}}:\begin{pmatrix}a & m\\ n & b\end{pmatrix}\mapsto a \quad \text{and} \quad \pi_{\mathcal{B}}:\begin{pmatrix}a & m\\ n & b\end{pmatrix}\mapsto b.$$

It is clear that, $\pi_{\mathcal{A}}(Z(\mathcal{G}))\subseteq Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathcal{G}))\subseteq Z(\mathcal{B})$. Further, similar to [4, Proposition 3] one can show that, if $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})-$module (i.e. $a\mathcal{M}=\{0\}$ necessities $a=0$ and $\mathcal{M}b=\{0\}$ necessities $b=0$) then there exists a unique algebra isomorphism

$$\varphi:\pi_{\mathcal{A}}(Z(\mathcal{G}))\longrightarrow\pi_{\mathcal{B}}(Z(\mathcal{G}))$$

such that $am=m\varphi(a)$ and $\varphi(a)n=na$ for all $m\in\mathcal{M}$, $n\in\mathcal{N}$; or equivalently, $a\oplus\varphi(a)\in Z(\mathcal{G})$ for all $a\in\mathcal{A}$.

Martindale [9] was the first one who investigated the Lie derivation property of certain primitive rings. Cheung [3] initiated the study of various mappings on triangular algebras; in particular, he studied the Lie derivation property for triangular algebras in [4]. Following his seminal work [4], Lie derivations on a wide variety of algebras have been studied by many authors (see [1, 2, 5–8, 10, 11, 13, 14] and the references therein). The main result of Cheung [4] has recently extended by Du and Wang [5] for a generalized matrix algebra. They have shown that [5, Theorem 1] a general matrix algebra $\mathcal{G}$ with $\mathcal{M}$ faithful enjoys the Lie derivation property if $\pi_{\mathcal{A}}(Z(\mathcal{G}))=Z(\mathcal{A}),\pi_{\mathcal{B}}(Z(\mathcal{G}))=Z(\mathcal{B})$, and either $\mathcal{A}$ or $\mathcal{B}$ does not contain nonzero central ideals. This result has been developed for Lie $n-$derivation in [13, Theorem 1], (see also [14, Theorem 2.1]).

In this paper, we use the construction of Lie derivations of a generalized matrix algebra $\mathcal{G}$ (Proposition 1) to give a criterion for properness of Lie derivations on $\mathcal{G}$ (see Theorem 1 and Corollary 1). We then deduce not only, as a byproduct, the aforementioned result of Du and Wang (Theorem 2), but also we provide some alternative sufficient conditions ensuring the Lie derivation property for $\mathcal{G}$ (Theorems 3, 4). We then come to our main result, Theorem 5, collecting some sufficient conditions assuring the Lie derivation property for a generalized matrix algebra. In the last section, we include some applications of our results to some special generalized matrix algebras such as: triangular algebras, the full matrix algebras and the algebras of operators on a Banach space.

2. Proper Lie derivations

From now on, we assume that the modules $\mathcal{M}$ and $\mathcal{N}$ appeared in the generalized matrix algebra $\mathcal{G}$ are 2−torion free; (M is called 2−torsion free if $2m=0$ implies $m=0$ for any $m\in\mathcal{M}$).

We commence with the following result providing the construction of (Lie) derivations of a generalized matrix algebra. It needs a standard argument, however Li and Wei [6, Propositions 4.1, 4.2] gave a complete proof for the first two parts of it.

Proposition 1. *Let* $\mathcal{G}=\begin{pmatrix}\mathcal{A} & \mathcal{M}\\ \mathcal{N} & \mathcal{B}\end{pmatrix}$ *be a generalized matrix algebra. Then*

• *A linear mapping* $\mathcal{L}$ *on* $\mathcal{G}$ *is a Lie derivation if and only if it has the presentation*

$$\mathcal{L}\begin{pmatrix}a & m\\ n & b\end{pmatrix}=\begin{pmatrix}P(a)-mn_0-m_0n+h_{\mathcal{B}}(b) & am_0-m_0b+f(m)\\ n_0a-bn_0+g(n) & h_{\mathcal{A}}(a)+n_0m+nm_0+Q(b)\end{pmatrix}, \tag{2.1}$$

for some $m_0\in\mathcal{M},n_0\in\mathcal{N}$ *and some linear maps* $P:\mathcal{A}\longrightarrow\mathcal{A},Q:\mathcal{B}\longrightarrow\mathcal{B},f:\mathcal{M}\longrightarrow\mathcal{M},g:\mathcal{N}\longrightarrow\mathcal{N},h_{\mathcal{B}}:\mathcal{B}\longrightarrow Z(\mathcal{A})$ *and* $h_{\mathcal{A}}:\mathcal{A}\longrightarrow Z(\mathcal{B})$ *satisfying the following conditions:*

(a) P,Q *are Lie derivations;*
(b) $h_{\mathcal{B}}([b,b'])=0,\ h_{\mathcal{A}}([a,a'])=0$*;*
(c) $f(am)=P(a)m-mh_{\mathcal{A}}(a)+af(m),\ f(mb)=mQ(b)-h_{\mathcal{B}}(b)m+f(m)b$*;*

(d) $g(na) = nP(a) - h_{\mathcal{A}}(a)n + g(n)a,\ g(bn) = Q(b)n - nh_{\mathcal{B}}(b) + bg(n)$;
(e) $P(mn) - h_{\mathcal{B}}(nm) = mg(n) + f(m)n,\ Q(nm) - h_{\mathcal{A}}(mn) = g(n)m + nf(m)$;

for all $a, a' \in \mathcal{A}, b, b' \in \mathcal{B}, m \in \mathcal{M}$ *and* $n \in \mathcal{N}$.

• *A linear mapping* $\mathcal{D}$ *on* $\mathcal{G}$ *is a derivation if and only if it has the presentation*

$$\mathcal{D}\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} P'(a) - mn_0 - m_0n & am_0 - m_0b + f'(m) \\ n_0a - bn_0 + g'(n) & n_0m + nm_0 + Q'(b) \end{pmatrix}, \tag{2.2}$$

where $m_0 \in \mathcal{M}, n_0 \in \mathcal{N}, P' : \mathcal{A} \longrightarrow \mathcal{A}, Q' : \mathcal{B} \longrightarrow \mathcal{B}, f' : \mathcal{M} \longrightarrow \mathcal{M}$ *and* $g' : \mathcal{N} \longrightarrow \mathcal{N}$ *are linear maps satisfying the following conditions:*

(a$'$) P', Q' *are derivations;*
(b$'$) $f'(am) = P'(a)m + af'(m),\ f'(mb) = mQ'(b) + f'(m)b$;
(c$'$) $g'(na) = nP'(a) + g'(n)a,\ g'(bn) = Q'(b)n + bg'(n)$;
(d$'$) $P'(mn) = mg'(n) + f'(m)n,\ Q'(nm) = g'(n)m + nf'(m)$;

for all $a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{M}$ *and* $n \in \mathcal{N}$.

• *A linear mapping* τ *on* $\mathcal{G}$ *is center valued and vanishes at commutators (i.e.* $\tau(\mathcal{G}) \subseteq \mathcal{G}$ *and* $\tau([\mathcal{G}, \mathcal{G}]) = 0$*) if and only if* τ *has the presentation*

$$\tau\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} \ell_{\mathcal{A}}(a) + h_{\mathcal{B}}(b) & 0 \\ 0 & h_{\mathcal{A}}(a) + \ell_{\mathcal{B}}(b) \end{pmatrix}, \tag{2.3}$$

where $\ell_{\mathcal{A}} : A \longrightarrow Z(A), h_{\mathcal{B}} : B \longrightarrow Z(A), h_{\mathcal{A}} : A \longrightarrow Z(B), \ell_{\mathcal{B}} : B \longrightarrow Z(B)$ *are linear maps vanishing at commutators, having the following properties:*

(a'') $\ell_{\mathcal{A}}(a) \oplus h_{\mathcal{A}}(a) \in Z(\mathcal{G})$ *and* $h_{\mathcal{B}}(b) \oplus \ell_{\mathcal{B}}(b) \in Z(\mathcal{G})$*, for all* $a \in \mathcal{A}, b \in \mathcal{B}$;
(b'') $\ell_{\mathcal{A}}(mn) = h_{\mathcal{B}}(nm)$ *and* $h_{\mathcal{A}}(mn) = \ell_{\mathcal{B}}(nm)$*, for all* $m \in \mathcal{M}, n \in \mathcal{N}$.

Following the method of Cheung [4, Theorem 6] in the next theorem, we give a necessary and sufficient condition for a Lie derivation on a generalized matrix algebra $\mathcal{G}$ to be proper.

Theorem 1. *Let* $\mathcal{G}$ *be a generalized matrix algebra. A Lie derivation* $\mathcal{L}$ *on* $\mathcal{G}$ *of the form presented in* (2.1) *is proper if and only if there exist linear mappings* $\ell_{\mathcal{A}} : \mathcal{A} \longrightarrow Z(\mathcal{A})$ *and* $\ell_{\mathcal{B}} : \mathcal{B} \longrightarrow Z(\mathcal{B})$ *satisfying the following conditions:*

(A) $P - \ell_{\mathcal{A}}$ *and* $Q - \ell_{\mathcal{B}}$ *are derivations on* $\mathcal{A}$ *and* $\mathcal{B}$*, respectively;*
(B) $\ell_{\mathcal{A}}(a) \oplus h_{\mathcal{A}}(a) \in Z(\mathcal{G})$ *and* $h_{\mathcal{B}}(b) \oplus \ell_{\mathcal{B}}(b) \in Z(\mathcal{G})$*, for all* $a \in \mathcal{A}, b \in \mathcal{B}$;
(C) $\ell_{\mathcal{A}}(mn) = h_{\mathcal{B}}(nm)$ *and* $\ell_{\mathcal{B}}(nm) = h_{\mathcal{A}}(mn)$*, for all* $m \in \mathcal{M}, n \in \mathcal{N}$.

Proof. Employing the characterization of $\mathcal{L}$ as in Proposition 1, for the sufficiency, we define $\mathcal{D}$ and τ by

$$\mathcal{D}\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} (P - \ell_{\mathcal{A}})(a) - mn_0 - m_0n & am_0 - m_0b + f(m) \\ n_0a - bn_0 + g(n) & n_0m + nm_0 + (Q - \ell_{\mathcal{B}})(b) \end{pmatrix}$$

and

$$\tau\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} \ell_{\mathcal{A}}(a) + h_{\mathcal{B}}(b) & 0 \\ 0 & h_{\mathcal{A}}(a) + \ell_{\mathcal{B}}(b) \end{pmatrix}.$$

Then a direct verification reveals that $\mathcal{D}$ is a derivation, τ is center valued and $\mathcal{L} = \mathcal{D} + \tau$.

For the converse, suppose that $\mathcal{L}$ is of the form $\mathcal{D} + \tau$, where $\mathcal{D}$ is a derivation and τ maps into $Z(\mathcal{G})$. Applying the presentations (2.1), (2.2) for $\mathcal{L}$ and $\mathcal{D}$, respectively, we get $\tau = \mathcal{L} - \mathcal{D}$ as

$$\tau\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} (P-P')(a) + h_{\mathcal{B}}(b) & 0 \\ 0 & h_{\mathcal{A}}(a) + (Q-Q')(b) \end{pmatrix}.$$

By setting $\ell_{\mathcal{A}} = P - P', \ell_{\mathcal{B}} = Q - Q'$, a direct verification shows that $\ell_{\mathcal{A}}, \ell_{\mathcal{B}}$ are our desired maps satisfying the required conditions. □

Remark 1. It is worthwhile mentioning that in the case where $\mathcal{M}$ is faithful as an $(\mathcal{A}, \mathcal{B})-$module then;

(i) In Proposition 1, the condition (b) becomes redundant as it can be followed by (a) and (c). Indeed, for $a, a' \in \mathcal{A}, m \in \mathcal{M}$, from (c) we get

$$f([a,a']m) = P([a,a'])m - mh_{\mathcal{A}}([a,a']) + [a,a']f(m). \tag{2.4}$$

On the other hand, employing (c) and then (a), we have

$$\begin{aligned} f([a,a']m) &= f(aa'm - a'am) \\ &= P(a)a'm - a'mh_{\mathcal{A}}(a) + af(a'm) - \big(P(a')am - amh_{\mathcal{A}}(a') + a'f(am)\big) \\ &= P(a)a'm - a'mh_{\mathcal{A}}(a) + a(P(a')m - mh_{\mathcal{A}}(a') + a'f(m)) \\ &\quad - \big(P(a')am - amh_{\mathcal{A}}(a') + a'(P(a)m - mh_{\mathcal{A}}(a) + af(m))\big) \\ &= [P(a),a']m + [a,P(a')]m + [a,a']f(m) \\ &= P([a,a'])m + [a,a']f(m). \end{aligned} \tag{2.5}$$

Comparing the equations (2.4) and (2.5) reveals that $mh_{\mathcal{A}}([a,a']) = 0$, for every $m \in \mathcal{M}$, and the faithfulness of $\mathcal{M}$ (as a right $\mathcal{B}-$module) implies that $h_{\mathcal{A}}([a,a']) = 0$, as claimed.

(ii) In Proposition 1, the condition (a$'$) can also be dropped as it can be derived from (b$'$) by a similar argument as in (i), (see [4, Page 303]).

(iii) In Theorem 1, the same reason as in (ii) shows that the condition (A) in Theorem 1, stating that $P - \ell_{\mathcal{A}}$ and $Q - \ell_{\mathcal{B}}$ are derivations, is superfluous.

As a consequence of Theorem 1, we have the following criterion characterizing Lie derivation property for a generalized matrix algebra $\mathcal{G}$.

Corollary 1. *Let $\mathcal{G}$ be a generalized matrix algebra and let $\mathcal{L}$ be a Lie derivation on $\mathcal{G}$ of the form presented in (2.1). If $\mathcal{L}$ is proper, then*

(A$'$) $h_{\mathcal{A}}(\mathcal{A}) \subseteq \pi_{\mathcal{B}}(Z(\mathcal{G}))$, $h_{\mathcal{B}}(\mathcal{B}) \subseteq \pi_{\mathcal{A}}(Z(\mathcal{G}))$, *and*

(B$'$) $h_{\mathcal{B}}(nm) \oplus h_{\mathcal{A}}(mn) \in Z(\mathcal{G})$, *for all* $m \in \mathcal{M}, n \in \mathcal{N}$.

The converse also holds in the case when $\mathcal{M}$ is faithful.

Proof. The necessity follows trivially from Theorem 1. For the sufficiency suppose that $\mathcal{M}$ is faithful. Let $\varphi : \pi_{\mathcal{A}}(Z(\mathcal{G})) \longrightarrow \pi_{\mathcal{B}}(Z(\mathcal{G}))$ be the isomorphism satisfying $a \oplus \varphi(a) \in Z(\mathcal{G})$ for all $a \in \mathcal{A}$, whose existence guaranteed by the faithfulness of $\mathcal{M}$. By virtue of (A′), we define $\ell_{\mathcal{A}} : \mathcal{A} \longrightarrow Z(\mathcal{A})$ and $\ell_{\mathcal{B}} : \mathcal{B} \longrightarrow Z(\mathcal{B})$ by $\ell_{\mathcal{A}} = \varphi^{-1} \circ h_{\mathcal{A}}$ and $\ell_{\mathcal{B}} = \varphi \circ h_{\mathcal{B}}$. It is now obvious that $\ell_{\mathcal{A}}(a) \oplus h_{\mathcal{A}}(a) \in Z(\mathcal{G})$ and $h_{\mathcal{B}}(b) \oplus \ell_{\mathcal{B}}(b) \in Z(\mathcal{G})$, for all $a \in \mathcal{A}, b \in \mathcal{B}$. Further, (B′) follows that

$$\ell_{\mathcal{A}}(mn) = \varphi^{-1}(h_{\mathcal{A}}(mn)) = h_{\mathcal{B}}(nm) \text{ and } \ell_{\mathcal{B}}(nm) = \varphi(h_{\mathcal{B}}(nm)) = h_{\mathcal{A}}(mn).$$

Now Theorem 1 together with Remark 1 (iii) confirm that $\mathcal{L}$ is proper, as required. □

3. Some sufficient conditions

We commence with the following result which employs Corollary 1 to give a proof for a modification of the main result of Du and Wang [5]. See also [13, Corollary 1] and examine [14, Theorem 2.1] for $n = 2$.

Theorem 2 ([5, Theorem 1])**.** *Let $\mathcal{G}$ be a generalized matrix algebra with faithful $\mathcal{M}$. Then $\mathcal{G}$ has Lie derivation property if*

(i) $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A}), \pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$, *and*
(ii) *either $\mathcal{A}$ or $\mathcal{B}$ does not contain nonzero central ideals.*

Proof. Let $\mathcal{L}$ be a Lie derivation of the form presented in (2.1). From Corollary 1, as (i) implies (A′), we only need to show that $h_{\mathcal{B}}(nm) \oplus h_{\mathcal{A}}(mn) \in Z(\mathcal{G})$ for all $m \in \mathcal{M}, n \in \mathcal{N}$. Without loss of generality suppose that $\mathcal{A}$ has no nonzero central ideals. Set

$$\gamma(a,b) = \ell_{\mathcal{A}}(a) + h_{\mathcal{B}}(b) \quad (a \in \mathcal{A}, b \in \mathcal{B}),$$

where, as in the proof of Corollary 1, $\ell_{\mathcal{A}} = \varphi^{-1} \circ h_{\mathcal{A}}$. Then $p_{\mathcal{A}} = P - \ell_{\mathcal{A}}$ is a derivation. Now Proposition 1 (e) implies that

$$p_{\mathcal{A}}(mn) = mg(n) + f(m)n - \gamma(mn, -nm) \quad (m \in \mathcal{M}, n \in \mathcal{N}),$$

so for each $a \in \mathcal{A}$, $p_{\mathcal{A}}(amn) - amg(n) - f(am)n = -\gamma(amn, -nam)$. By the latter identity and the fact that $p_{\mathcal{A}}$ is a derivation, we get

$$p_{\mathcal{A}}(a)mn + ap_{\mathcal{A}}(mn) - amg(n) - p_{\mathcal{A}}(a)mn - af(m)n = -\gamma(amn, -nam).$$

These relations follow that $a\gamma(mn, -nm) = \gamma(amn, -nam)$, for $a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}$.

Therefore for any two elements $m \in \mathcal{M}$ and $n \in \mathcal{N}$, the set $\mathcal{A}\gamma(mn, -nm)$ is a central ideal of $\mathcal{A}$. Thus $\ell_{\mathcal{A}}(mn) - h_{\mathcal{B}}(nm) = \gamma(mn, -nm) = 0$ and so $h_{\mathcal{B}}(nm) \oplus h_{\mathcal{A}}(mn) = \ell_{\mathcal{A}}(mn) \oplus h_{\mathcal{A}}(mn) \in Z(\mathcal{G})$, as claimed. □

For an algebra $\mathcal{A}$ satisfying condition (ii) of Theorem 2, one can directly show that every noncommutative unital prime algebra with a nontrivial idempotent does not contain nonzero central ideals. In particular, B(X), the algebra of operators on a Banach space X when the dimension X is grater that 1, does not contain central ideal. The full matrix matrix algebra $M_n(A)$, $n \geq 2$, has also no nonzero central ideal, [5, Lemma 1]. In [5, Theorem 2], they also showed that in a generalized matrix algebra $\mathcal{G}$ with loyal $\mathcal{M}$ if $\mathcal{A}$ is noncommutative, then $\mathcal{A}$ has no central ideals.

In the next result, we provide some new sufficient conditions assuring the Lie derivation property for $\mathcal{G}$. We say that an algebra $\mathcal{A}$ is a domain if it has no zero devisors if $aa' = 0$ implies $a = 0$ or $a' = 0$ for any two elements $a, a' \in \mathcal{A}$.

Theorem 3. *Let $\mathcal{G}$ be a generalized matrix algebra with faithful $\mathcal{M}$. Then $\mathcal{G}$ has Lie derivation property if*

(i) $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A}), \pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$*, and*

(ii) *$\mathcal{A}$ and $\mathcal{B}$ are domain.*

Proof. Let $\mathcal{L}$ be a Lie derivation of the form presented in (2.1). From Corollary 1, we only need to show that $h_{\mathcal{B}}(nm) \oplus h_{\mathcal{A}}(mn) \in Z(\mathcal{G})$.

Let $m \in \mathcal{M}, n \in \mathcal{N}$. Using the identities in (c) of Proposition 1 for $a = mn$ and $b = nm$, we get

$$P(mn)m - mh_{\mathcal{A}}(mn) + mnf(m) = mQ(nm) - h_{\mathcal{B}}(nm)m + f(m)nm. \tag{3.1}$$

Multiplying by m the identities in (e) of Proposition 1, we get

$$\begin{aligned} P(mn)m - h_{\mathcal{B}}(nm)m &= mg(n)m + f(m)nm, \\ mQ(nm) - mh_{\mathcal{A}}(mn) &= mg(n)m + mnf(m). \end{aligned} \tag{3.2}$$

Combining the equations in (3.1) and (3.2), we get $2(h_{\mathcal{B}}(nm)m - mh_{\mathcal{A}}(mn)) = 0$, so the 2−torsion freeness of $\mathcal{M}$ implies that

$$h_{\mathcal{B}}(nm)m = mh_{\mathcal{A}}(mn). \tag{3.3}$$

The faithfulness of $\mathcal{M}$ guaranties the existence of an isomorphism φ from $\pi_{\mathcal{A}}(Z(\mathcal{G}))$ to $\pi_{\mathcal{B}}(Z(\mathcal{G}))$ satisfying $a \oplus \varphi(a) \in Z(\mathcal{G})$ for all $a \in \mathcal{A}$. As $\pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$ we define $\ell_{\mathcal{A}} : \mathcal{A} \longrightarrow Z(\mathcal{A})$ and $\ell_{\mathcal{B}} : \mathcal{B} \longrightarrow Z(\mathcal{B})$ by $\ell_{\mathcal{A}} = \varphi^{-1} \circ h_{\mathcal{A}}$ and $\ell_{\mathcal{B}} = \varphi \circ h_{\mathcal{B}}$, respectively. It follows that $\ell_{\mathcal{A}}(a) \oplus h_{\mathcal{A}}(a) \in Z(\mathcal{G})$ and $h_{\mathcal{B}}(b) \oplus \ell_B(b) \in Z(\mathcal{G})$ for all $a \in \mathcal{A}, b \in \mathcal{B}$. By (3.3), we then have

$$(h_{\mathcal{B}}(nm) - \ell_{\mathcal{A}}(mn))m = 0; \tag{3.4}$$

or equivalently,

$$m(\ell_B(nm) - h_A(mn)) = 0. \tag{3.5}$$

If $\mathcal{N}m = 0$ and $m\mathcal{N} = 0$, then trivially $h_{\mathcal{B}}(nm) - \ell_{\mathcal{A}}(mn) = 0$, otherwise either $m\mathcal{N} \neq 0$ or $\mathcal{N}m \neq 0$. Take $0 \neq n_0 \in \mathcal{N}$. If $mn_0 \neq 0$, then as $\mathcal{A}$ is a domain, (3.4) implies that $h_{\mathcal{B}}(nm) - \ell_{\mathcal{A}}(mn) = 0$; so

$$h_{\mathcal{B}}(nm) \oplus h_{\mathcal{A}}(mn) = \ell_{\mathcal{A}}(mn) \oplus h_{\mathcal{A}}(mn) \in Z(\mathcal{G}).$$

If $n_0 m \neq 0$, then as B is a domain, (3.5) implies that $\ell_B(nm) - h_A(mn) = 0$; so

$$h_{\mathcal{B}}(nm) \oplus h_{\mathcal{A}}(mn) = h_{\mathcal{B}}(nm) \oplus \ell_B(nm) \in Z(\mathcal{G}).$$

We thus have $h_{\mathcal{B}}(nm)\oplus h_{\mathcal{A}}(mn)\in Z(\mathcal{G})$ for all $m\in\mathcal{M}$ and $n\in\mathcal{N}$, as required. □

Here we impose a condition on $\mathcal{M}$ which is slightly stronger than the faithfulness of $\mathcal{M}$. We say $\mathcal{M}$ as an $(\mathcal{A},\mathcal{B})-$module is strongly faithful if $\mathcal{M}$ satisfies in one of the following conditions:

$\mathcal{M}$ is faithful as a right $\mathcal{B}-$module and $am=0$ implies $a=0$ or $m=0$ for any $a\in\mathcal{A}, m\in\mathcal{M}$.

$\mathcal{M}$ is faithful as a left $\mathcal{A}-$module and $mb=0$ implies $m=0$ or $b=0$ for any $m\in\mathcal{M}, b\in\mathcal{B}$.

It is evident that if $\mathcal{M}$ is strongly faithful, then $\mathcal{M}$ is faithful and either $\mathcal{A}$ or $\mathcal{B}$ has no zero devisors. As an immediate consequence of the above theorem, we have the following corollary.

Theorem 4. *A generalized matrix algebra $\mathcal{G}$ has Lie derivation property, if*

(i) $\pi_{\mathcal{A}}(Z(\mathcal{G}))=Z(\mathcal{A}), \pi_{\mathcal{B}}(Z(\mathcal{G}))=Z(\mathcal{B})$, *and*

(ii) $\mathcal{M}$ *is strongly faithful.*

Proof. Following the proof of Theorem 3, the equations (3.4) and (3.5) together with strong faithfulness of $\mathcal{M}$ ensure that $h_{\mathcal{B}}(nm)-\ell_{\mathcal{A}}(mn)=0$, so $h_{\mathcal{B}}(nm)\oplus h_{\mathcal{A}}(mn)=\ell_{\mathcal{A}}(mn)\oplus h_{\mathcal{A}}(mn)\in Z(\mathcal{G})$. The conclusion now follows from Corollary 1. □

We remark that, to the best of our knowledge, we do not know when one can withdraw the strong faithfulness in Theorem 4.

Similar to what Cheung has introduced in [4, Section 3], we also introduce a critical subalgebra $\mathcal{W}_{\mathcal{A}}$ of an algebra $\mathcal{A}$. With the same notations as in Theorem 1, suppose that P is a Lie derivation on $\mathcal{A}$ and $\ell_{\mathcal{A}}:\mathcal{A}\to Z(\mathcal{A})$ is a linear map such that $P-\ell_{\mathcal{A}}$ is a derivation on $\mathcal{A}$. Set

$$\mathcal{V}_{\mathcal{A}}=\{a\in\mathcal{A}:\ell_{\mathcal{A}}(a)\oplus h_{\mathcal{A}}(a)\in Z(\mathcal{G})\}.$$

In other words, $\mathcal{V}_{\mathcal{A}}$ consists those elements $a\in\mathcal{A}$ such that

$$\ell_{\mathcal{A}}(a)m=mh_{\mathcal{A}}(a),\ n\ell_{\mathcal{A}}(a)=h_{\mathcal{A}}(a)n, \text{for all } m\in\mathcal{M}, n\in\mathcal{N}.$$

It is also easy to verify that $\mathcal{V}_{\mathcal{A}}\subseteq h_{\mathcal{A}}^{-1}(\pi_{\mathcal{B}}(Z(\mathcal{G})))$, with equality holds in the case where $\mathcal{M}$ is faithful. With some modifications in the proof of [4, Proposition 10], one can show that $\mathcal{V}_{\mathcal{A}}$ is a subalgebra of $\mathcal{A}$ containing all commutators and idempotents. More properties of $\mathcal{V}_{\mathcal{A}}$ are investigated in [10].

We denote by $\mathcal{W}_{\mathcal{A}}$ the smallest subalgebra of $\mathcal{A}$ contains all commutators and idempotents. We are especially dealing with those algebras satisfying $\mathcal{W}_{\mathcal{A}}=\mathcal{A}$. If $\mathcal{W}_{\mathcal{A}}=\mathcal{A}$, then trivially $h_{\mathcal{A}}(\mathcal{A})\subseteq\pi_{\mathcal{B}}(Z(\mathcal{G}))$, or equivalently, $\pi_{\mathcal{B}}(\mathcal{L}(\mathcal{A}))\subseteq\pi_{\mathcal{B}}(Z(G))$. Some examples of algebras satisfying $\mathcal{W}_{\mathcal{A}}=\mathcal{A}$ are: the full matrix algebra $\mathcal{A}=M_n(A), n\geq 2$, where A is a unital algebra, and also every simple unital algebra $\mathcal{A}$ with a nontrivial idempotent.

Motivated by the Cheung's idea [4], regarding the latter observations and some suitable combinations of the various assertions in the results 2, 3 and 4, we apply Theorem 1 and Corollary 1 to arrive the main result of the paper providing several sufficient conditions ensuring the Lie derivation property for a generalized matrix algebra $\mathcal{G}$, which is a generalization of [4, Theorem 11].

Theorem 5. *A generalized matrix algebra $\mathcal{G}$ has Lie derivation property, if the following three conditions hold:*

- (I) $\pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$ *and* $\mathcal{M}$ *is a faithful left* $\mathcal{A}$*-module, or* $\mathcal{A} = \mathcal{W}_{\mathcal{A}}$ *and* $\mathcal{M}$ *is a faithful left* $\mathcal{A}$*-module, or* $\mathcal{A}$ *has Lie derivation property and* $\mathcal{A} = \mathcal{W}_{\mathcal{A}}$.
- (II) $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A})$ *and* $\mathcal{M}$ *is a faithful right* $\mathcal{B}$*-module, or* $\mathcal{B} = \mathcal{W}_{\mathcal{B}}$ *and* $\mathcal{M}$ *is a faithful right* $\mathcal{B}$*-module, or* B *has Lie derivation property and* $\mathcal{B} = \mathcal{W}_{\mathcal{B}}$.
- (III) *One of the following assertions holds:*
  - (i) *Either* $\mathcal{A}$ *or* $\mathcal{B}$ *does not contain nonzero central ideals.*
  - (ii) $\mathcal{A}$ *and* $\mathcal{B}$ *are domain.*
  - (iii) *Either* $\mathcal{M}$ *or* $\mathcal{N}$ *is strongly faithful.*

4. Some applications

The main examples of generalized matrix algebras are: a triangular algebra $\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$, a unital algebra with a nontrivial idempotent, the algebra $B(X)$ of operators on a Banach space X and the full matrix algebra $M_n(\mathcal{A})$ on a unital algebra $\mathcal{A}$.

Lie derivations on trivial generalized matrix algebras and $\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$

We say $\mathcal{G}$ is a trivial generalized matrix algebra when $\mathcal{M}\mathcal{N} = 0$ and $\mathcal{N}\mathcal{M} = 0$. The main example of trivial generalized matrix algebra is the so-called triangular algebra $\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$ whose Lie derivation property has been extensively investigated by Cheung [3, 4]. As an immediate consequence of Corollary 1 and Theorem 5, we get the following corollary which generalizes [4, Theorem 11] to trivial generalized matrix algebras.

Corollary 2. *Let* $\mathcal{G}$ *be a trivial generalized matrix algebra and let* $\mathcal{L}$ *be a Lie derivation on* $\mathcal{G}$ *of the form presented in* (2.1). *If* $\mathcal{L}$ *is proper then* $h_{\mathcal{A}}(\mathcal{A}) \subseteq \pi_{\mathcal{B}}(Z(\mathcal{G}))$ *and* $h_{\mathcal{B}}(\mathcal{B}) \subseteq \pi_{\mathcal{A}}(Z(\mathcal{G}))$. *The converse is also hold in the case when* $\mathcal{M}$ *is faithful.*

In particular, a trivial generalized matrix algebra $\mathcal{G}$ *has Lie derivation property, if the following two conditions hold:*

- (I) $\pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$ *and* $\mathcal{M}$ *is a faithful left* $\mathcal{A}$*-module, or* $\mathcal{A} = \mathcal{W}_{\mathcal{A}}$ *and* $\mathcal{M}$ *is a faithful left* $\mathcal{A}$*-module, or* $\mathcal{A}$ *has Lie derivation property and* $\mathcal{A} = \mathcal{W}_{\mathcal{A}}$.
- (II) $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A})$ *and* $\mathcal{M}$ *is a faithful right* $\mathcal{B}$*-module, or* $\mathcal{B} = \mathcal{W}_{\mathcal{B}}$ *and* $\mathcal{M}$ *is a faithful right* $\mathcal{B}$*-module, or* B *has Lie derivation property and* $\mathcal{B} = \mathcal{W}_{\mathcal{B}}$.

The following example which has been adapted from [1, Example 3.8] presents a trivial generalized matrix algebra (which is not a triangular algebra) without the Lie derivation property.

Example 1. Let $\mathcal{M}$ be a commutative unital algebra of dimension 3 (on the commutative unital ring R) with base $\{1, a_0, b_0\}$ such that ${a_0}^2 = {b_0}^2 = a_0b_0 = b_0a_0 = 0$. Set $\mathcal{N} = \mathcal{M}$ and let $\mathcal{A}$ and $\mathcal{B}$ be the subalgebras of $\mathcal{M}$ generated by $\{1, a_0\}$ and $\{1, b_0\}$, respectively. Then $\mathcal{M}\mathcal{N} = 0 = \mathcal{N}\mathcal{M}$, so the generalized matrix algebra $\mathcal{G} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{pmatrix}$ is trivial, and a direct calculation reveals that the map $\mathcal{L} : \mathcal{G} \longrightarrow \mathcal{G}$ defined as

$$\mathcal{L}\begin{pmatrix} r_1 + r_2a_0 & s_1 + s_2a_0 + s_3b_0 \\ t_1 + t_2a_0 + t_3b_0 & u_1 + u_2b_0 \end{pmatrix} = \begin{pmatrix} u_2a_0 & -s_3a_0 - s_2b_0 \\ -t_3a_0 - t_2b_0 & r_2b_0 \end{pmatrix},$$

(where the coefficients are taken from the ring R) is a non-proper Lie derivation. It is worthwhile mentioning that $Z(\mathcal{G}) = R \cdot 1_{\mathcal{G}}$ and

$$\pi_{\mathcal{A}}(Z(\mathcal{G})) = R \cdot 1_{\mathcal{A}} \neq Z(\mathcal{A}), \pi_{\mathcal{B}}(Z(\mathcal{G})) = R \cdot 1_{\mathcal{B}} \neq Z(\mathcal{B}).$$

Furthermore, it is also easy to verify that $\mathcal{A} \neq \mathcal{W}_{\mathcal{A}}, \mathcal{B} \neq \mathcal{W}_{\mathcal{B}}$. Therefore, none of the conditions (I) and (II) of Corollary 2 is hold. Nevertheless, $\mathcal{A}$ and $\mathcal{B}$ have Lie derivation property. It should be also noted that $h_{\mathcal{A}}(\mathcal{A}) \not\subseteq \pi_{\mathcal{B}}(Z(\mathcal{G}))$, $h_{\mathcal{B}}(\mathcal{B}) \not\subseteq \pi_{\mathcal{A}}(Z(\mathcal{G}))$; as, $h_{\mathcal{A}}(a_0) = b_0$ and $h_{\mathcal{B}}(b_0) = a_0$.

This example can be compared to that was given by Cheung in [4, Example 8]. They clarify the same discipline, however, $\mathcal{G}$ is not a triangular algebra.

Lie derivations on unital algebras with a nontrivial idempotent

In the following, we investigate the Lie derivation property for some unital algebras with a nontrivial idempotent. Let $\mathcal{A}$ be a unital algebra with a nontrivial idempotent p and $q = 1 - p$. Then $\mathcal{A}$ enjoys the Peirce decomposition $\mathcal{G} = \begin{pmatrix} p\mathcal{A}q & p\mathcal{A}q \\ q\mathcal{A}p & q\mathcal{A}q \end{pmatrix}$, as a generalized matrix algebra. Applying Theorem 5 for this generalized matrix algebra $\mathcal{G}$, we obtain the following result which partly improves the case $n = 2$ of a result given by Wang [14, Theorem 2.1] and Benkovič [1, Theorem 5.3].

Corollary 3. *Let $\mathcal{A}$ be a $2-$torsion free unital algebra with a nontrivial idempotent p satisfying*

$$pap \cdot p\mathcal{A}q = 0 \; implies \; pap = 0, \quad and \quad p\mathcal{A}q \cdot qaq = 0 \; implies \; qaq = 0, \tag{4.1}$$

for any $a \in \mathcal{A}$, where $q = 1 - p$. Then $\mathcal{A}$ has Lie derivation property, if the following three conditions hold:

(I) *$Z(q\mathcal{A}q) = Z(\mathcal{A})q$, or $p\mathcal{A}p = \mathcal{W}_{p\mathcal{A}p}$, or $p\mathcal{A}p$ has Lie derivation property and $p\mathcal{A}p = \mathcal{W}_{p\mathcal{A}p}$.*

(II) *$Z(p\mathcal{A}p) = Z(\mathcal{A})p$, or $q\mathcal{A}q = \mathcal{W}_{q\mathcal{A}q}$, or $q\mathcal{A}q$ has Lie derivation property and $q\mathcal{A}q = \mathcal{W}_{q\mathcal{A}q}$.*

(III) *One of the following assertions holds:*
  (i) *Either $p\mathcal{A}p$ or $q\mathcal{A}q$ does not contain nonzero central ideals.*
  (ii) *$p\mathcal{A}p$ and $q\mathcal{A}q$ are domain.*
  (iii) *Either $p\mathcal{A}q$ or $q\mathcal{A}p$ is strongly faithful.*

As a consequence of the above corollary, we bring the following result showing that the algebra $B(X)$ of bounded operators on a Banach space X enjoys Lie derivation property. The same result has already proved by Lu and Jing [7] for Lie derivable map at zero and idempotents by a completely different method. See also [8] for the properness of nonlinear derivations on $B(X)$.

Corollary 4. *Let X be a Banach space of dimension greater than 2. Then $B(X)$ has Lie derivation property.*

Proof. Set $\mathcal{A} = B(X)$. Consider a nonzero element $x_0 \in X$ and $f_0 \in X^*$ such that $f_0(x_0) = 1$, then $p = x_0 \otimes f_0$ defined by $y \longmapsto f_0(y)x_0$ is a nontrivial idempotent. A direct verification reveals that $\mathcal{A}$ satisfies the implications (4.1). Indeed, if $pTp \cdot p\mathcal{A}q = \{0\}$ for some $T \in \mathcal{A}$, then choose a nonzero element $y \in q(X)$ such that $q(y) = y$. Let $x \in X$, there exists an operator $S \in B(X)$ such that $S(y) = x$ (e.g. $S := x \otimes g$ for some $g \in X^*$ with $g(y) = 1$). We then get $pTp(x) = pTp(S(q(y))) = pTp \cdot pSq(y) = 0$. Further, it can be readily verified that $Z(\mathcal{A}) = \mathbb{C}I_X = \mathbb{C}(p+q)$, $Z(p\mathcal{A}p) = \mathbb{C}p$ and $Z(q\mathcal{A}q) = \mathbb{C}q$. In particular, $Z(p\mathcal{A}p) = Z(\mathcal{A})p$, $Z(q\mathcal{A}q) = Z(\mathcal{A})q$. These also imply that neither $p\mathcal{A}p$ nor $q\mathcal{A}q$ has no central ideals. By Corollary 3 $\mathcal{A} = B(X)$ has Lie derivation property. □

We conclude this section with an application of Corollary 3 to the full matrix algebra $\mathcal{A} = M_n(A)$, $n \geq 2$, where A is a unital algebra. Then $p = e_{11}$ is a nontrivial idempotent and $q = e_{11} + \cdots e_{nn}$. It is easy to verify that $p\mathcal{A}p = A, q\mathcal{A}q = M_{n-1}(A)$. As $Z(\mathcal{A}) = Z(A)1_{\mathcal{A}}$, we get $Z(p\mathcal{A}p) = Z(\mathcal{A})p, Z(q\mathcal{A}q) = Z(\mathcal{A})q$, so both conditions (I) and (II) in Corollary 3 are fulfilled. Further, in the case where $n \geq 3$, the algebra $q\mathcal{A}q = M_{n-1}(A)$ does not contain nonzero central ideals, so $\mathcal{A} = M_n(A)$ also satisfies assumption (i) of the aforementioned corollary. Therefore $M_n(A)$ has Lie derivation property for $n \geq 3$. This is an adaptation of [5, Corollary 1]. For $n = 2$, the same result was treated in [1, Corollary 5.7].

References

[1] D. Benkovič, "Lie triple derivations of unital algebras with idempotents," *Linear Multilinear Algebra*, vol. 63, no. 1, pp. 141–165, 2015, doi: 10.1080/03081087.2013.851200.

[2] M. Brešar, "Commuting traces of biadditive mappings, commutativity-preserving mappings and lie mappings," *Trans. Amer. Math. Soc.*, vol. 335, no. 2, pp. 525–546, 1993, doi: 10.1090/S0002-9947-1993-1069746-X.

[3] W.-S. Cheung, *Mappings on triangular algebras*. PhD Thesis, University of Victoria, 2000.

[4] W.-S. Cheung, "Lie derivations of triangular algebras," *Linear Multilinear Algebra*, vol. 51, no. 3, pp. 299–310, 2003, doi: 10.1080/0308108031000096993.
[5] Y. Du and Y. Wang, "Lie derivations of generalized matrix algebras," *Linear Algebra Appl.*, vol. 437, no. 11, pp. 2719–2726, 2012, doi: 10.1016/j.laa.2012.06.013.
[6] Y. Li and F. Wei, "Semi-centralizing maps of generalized matrix algebras," *Linear Algebra Appl.*, vol. 436, no. 5, pp. 1122–1153, 2012, doi: 10.1016/j.laa.2011.07.014.
[7] F. Lu and W. Jing, "Characterizations of lie derivations of b(x)," *Linear Algebra Appl.*, vol. 1, no. 432, pp. 89–99, 2010, doi: 10.1016/j.laa.2009.07.026.
[8] F. Lu and B. Liu, "Lie derivable maps on $b(x)$," *J. Math. Anal. Appl.*, vol. 2, no. 372, pp. 369–376, 2010, doi: 10.1016/j.jmaa.2010.07.002.
[9] W. S. Martindale *et al.*, "Lie derivations of primitive rings." *Michigan Math. J.*, vol. 11, no. 2, pp. 183–187, 1964, doi: doi:10.1307/mmj/1028999091.
[10] A. Mokhtari, *Lie derivations on module extension Banach algebras*. PhD Thesis, Ferdowsi University of Mashhad, 2015.
[11] A. H. Mokhtari, F. Moafian, and H. R. Ebrahimi Vishki, "Lie derivations on trivial extension algebras," *Annales Mathematicae Silesianae*, vol. 31, no. 1, pp. 141–153, 2017, doi: 10.1515/amsil-2016-0017.
[12] A. Sands, "Radicals and morita contexts," *J. Algebra*, vol. 24, no. 2, pp. 335–345, 1973, doi: 10.1016/0021-8693(73)90143-9.
[13] Y. Wang and Y. Wang, "Multiplicative lie n-derivations of generalized matrix algebras," *Linear Algebra Appl.*, vol. 438, no. 5, pp. 2599–2616, 2013, doi: 10.1016/j.laa.2012.10.052.
[14] Y. Wang, "Lie n-derivations of unital algebras with idempotents," *Linear Algebra Appl.*, vol. 458, pp. 512–525, 2014, doi: 10.1016/j.laa.2014.06.029.

Authors' addresses

A.H. Mokhtari
Technical Faculty of Ferdows, University of Birjand, Iran
E-mail address: a.mokhtari@birjand.ac.ir

H.R. Ebrahimi Vishki
Department of Pure Mathematics and Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran
E-mail address: vishki@um.ac.ir