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SIMPSON TYPE QUANTUM INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. In this paper we establish some new Simpson type quantum integral inequalities for convex functions. Moreover, we obtain some inequalities for special means.

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1. INTRODUCTION

A function $f : J \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on J if the inequality

$$f(tu + (1-t)v) \leq tf(u) + (1-t)f(v) \tag{1.1}$$

holds for all $u, v \in J$ and $t \in [0,1]$. We say that f is concave if $(-f)$ is convex.

Convex functions play an important role in mathematical inequalities. The most famous inequality have been used with convex functions is Hermite-Hadamard, which is stated as follows:

Let $f : J \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $u, v \in J$ with $u < v$. Then the following double inequalities hold:

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v f(x)\,dx \leq \frac{f(u)+f(v)}{2}. \tag{1.2}$$

In recent years quantum calculus has been actively studied. There are numerous applications in many mathematical areas like special functions, integral transforms, quantum mechanics, information technology and mathematical inequalities. At present q analogous of many inequalities have been established. In the view of these developments q-convexity and convexity of q analogous of the inequalities has also been considered, see [3–5, 7, 9–11].

The inequality given below is well known in the literature as Simpson's inequality:

$$\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] \leq \frac{1}{2880}\left\| f^{(4)} \right\|_\infty (b-a)^4 \tag{1.3}$$

where the mapping $f:[a,b]\to\mathbb{R}$ is assumed to be four times continuously differentiable on the interval and $f^{(4)}$ to be bounded on (a,b), that is,

$$\left\| f^{(4)} \right\|_{\infty} = \sup_{t\in(a,b)} \left| f^{(4)} \right| < \infty.$$

Inequality (1.3) have been studied by many authors. For more details see [1, 2, 6, 8]. The aim of this paper is to establish q analogues of Simpson type inequalities based on convexity. The consequences of Simpson type inequalities for convex functions are given as special cases when $q\to 1$.

2. Preliminaries

In this section, we recall some previously known concepts and basic results.

Let $J=[a,b]\subset\mathbb{R}$ be an interval and $0<q<1$ be a constant. We define q-derivative of a function $f:J\to\mathbb{R}$ at a point $x\in J$ on $[a,b]$ as follows.

Definition 1. Assume $f:J\to\mathbb{R}$ is a continuous function and let $x\in J$. Then the expression

$$_aD_q f(x) = \frac{f(x)-f(qx+(1-q)a)}{(1-q)(x-a)},\ x\neq a, \quad _aD_q f(a) = \lim_{x\to a} {_aD_q f(x)}, \tag{2.1}$$

is called the q-derivative on J of function f at x.

Also if $a=0$ in (2.1), then $_0D_q f(a) = D_q f$, where D_q is the q-derivative of the function $f(x)$ defined by

$$D_q f(x) = \frac{f(x)-f(qx)}{(1-q)x}.$$

For more details, see [7].

Lemma 1 ([10])**.** *Let $\alpha\in\mathbb{R}$, then we have*

$$_aD_q (x-a)^{\alpha} = \left(\frac{1-q^{\alpha}}{1-q}\right)(x-a)^{\alpha-1}. \tag{2.2}$$

Definition 2. Let $f:J\subset\mathbb{R}\to\mathbb{R}$ be a continuous function. Then q-integral on J is defined as

$$\int_a^x f(t)\, _ad_qt = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f\left(q^n x + \left(1-q^n\right)a\right) \tag{2.3}$$

for $x\in J$. If $a=0$ in (2.2), then we have the classical q-integral [7].

Moreover, if $v\in(a,x)$ then the definite q-integral on J is defined by

$$\int_v^x f(t)\, _ad_qt = \int_a^x f(t)\, _ad_qt - \int_a^v f(t)\, _ad_qt$$

$$= (1-q)(x-a)\sum_{n=0}^{\infty} q^n f\left(q^n x + \left(1-q^n\right)a\right)$$
$$-(1-q)(v-a)\sum_{n=0}^{\infty} q^n f\left(q^n v + \left(1-q^n\right)a\right).$$

Lemma 2 ([11]). *For $\alpha \in \mathbb{R}\setminus\{-1\}$, the following formula holds:*

$$\int_a^x (t-a)^{\alpha}\ {}_a d_q t = \left(\frac{1-q}{1-q^{\alpha+1}}\right)(x-a)^{\alpha+1}. \tag{2.4}$$

3. Results

We begin with the following lemma.

Lemma 3. *Let $f : J \to \mathbb{R}$ be a continuous function and $0<q<1$. If ${}_aD_q f$ is an integrable function on J° (the interior of J), then the following inequality holds:*

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\ {}_a d_q x$$
$$= (b-a)\int_0^1 p(t)\ {}_aD_q f((1-t)a+tb)\ {}_0 d_q t \tag{3.1}$$

where

$$p(t) = \begin{cases} qt-\frac{1}{6} & ,\quad t\in\left[0,\frac{1}{2}\right) \\ qt-\frac{5}{6} & ,\quad t\in\left[\frac{1}{2},1\right) \end{cases}.$$

Proof. From Definition 1 and Definition 2, we have

$$\int_0^{\frac{1}{2}}\left(qt-\frac{1}{6}\right){}_aD_q f((1-t)a+tb)\ {}_0d_q t$$
$$= \int_0^{\frac{1}{2}} qt\ {}_aD_q f((1-t)a+tb)\ {}_0d_q t - \frac{1}{6}\int_0^{\frac{1}{2}} {}_aD_q f((1-t)a+tb)\ {}_0d_q t$$
$$= \int_0^{\frac{1}{2}} q\,\frac{f((1-t)a+tb)-f((1-qt)a+qtb)}{(1-q)(b-a)}\ {}_0d_q t$$
$$-\frac{1}{6}\int_0^{\frac{1}{2}} \frac{f((1-t)a+tb)-f((1-qt)a+qtb)}{(1-q)(b-a)t}\ {}_0d_q t$$
$$= \frac{1}{2}\sum_{n=0}^{\infty} q^{n+1}\frac{f\left(\left(1-\frac{1}{2}q^n\right)a+\frac{1}{2}q^n b\right)}{(b-a)} - \frac{1}{2}\sum_{n=0}^{\infty} q^{n+1}\frac{f\left(\left(1-\frac{1}{2}q^{n+1}\right)a+\frac{1}{2}q^{n+1} b\right)}{(b-a)}$$
$$-\frac{1}{6}\sum_{n=0}^{\infty}\frac{f\left(\left(1-\frac{1}{2}q^n\right)a+\frac{1}{2}q^n b\right)}{(b-a)} + \frac{1}{6}\sum_{n=0}^{\infty}\frac{f\left(\left(1-\frac{1}{2}q^{n+1}\right)a+\frac{1}{2}q^{n+1} b\right)}{(b-a)}$$

$$= \frac{1}{2}q\sum_{n=0}^{\infty} q^n \frac{f\left(\left(1-\frac{1}{2}q^n\right)a+\frac{1}{2}q^n b\right)}{(b-a)} - \frac{1}{2}\sum_{n=1}^{\infty} q^n \frac{f\left(\left(1-\frac{1}{2}q^n\right)a+\frac{1}{2}q^n b\right)}{(b-a)}$$

$$-\frac{1}{6}\frac{1}{b-a}\sum_{n=0}^{\infty} f\left(\left(1-\frac{1}{2}q^n\right)a+\frac{1}{2}q^n b\right)$$

$$+\frac{1}{6}\frac{1}{b-a}\sum_{n=1}^{\infty} f\left(\left(1-\frac{1}{2}q^n\right)a+\frac{1}{2}q^n b\right)$$

$$= -\frac{1}{b-a}(1-q)\frac{1}{2}\sum_{n=0}^{\infty} q^n f\left(\left(1-\frac{1}{2}q^n\right)a+\frac{1}{2}q^n b\right)$$

$$+\frac{1}{2}\frac{1}{b-a} f\left(\frac{a+b}{2}\right) - \frac{1}{6}\frac{1}{b-a}\left\{f\left(\frac{a+b}{2}\right)-f(a)\right\}$$

$$= -\frac{1}{b-a}\int_0^{\frac{1}{2}} f((1-t)a+tb)\ {}_0d_qt + \frac{1}{3(b-a)} f\left(\frac{a+b}{2}\right) + \frac{1}{6(b-a)} f(a).$$

For the second part of the integral, we have

$$\int_{\frac{1}{2}}^{1}\left(qt-\frac{5}{6}\right) {}_aD_q f((1-t)a+tb)\ {}_0d_qt$$

$$= \int_0^1 \left(qt-\frac{5}{6}\right) {}_aD_q f((1-t)a+tb)\ {}_0d_qt$$

$$-\int_0^{\frac{1}{2}} \left(qt-\frac{5}{6}\right) {}_aD_q f((1-t)a+tb)\ {}_0d_qt$$

and similarly we obtain

$$\int_0^1 \left(qt-\frac{5}{6}\right) {}_aD_q f((1-t)a+tb)\ {}_0d_qt$$

$$= -\frac{1}{b-a}\int_0^1 f((1-t)a+tb)\ {}_0d_qt + \frac{1}{6(b-a)} f(b) + \frac{5}{6(b-a)} f(a)$$

and

$$\int_0^{\frac{1}{2}} \left(qt-\frac{5}{6}\right) {}_aD_q f((1-t)a+tb)\ {}_0d_qt$$

$$= -\frac{1}{b-a}\int_0^{\frac{1}{2}} f((1-t)a+tb)\ {}_0d_qt - \frac{1}{3(b-a)} f\left(\frac{a+b}{2}\right) + \frac{5}{6(b-a)} f(a).$$

Thus, we have

$$\int_0^1 p(t)\ {}_aD_q f((1-t)a+tb)\ {}_0d_qt$$

$$= -\frac{1}{b-a}\int_0^1 f((1-t)a+tb)\ {}_0d_qt + \frac{2}{3(b-a)}f\left(\frac{a+b}{2}\right)$$
$$+\frac{1}{6(b-a)}\{f(a)+f(b)\}$$
$$= -\frac{1}{(b-a)^2}\int_a^b f(x)\ {}_ad_qx + \frac{2}{3(b-a)}f\left(\frac{a+b}{2}\right) + \frac{1}{6(b-a)}\{f(a)+f(b)\}$$

We complete the proof. □

Remark 1. If $q \to 1$, then (3.1) reduces to

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx$$
$$=(b-a)\int_0^1 p(t)\,f'(tb+(1-t)a)\,dt,$$

where

$$p(t)=\begin{cases} t-\frac{1}{6}, & t\in\left[0,\frac{1}{2}\right) \\ t-\frac{5}{6} & t\in\left[\frac{1}{2},1\right) \end{cases}.$$

See also [1, Lemma 1].

Lemma 4. *Let* $0<q<1$ *be a constant. Then,*

$$\int_0^{\frac{1}{2}}(1-t)\left|qt-\frac{1}{6}\right|\ {}_0d_qt = \frac{1}{216}\frac{36q^3+12q^2+12q+1}{q^3+2q^2+2q+1}. \tag{3.2}$$

Proof. By computing directly and using (2.4), we have

$$\int_0^{\frac{1}{2}}(1-t)\left|qt-\frac{1}{6}\right|\ {}_0d_qt$$
$$=\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|\ {}_0d_qt-\int_0^{\frac{1}{2}}t\left|qt-\frac{1}{6}\right|\ {}_0d_qt$$
$$=\int_0^{\frac{1}{6q}}\left(\frac{1}{6}-qt\right)\ {}_0d_qt+\int_{\frac{1}{6q}}^{\frac{1}{2}}\left(qt-\frac{1}{6}\right)\ {}_0d_qt$$
$$-\left(\int_0^{\frac{1}{6q}}t\left(\frac{1}{6}-qt\right)\ {}_0d_qt+\int_{\frac{1}{6q}}^{\frac{1}{2}}t\left(qt-\frac{1}{6}\right)\ {}_0d_qt\right)$$
$$=\int_0^{\frac{1}{6q}}\left(\frac{1}{6}-qt\right)\ {}_0d_qt+\int_0^{\frac{1}{2}}\left(qt-\frac{1}{6}\right)\ {}_0d_qt-\int_0^{\frac{1}{6q}}\left(qt-\frac{1}{6}\right)\ {}_0d_qt$$
$$-\left(\int_0^{\frac{1}{6q}}t\left(\frac{1}{6}-qt\right)\ {}_0d_qt+\int_0^{\frac{1}{2}}t\left(qt-\frac{1}{6}\right)\ {}_0d_qt-\int_0^{\frac{1}{6q}}t\left(qt-\frac{1}{6}\right)\ {}_0d_qt\right)$$

$$= \frac{1}{36}\frac{6q-1}{q+1} - \frac{1}{216}\frac{18q^2+18q-7}{q^3+2q^2+2q+1}$$
$$= \frac{1}{216}\frac{36q^3+12q^2+12q+1}{q^3+2q^2+2q+1}.$$

□

Lemma 5. *Let* $0 < q < 1$ *be a constant. Then,*

$$\int_{\frac{1}{2}}^{1} (1-t)\left|qt-\frac{5}{6}\right| {}_0d_qt = \frac{1}{216}\frac{12q^2+12q+5}{q^3+2q^2+2q+1}. \tag{3.3}$$

Proof. By computing directly and using (2.4), we have

$$\int_{\frac{1}{2}}^{1} (1-t)\left|qt-\frac{5}{6}\right| {}_0d_qt$$
$$= \int_{\frac{1}{2}}^{1} \left|qt-\frac{5}{6}\right| {}_0d_qt - \int_{\frac{1}{2}}^{1} t\left|qt-\frac{5}{6}\right| {}_0d_qt$$
$$= \int_{\frac{1}{2}}^{\frac{5}{6q}} \left(\frac{5}{6}-qt\right) {}_0d_qt + \int_{\frac{5}{6q}}^{1} \left(qt-\frac{5}{6}\right) {}_0d_qt$$
$$-\left(\int_{\frac{1}{2}}^{\frac{5}{6q}} t\left(\frac{5}{6}-qt\right) {}_0d_qt + \int_{\frac{5}{6q}}^{1} t\left(qt-\frac{5}{6}\right) {}_0d_qt\right)$$
$$= \frac{5}{36(q+1)} - \frac{1}{216}\frac{18q^2+18q+25}{q^3+2q^2+2q+1}$$
$$= \frac{1}{216}\frac{12q^2+12q+5}{q^3+2q^2+2q+1}.$$

□

Theorem 1. *Let* $f : J \to \mathbb{R}$ *be a continuous function and* $0 < q < 1$. *If* $\left|{}_aD_qf\right|$ *is convex and integrable function on* J°, *then the following inequality holds:*

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx \tag{3.4}$$
$$\leq \frac{(b-a)}{12}\left[\frac{2q^2+2q+1}{q^3+2q^2+2q+1}\left|{}_aD_qf(b)\right| + \frac{1}{3}\frac{6q^3+4q^2+4q+1}{q^3+2q^2+2q+1}\left|{}_aD_qf(a)\right|\right].$$

Proof. Using Lemma 3 and the convexity of $\left|{}_aD_qf\right|$ on J°, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\right|$$

$$= (b-a)\left|\int_0^{\frac{1}{2}}\left(qt-\frac{1}{6}\right){}_aD_qf(tb+(1-t)a)\ {}_0d_qt\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left(qt-\frac{5}{6}\right){}_aD_qf(tb+(1-t)a)\ {}_0d_qt\right|$$
$$\leq (b-a)\left(\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|\left|{}_aD_qf(tb+(1-t)a)\right|\ {}_0d_qt\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left|qt-\frac{5}{6}\right|\left|{}_aD_qf(tb+(1-t)a)\right|\ {}_0d_qt\right)$$
$$\leq (b-a)\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|\left(t\left|{}_aD_qf(b)\right|+(1-t)\left|{}_aD_qf(a)\right|\right)\ {}_0d_qt$$
$$+(b-a)\int_{\frac{1}{2}}^1\left|qt-\frac{5}{6}\right|\left(t\left|{}_aD_qf(b)\right|+(1-t)\left|{}_aD_qf(a)\right|\right)\ {}_0d_qt$$
$$= \left|{}_aD_qf(b)\right|(b-a)\int_0^{\frac{1}{2}}t\left|qt-\frac{1}{6}\right|\ {}_0d_qt+\left|{}_aD_qf(a)\right|(b-a)\int_0^{\frac{1}{2}}(1-t)\left|qt-\frac{1}{6}\right|\ {}_0d_qt$$
$$+\left|{}_aD_qf(b)\right|(b-a)\int_{\frac{1}{2}}^1t\left|qt-\frac{5}{6}\right|\ {}_0d_qt+\left|{}_aD_qf(a)\right|(b-a)\int_{\frac{1}{2}}^1(1-t)\left|qt-\frac{5}{6}\right|\ {}_0d_qt$$

Applying Lemma 4 and Lemma 5, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx\right|$$
$$\leq \frac{(b-a)}{216}\frac{18q^2+18q-7}{q^3+2q^2+2q+1}\left|{}_aD_qf(b)\right|+\frac{(b-a)}{216}\frac{36q^3+12q^2+12q+1}{q^3+2q^2+2q+1}\left|{}_aD_qf(a)\right|$$
$$+\frac{(b-a)}{216}\frac{18q^2+18q+25}{q^3+2q^2+2q+1}\left|{}_aD_qf(b)\right|+\frac{(b-a)}{216}\frac{12q^2+12q+5}{q^3+2q^2+2q+1}\left|{}_aD_qf(a)\right|$$
$$= \frac{(b-a)}{12}\frac{2q^2+2q+1}{q^3+2q^2+2q+1}\left|{}_aD_qf(b)\right|+\frac{(b-a)}{36}\frac{6q^3+4q^2+4q+1}{q^3+2q^2+2q+1}\left|{}_aD_qf(a)\right|.$$

The proof is complete. □

Remark 2. If $q\to 1$, then (3.4) reduces to

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{5(b-a)}{72}\left[f'(a)+f'(b)\right]$$

See also [1, Corollary 1].

Corollary 1. *In Theorem 1, if* $f(a)=f\left(\frac{a+b}{2}\right)=f(b)$*, then we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx\right|$$

$$\leq \frac{(b-a)}{12}\left[\frac{2q^2+2q+1}{q^3+2q^2+2q+1}\left|{}_aD_q f(b)\right| + \frac{1}{3}\frac{6q^3+4q^2+4q+1}{q^3+2q^2+2q+1}\left|{}_aD_q f(a)\right|\right].$$

This inequality can be considered a product of midpoint q-Hadamard type inequality.

Remark 3. In Corollary 1, if $q \to 1$, then we obtain

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{5(b-a)}{72}\left[f'(a)+f'(b)\right].$$

See also [1, Corollary 3].

Theorem 2. *Let $f : J = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on J° with ${}_aD_q$ be continuous and integrable on J where $0 < q < 1$. If $\left|{}_aD_q f\right|^r$ is convex function where $p,r > 1$, $\frac{1}{p}+\frac{1}{r}=1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\right| \tag{3.5}$$

$$\leq \frac{(b-a)}{2^{\frac{1}{r}}}\left(\frac{(1-q)}{6^{p+1}q\left(1-q^{p+1}\right)}\right)^{\frac{1}{p}}$$

$$\Bigg\{\left(1+(3q-1)^{p+1}\right)^{\frac{1}{p}}\left(\left|{}_aD_q f(a)\right|^r + \left|{}_aD_q f\left(\frac{a+b}{2}\right)\right|^r\right)^{\frac{1}{r}}$$

$$+\left[(5-3q)^{p+1}+(6q-5)^{p+1}\right]^{\frac{1}{p}}\left(\left|{}_aD_q f(b)\right|^r + \left|{}_aD_q f\left(\frac{a+b}{2}\right)\right|^r\right)^{\frac{1}{r}}\Bigg\}.$$

Proof. From Lemma 3, using the well known Hölder integral inequality, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\right|$$

$$\leq (b-a)\left(\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|\left|{}_aD_q f(tb+(1-t)a)\right|\,{}_0d_qt\right.$$

$$\left.+\int_{\frac{1}{2}}^1\left|qt-\frac{5}{6}\right|\left|{}_aD_q f(tb+(1-t)a)\right|\,{}_0d_qt\right)$$

$$\leq (b-a)\left(\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|^p\,{}_0d_qt\right)^{1/p}\left(\int_0^{\frac{1}{2}}\left|{}_aD_q f(tb+(1-t)a)\right|^r\,{}_0d_qt\right)^{1/r}$$

$$+(b-a)\left(\int_{\frac{1}{2}}^1\left|qt-\frac{5}{6}\right|^p\,{}_0d_qt\right)^{1/p}\left(\int_{\frac{1}{2}}^1\left|{}_aD_q f(tb+(1-t)a)\right|^r\,{}_0d_qt\right)^{1/r}.$$

From (2.4), it is easy to see that

$$\begin{aligned}\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|^p {}_0d_qt &= \int_0^{\frac{1}{6q}}\left(\frac{1}{6}-qt\right)^p {}_0d_qt + \int_{\frac{1}{6q}}^{\frac{1}{2}}\left(qt-\frac{1}{6}\right)^p {}_0d_qt\\ &= (-1)^{p+1}q^p\int_{\frac{1}{6q}}^{0}\left(t-\frac{1}{6q}\right)^p {}_0d_qt + q^p\int_{\frac{1}{6q}}^{\frac{1}{2}}\left(t-\frac{1}{6q}\right)^p {}_0d_qt\\ &= q^p\left(\frac{1-q}{1-q^{p+1}}\left(\frac{1}{6q}\right)^{p+1}\right)+q^p\left(\frac{1-q}{1-q^{p+1}}\left(\frac{1}{2}-\frac{1}{6q}\right)^{p+1}\right)\\ &= \frac{\left(1+(3q-1)^{p+1}\right)(1-q)}{6^{p+1}q\left(1-q^{p+1}\right)},\end{aligned}$$

analogously

$$\begin{aligned}\int_{\frac{1}{2}}^{1}\left|qt-\frac{5}{6}\right|^p {}_0d_qt &= \int_{\frac{1}{2}}^{\frac{5}{6q}}\left(\frac{1}{6}-qt\right)^p {}_0d_qt + \int_{\frac{5}{6q}}^{1}\left(qt-\frac{1}{6}\right)^p {}_0d_qt\\ &= (-1)^{p+1}q^p\int_{\frac{5}{6q}}^{\frac{1}{2}}\left(t-\frac{5}{6q}\right)^p {}_0d_qt + q^p\int_{\frac{5}{6q}}^{1}\left(t-\frac{5}{6q}\right)^p {}_0d_qt\\ &= \frac{\left[(5-3q)^{p+1}+(6q-5)^{p+1}\right](1-q)}{6^{p+1}q\left(1-q^{p+1}\right)}.\end{aligned}$$

Since $\left|{}_aD_qf\right|$ is convex by (1.2), we have

$$\int_0^{\frac{1}{2}}\left|{}_aD_qf\left(tb+(1-t)a\right)\right|^r {}_0d_qt \leq \frac{\left|{}_aD_qf(a)\right|^r+\left|{}_aD_qf\left(\frac{a+b}{2}\right)\right|^r}{2}$$

and

$$\int_{\frac{1}{2}}^{1}\left|{}_aD_qf\left(tb+(1-t)a\right)\right|^r {}_0d_qt \leq \frac{\left|{}_aD_qf(b)\right|^r+\left|{}_aD_qf\left(\frac{a+b}{2}\right)\right|^r}{2}.$$

So, we obtain

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\right|\\ &\leq (b-a)\left(\frac{\left(1+(3q-1)^{p+1}\right)(1-q)}{6^{p+1}q\left(1-q^{p+1}\right)}\right)^{\frac{1}{p}}\left(\frac{\left|{}_aD_qf(a)\right|^r+\left|{}_aD_qf\left(\frac{a+b}{2}\right)\right|^r}{2}\right)^{\frac{1}{r}}\\ &\quad+(b-a)\left(\frac{\left[(5-3q)^{p+1}-(6q-5)^{p+1}\right](1-q)}{6^{p+1}q\left(1-q^{p+1}\right)}\right)^{\frac{1}{p}}\left(\frac{\left|{}_aD_qf(b)\right|^r+\left|{}_aD_qf\left(\frac{a+b}{2}\right)\right|^r}{2}\right)^{\frac{1}{r}}.\end{aligned}$$

The proof is completed. □

Remark 4. If $q \to 1$, then (3.5) reduces to

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{2^{\frac{1}{r}}}\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\left|f'(a)\right|^r+\left|f'\left(\frac{a+b}{2}\right)\right|^r\right)^{1/r}\right.$$
$$\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^r+\left|f'(b)\right|^r\right)^{1/r}\right].$$

See also [1, Corollary 4].

Corollary 2. *In Theorem 2, if* $f(a)=f\left(\frac{a+b}{2}\right)=f(b)$*, then we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\right|$$
$$\leq \frac{(b-a)}{2^{\frac{1}{r}}}\left(\frac{(1-q)}{6^{p+1}q\left(1-q^{p+1}\right)}\right)^{\frac{1}{p}}$$
$$\left\{\left(1+(3q-1)^{p+1}\right)^{\frac{1}{p}}\left(\left|{}_aD_qf(a)\right|^r+\left|{}_aD_qf\left(\frac{a+b}{2}\right)\right|^r\right)^{\frac{1}{r}}\right.$$
$$\left.+\left[(5-3q)^{p+1}+(6q-5)^{p+1}\right]^{\frac{1}{p}}\left(\left|{}_aD_qf(b)\right|^r+\left|{}_aD_qf\left(\frac{a+b}{2}\right)\right|^r\right)^{\frac{1}{r}}\right\}.$$

Remark 5. In Corollary 2, if $q \to 1$, then we have

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{2^{\frac{1}{r}}}\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\left|f'(a)\right|^r+\left|f'\left(\frac{a+b}{2}\right)\right|^r\right)^{1/r}\right.$$
$$\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^r+\left|f'(b)\right|^r\right)^{1/r}\right].$$

See also [1, Corollary 6] and take $s=1$.

Theorem 3. *Let* $f: J=[a,b]\subset\mathbb{R}\to\mathbb{R}$ *be a* q*-differentiable function on* J° *with* ${}_aD_q$ *be continuous and integrable on* J *where* $0<q<1$*. If* $\left|{}_aD_qf\right|^r$ *is convex* *function, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\ {}_a d_q x\right| \tag{3.6}$$
$$\leq (b-a)\left(\frac{1}{216}\right)^{\frac{1}{r}}\left(\frac{6q-1}{36(q+1)}\right)^{1-\frac{1}{r}}$$
$$\times\left(\left|{}_aD_q f(b)\right|^r \frac{18q^2+18q-7}{q^3+2q^2+2q+1}+\left|{}_aD_q f(a)\right|^r \frac{36q^3+12q^2+12q+1}{q^3+2q^2+2q+1}\right)^{1/r}$$
$$+(b-a)\left(\frac{1}{216}\right)^{\frac{1}{r}}\left(\frac{5}{36(q+1)}\right)^{1-\frac{1}{r}}$$
$$\times\left(\left|{}_aD_q f(b)\right|^r \frac{18q^2+18q+25}{q^3+2q^2+2q+1}+\left|{}_aD_q f(a)\right|^r \frac{12q^2+12q+5}{q^3+2q^2+2q+1}\right)^{1/r}.$$

Proof. From Lemma 3 and using the well known power mean integral inequality and convexity of $\left|{}_aD_q f\right|^r$, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\ {}_a d_q x\right|$$
$$\leq (b-a)\left[\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|\left|{}_aD_q f(tb+(1-t)a)\right|\ {}_0d_q t\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left|qt-\frac{5}{6}\right|\left|{}_aD_q f(tb+(1-t)a)\right|\ {}_0d_q t\right]$$
$$\leq (b-a)\left(\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|\ {}_0d_q t\right)^{1-\frac{1}{r}}\left(\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|\left|{}_aD_q f(tb+(1-t)a)\right|^r\ {}_0d_q t\right)^{1/r}$$
$$+(b-a)\left(\int_{\frac{1}{2}}^1\left|qt-\frac{5}{6}\right|\ {}_0d_q t\right)^{1-\frac{1}{r}}\left(\int_{\frac{1}{2}}^1\left|qt-\frac{5}{6}\right|\left|{}_aD_q f(tb+(1-t)a)\right|^r\ {}_0d_q t\right)^{1/r}$$
$$\leq (b-a)\left(\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|\ {}_0d_q t\right)^{1-\frac{1}{r}}$$
$$\times\left(\left|{}_aD_q f(b)\right|^r\int_0^{\frac{1}{2}} t\left|qt-\frac{1}{6}\right|\ {}_0d_q t+\left|{}_aD_q f(a)\right|^r\int_0^{\frac{1}{2}}(1-t)\left|qt-\frac{1}{6}\right|\ {}_0d_q t\right)^{1/r}$$
$$+(b-a)\left(\int_{\frac{1}{2}}^1\left|qt-\frac{5}{6}\right|\ {}_0d_q t\right)^{1-\frac{1}{r}}$$

$$\times\left(\left|{}_aD_qf(b)\right|^r\int_{\frac{1}{2}}^1 t\left|qt-\frac{5}{6}\right|{}_0d_qt+\left|{}_aD_qf(a)\right|^r\int_{\frac{1}{2}}^1 (1-t)\left|qt-\frac{5}{6}\right|{}_0d_qt\right)^{1/r}.$$

From Lemmas 4 and 5, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\right|$$
$$\leq (b-a)\left(\frac{1}{36}\frac{6q-1}{q+1}\right)^{1-\frac{1}{r}}$$
$$\times\left(\left|{}_aD_qf(b)\right|^r\frac{1}{216}\frac{18q^2+18q-7}{q^3+2q^2+2q+1}+\left|{}_aD_qf(a)\right|^r\frac{1}{216}\frac{36q^3+12q^2+12q+1}{q^3+2q^2+2q+1}\right)^{1/r}$$
$$+(b-a)\left(\frac{5}{36(q+1)}\right)^{1-\frac{1}{r}}$$
$$\times\left(\left|{}_aD_qf(b)\right|^r\frac{1}{216}\frac{18q^2+18q+25}{q^3+2q^2+2q+1}+\left|{}_aD_qf(a)\right|^r\frac{1}{216}\frac{12q^2+12q+5}{q^3+2q^2+2q+1}\right)^{1/r}.$$

The proof is completed. □

Remark 6. If $q\to 1$, then (3.6) reduces to

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{(b-a)}{216^{\frac{1}{r}}}\left(\frac{5}{72}\right)^{1-\frac{1}{r}}\left\{\left(\frac{29}{6}\left|f'(b)\right|^r+\frac{61}{6}\left|f'(a)\right|^r\right)^{1/r}\right.$$
$$\left.+\left(\frac{29}{6}\left|f'(a)\right|^r+\frac{61}{6}\left|f'(b)\right|^r\right)^{1/r}\right\}.$$

See also [1, Theorem 7] and take $s=1$.

Corollary 3. *In Theorem 2, if $f(a)=f\left(\frac{a+b}{2}\right)=f(b)$, then we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\right|$$
$$\leq (b-a)\left(\frac{1}{216}\right)^{\frac{1}{r}}\left(\frac{6q-1}{36(q+1)}\right)^{1-\frac{1}{r}}$$
$$\times\left(\left|{}_aD_qf(b)\right|^r\frac{18q^2+18q-7}{q^3+2q^2+2q+1}+\left|{}_aD_qf(a)\right|^r\frac{36q^3+12q^2+12q+1}{q^3+2q^2+2q+1}\right)^{1/r}$$
$$+(b-a)\left(\frac{1}{216}\right)^{\frac{1}{r}}\left(\frac{5}{36(q+1)}\right)^{1-\frac{1}{r}}$$

$$\times\left(\left|{}_aD_qf(b)\right|^r\frac{18q^2+18q+25}{q^3+2q^2+2q+1}+\left|{}_aD_qf(a)\right|^r\frac{12q^2+12q+5}{q^3+2q^2+2q+1}\right)^{1/r}.$$

Corollary 4. *In Corollary 3, if $q\to 1$, then we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\le\frac{(b-a)}{216^{\frac{1}{r}}}\left(\frac{5}{72}\right)^{1-\frac{1}{r}}\left\{\left(\frac{29}{6}\left|f'(b)\right|^r+\frac{61}{6}\left|f'(a)\right|^r\right)^{1/r}\right.$$
$$\left.+\left(\frac{29}{6}\left|f'(a)\right|^r+\frac{61}{6}\left|f'(b)\right|^r\right)^{1/r}\right\}.$$

4. Applications

For arbitrary real numbers, we consider the following means:

$$\text{The arithmetic mean}: A(a,b)=\frac{a+b}{2},$$

$$\text{The generalized log-mean}: L_p(a,b)=\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}$$

where $p\in\mathbb{R}\setminus\{-1,0\}$, $a,b\in\mathbb{R}$, $a\neq b$.

We derive some new inequalities for the above means in the following.

Proposition 1. *Let $0<a<b$, $n\in\mathbb{N}$, $0<q<1$, then*

$$\left|\frac{1}{3}A(a^n,b^n)+\frac{2}{3}A^n(a,b)-\frac{(n+1)(1-q)}{1-q^{n+1}}L_n^n(a,b)\right| \tag{4.1}$$
$$\le\frac{(b-a)}{12}\left[\frac{2q^2+2q+1}{q^3+2q^2+2q+1}\frac{b^n-(qb+(1-q)a)^n}{(b-a)(1-q)}+\frac{1}{3}\frac{6q^3+4q^2+4q+1}{q^3+2q^2+2q+1}na^{n-1}\right].$$

Proof. The proof is obvious from Theorem 1 applied $f(x)=x^n$. □

Corollary 5. *Let $0<a<b$, $n\in\mathbb{N}$, $q\to 1$, then (4.1) reduces to*

$$\left|\frac{1}{3}A\left(a^n,b^n\right)+\frac{2}{3}A^n(a,b)-L_n^n(a,b)\right|\le\frac{5(b-a)}{72}n\left[b^{n-1}+a^{n-1}\right].$$

Remark 7. If $q\to 1$ and $n=1$ then (4.1) reduces to

$$|A(a,b)-L(a,b)|\le\frac{5}{72}(b-a).$$

See also [1, Page 13].

Proposition 2. *Let* $0<a<b$, $n\in\mathbb{N}$, $0<q<1$, *then*

$$\left|\frac{1}{3}A\left(a^n,b^n\right)+\frac{2}{3}A^n\left(a,b\right)-\frac{(n+1)(1-q)}{1-q^{n+1}}L_n^n\left(a,b\right)\right| \tag{4.2}$$
$$\leq\frac{(b-a)}{2^{1/r}}\left(\frac{(1-q)}{6^{p+1}q\left(1-q^{p+1}\right)}\right)^{\frac{1}{p}}$$
$$\times\left\{\left(1+(3q-1)^{p+1}\right)^{\frac{1}{p}}\left(\left|na^{n-1}\right|^r+\left|\frac{\left(\frac{a+b}{2}\right)^n-\left(\left(\frac{b-a}{2}\right)q+a\right)^n}{\left(\frac{b-a}{2}\right)(1-q)}\right|^r\right)^{\frac{1}{r}}\right.$$
$$+\left((5-3q)^{p+1}+(6q-5)^{p+1}\right)^{\frac{1}{p}}$$
$$\left.\times\left(\left|\frac{b^n-(qb+(1-q)a)^n}{(b-a)(1-q)}\right|^r+\left|\frac{\left(\frac{a+b}{2}\right)^n-\left(\left(\frac{b-a}{2}\right)q+a\right)^n}{\left(\frac{b-a}{2}\right)(1-q)}\right|^r\right)^{\frac{1}{r}}\right\}.$$

Proof. The proof is obvious from Theorem 2 applied $f(x)=x^n$. □

Corollary 6. *Let* $0<a<b$, $n\in\mathbb{N}$, $q\to 1$, *then* (4.2) *reduces to*

$$\left|\frac{1}{3}A\left(a^n,b^n\right)+\frac{2}{3}A^n\left(a,b\right)-L_n^n\left(b,a\right)\right|$$
$$\leq\frac{(b-a)}{2^{1/r}}n\left(\frac{1}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(1+2^{p+1}\right)^{1/p}$$
$$\times\left\{\left(\left|a^{n-1}\right|^r+\left|\left(\frac{a+b}{2}\right)^{n-1}\right|^r\right)^{\frac{1}{r}}+\left(\left|b^{n-1}\right|^r+\left|\left(\frac{a+b}{2}\right)^{n-1}\right|^r\right)^{\frac{1}{r}}\right\}.$$

Proposition 3. *Let* $0<a<b$, $n\in\mathbb{N}$, $0<q<1$, *then*

$$\left|\frac{1}{3}A\left(a^n,b^n\right)+\frac{2}{3}A^n\left(a,b\right)-\frac{(n+1)(1-q)}{1-q^{n+1}}L_n^n\left(a,b\right)\right| \tag{4.3}$$
$$\leq(b-a)\left(\frac{1}{216}\right)^{\frac{1}{r}}\left(\frac{6q-1}{36(q+1)}\right)^{1-\frac{1}{r}}$$
$$\times\left(\left|\frac{b^n-(qb+(1-q)a)^n}{(b-a)(1-q)}\right|^r\frac{18q^2+18q-7}{q^3+2q^2+2q+1}+\left|na^{n-1}\right|^r\frac{36q^3+12q^2+12q+1}{q^3+2q^2+2q+1}\right)^{1/r}$$
$$+(b-a)\left(\frac{1}{216}\right)^{\frac{1}{r}}\left(\frac{5}{36(q+1)}\right)^{1-\frac{1}{r}}$$

$$\times\left(\left|\frac{b^n-(qb+(1-q)a)^n}{(b-a)(1-q)}\right|^r \frac{18q^2+18q+25}{q^3+2q^2+2q+1}+\left|na^{n-1}\right|^r \frac{12q^2+12q+5}{q^3+2q^2+2q+1}\right)^{1/r}.$$

Proof. The proof is obvious from Theorem 3 applied $f(x)=x^n$. □

Corollary 7. *Let* $0<a<b$*,* $n\in\mathbb{N}$*,* $q\to 1$*, then (4.3) reduces to*

$$\begin{aligned}&\left|\frac{1}{3}A\left(a^n,b^n\right)+\frac{2}{3}A^n(a,b)-L_n^n(b,a)\right|\\&\leq\frac{n(b-a)}{(216)^{\frac{1}{r}}}\left(\frac{5}{72}\right)^{1-\frac{1}{r}}\\&\times\left\{\left(\frac{29}{6}\left|b^{n-1}\right|^r+\frac{61}{6}\left|a^{n-1}\right|^r\right)^{1/r}+\left(\frac{61}{6}\left|b^{n-1}\right|^r+\frac{29}{6}\left|a^{n-1}\right|^r\right)^{1/r}\right\}.\end{aligned}$$

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