ON THE BARR EXACTNESS PROPERTY OF BXMOD/R

H. GÜLSÜN AKAY AND U. EGE ARSLAN

Abstract. In this work, it is shown that the category BXMod/R of braided crossed modules over a fixed commutative algebra R is an exact category in the sense of Barr.
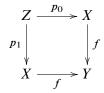
1. INTRODUCTION

Exact categories for additive categories were introduced by Quillen [9]. Barr defined exact categories for non-additive categories [1]. He introduced exact categories in order to define a good notion of non-abelian cohomology.

We will mention regular categories closely related by exact categories. There exist in the literature many different definitions of regular categories, which are all equivalent under the assumptions that finite limits and coequalisers exist. We will recall the following definition from [7], but a weaker version of it is given in [1].

A category C is called regular if it satisfies the following three properties:

i: C is finitely complete,
ii: If $f : X \longrightarrow Y$ is a morphism in C, and

$$
\begin{array}{ccc}
Z & \xrightarrow{p_0} & X \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{f} & Y
\end{array}
$$

is a pullback (then $Z \underset{p_1}{\overset{p_0}{\rightrightarrows}} X$ is called the kernel pair of f), the coequaliser of p_0, p_1 exists;

---

iii: If $f : X \longrightarrow Y$ is a morphism in C, and

$$
\begin{array}{ccc}
W & \longrightarrow & X \\
g \downarrow & & \downarrow f \\
Z & \longrightarrow & Y
\end{array}
$$

is a pullback, and if f is a regular epimorphism, then g is a regular epimorphism as well.

If a regular category additionally has the property that every equivalence relation is effective that is every equivalence relation is a kernel pair, then it is called a Barr exact category.

Examples of exact categories are:

(1) The category Set of sets.
(2) The category of non empty sets.
(3) Any abelian category.
(4) Every partially ordered set considered as a category.
(5) For any small category C, the functor category $(\mathcal{C}^{op}, Set)$.

Brown and Gilbert introduced in [3] the notion of braided regular crossed module of groupoids and groups as an algebraic model for homotopy 3-types equivalent to Conduché's 2-crossed module. The reduced case of braided regular crossed module is called a braided crossed module of groups. Braided crossed modules in the category of commutative algebras were defined by Ulualan in [10].

The purpose of this paper is to answer the question whether the category of braided crossed modules of commutative algebras is exact. We prove that the category of braided crossed modules of commutative algebras is an exact category.

Throughout this paper k will be a fixed commutative ring with $0 \neq 1$. All k-algebras will be commutative and associative.

2. BRAIDED CROSSED MODULES

In order to give the notion of braided crossed modules of commutative algebras, we will recall the concept of crossed modules of commutative algebras. Crossed modules of groups originate in algebraic topology and more particularly in homotopy theory. Mac Lane and Whitehead showed in [6] that crossed modules of groups modelled homotopy 2-types (3-types in their notation). The commutative algebra case of crossed modules is contained in the paper of Lichtenbaum and Schlessinger [5] and also in the work of Gerstenhaber [4] under different names. Some categorical results and Koszul complex link are also given by Porter [8].

Definition 1 ([8])**.** A crossed module of commutative algebras, (C, R, ∂), is an R-algebra, C, together with an R-algebra morphism $\partial : C \longrightarrow R$ such that for all

$c, c' \in C$

$$\partial(c) \cdot c' = cc'$$

where R is a k-algebra. A morphism of crossed modules from (C, R, ∂) to (C', R', ∂') is a pair of k-algebra morphisms, $\phi : C \longrightarrow C'$ and $\psi : R \longrightarrow R'$ such that

$$(i)\ \partial'\phi = \psi\partial \quad \text{and} \quad (ii)\ \phi(r \cdot c) = \psi(r) \cdot \phi(c)$$

for all $r \in R$ and $c \in C$. We thus get the category XMod of crossed modules.

There is, for a fixed algebra R, a subcategory XMod/R of the category of crossed modules, which has as objects those crossed modules with R as the "base", i.e., all (C, R, ∂) for this fixed R, and having as morphisms from (C, R, ∂_1) to (C', R, ∂_2) just those (f_1, f_0) in XMod in which $f_0 : R \longrightarrow R$ is the identity homomorphism on R.

Definition 2 ([10]). A braided crossed module of commutative algebras $\partial : C \to R$ is a crossed module with the braiding function $\{-, -\} : R \times R \longrightarrow C$ satisfying the following axioms:

$BCM1)\ \partial\{r, r'\} = rr'$,
$BCM2)\ \{\partial c, \partial c'\} = cc'$,
$BCM3)\ \{\partial c, r\} = r \cdot c$ and $\{r, \partial c\} = r \cdot c$,
$BCM4)\ \{rr', r''\} - \{r, r'r''\} = 0$,

for all $r, r', r'' \in R$ and $c, c' \in C$.

We denote such a braided crossed module of commutative algebras by $\{C, R, \partial\}$.

If $\{C, R, \partial_1\}$ and $\{C', R', \partial_2\}$ are braided crossed modules, a morphism

$$(f_1, f_0) : \{C, R, \partial_1\} \longrightarrow \{C', R', \partial_2\},$$

of braided crossed modules is given by a morphism of crossed modules such that

$$\{-, -\}(f_0 \times f_0) = f_1\{-, -\}.$$

We thus get the category BXMod of braided crossed modules of commutative algebras.

In the case of a morphism (f_1, f_0) between braided crossed modules with the same base R, i.e. where f_0 is the identity on R with $f_1\{-, -\} = \{-, -\}$, then we say that f_1 is a morphism of braided crossed R-modules, $\partial_1 : C \longrightarrow R$ is a braided crossed R-module and we use $\{C, \partial_1\}$ instead of $\{C, R, \partial_1\}$. This gives a subcategory BXMod/R of BXMod. Our results are obtained for this subcategory.

Several well known examples of crossed modules give rise to braided crossed modules as follows.

Examples of braided crossed modules

(1) Any identity map of k-algebras $\partial : X \longrightarrow X$ is a braided crossed module with $\{x, y\} = xy$.

(2) If C is a k-algebra and C^2 is an ideal generated by $\{c_1 c_2 \mid c_1, c_2 \in C\}$. Then $\partial : C^2 \longrightarrow C$ is a braided crossed module with $\{c_1, c_2\} = c_1 c_2$, for $c_1, c_2 \in C$.

(3) Any R-module M can be considered as an R-algebra with zero multiplication and hence the zero morphism $0 : M \longrightarrow R$ is a braided crossed module with $\{r, r'\} = 0$.
(4) Let $\{C, R, \partial_1\}$ and $\{C', R', \partial_2\}$ be two braided crossed modules, then

$$\{C \times C', R \times R', \partial\}$$

is a braided crossed module.

3. THE BARR EXACTNESS PROPERTY OF BRAIDED CROSSED MODULES

Our aim now is to obtain that BXMod/R is an exact category. For this purpose, we have to prove some statements in this section.

Proposition 1. *In* BXMod/R *every pair of morphisms with common domain and codomain has an equaliser.*

Proof. Let $f, g : \{C, \partial\} \longrightarrow \{D, \delta\}$ be two morphisms of braided crossed R-modules. Let E denotes the set $E = \{c \in C \mid f(c) = g(c)\}$. It can be easily checked that $\{E, \varepsilon\}$ has the structure of a braided crossed R-module, and the inclusion $u : \{E, \varepsilon\} \longrightarrow \{C, \partial\}$ is a morphism of braided crossed R-modules and clearly $fu = gu$.
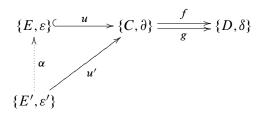
Suppose that there exist a braided crossed R-module, $\{E', \varepsilon'\}$ and a morphism $u' : \{E', \varepsilon'\} \longrightarrow \{C, \partial\}$ of braided crossed R-modules such that $fu' = gu'$. Then for all $x \in E'$, $f(u'(x)) = g(u'(x))$, and hence $u'(x) \in E$. Thus, we have $\alpha : \{E', \varepsilon'\} \longrightarrow \{E, \varepsilon\}$ by $\alpha(x) = u'(x)$ from which we get

$$\varepsilon\alpha(x) = \varepsilon u'(x) = \partial u'(x) = \varepsilon'(x),$$

$$\alpha(r \cdot x) = u'(r \cdot x) = r \cdot u'(x) = r \cdot \alpha(x),$$

for all $x \in E', r \in R$. Since u, u' are braided crossed R-module morphisms, it is clear that $\alpha\{r, r'\} = \{r, r'\}$ for all $r, r' \in R$.

Let $\alpha' : E' \longrightarrow E$ be a morphism of braided crossed R-module such that $u\alpha' = u'$. Since $\alpha(x) = u'(x) = u\alpha'(x) = \alpha'(x)$ for all $x \in E'$, we have that α is the unique morphism which makes the diagram

$$\{E, \varepsilon\} \overset{u}{\hookrightarrow} \{C, \partial\} \underset{g}{\overset{f}{\rightrightarrows}} \{D, \delta\}$$

$$\alpha \uparrow \quad \nearrow u'$$

$$\{E', \varepsilon'\}$$

commutative. Hence u is the equaliser of (f, g), as required. $\qquad\square$

Proposition 2. BXMod/R *has finite products.*

Proof. Let $\{C,\partial\}$ and $\{D,\delta\}$ be braided crossed R-modules. The product $\{C \sqcap D, \tau\}$ is the pullback over the terminal object $\{R, i_R\}$,

$$
\begin{array}{ccc}
\{C \sqcap D, \tau\} & \xrightarrow{\ \partial'\ } & \{C, \partial\} \\
{\scriptstyle \delta'}\Big\downarrow & & \Big\downarrow{\scriptstyle \partial} \\
\{D, \delta\} & \xrightarrow[\ \delta\ ]{} & \{R, i_R\}
\end{array}
$$

where $C \sqcap D = \{(c,d) \mid \partial(c) = \delta(d)\}$ and $\tau : C \sqcap D \longrightarrow R$ is defined by $\tau = \partial\partial' = \delta\delta'$. Then by induction, BXMod/R has finite products. $\square$

Proposition 3. BXMod/R *is finitely complete.*

Proof. Follows from Propositions 1 and 2. $\square$

Proposition 4. *In* BXMod/R *every morphism has a kernel pair, and the kernel pair has a coequaliser.*

Proof. Let $\{A, \partial\}$ and $\{B, \beta\}$ be two braided crossed R-modules. Let $f : \{A, \partial\} \longrightarrow \{B, \beta\}$ be a morphism of braided crossed R-modules. Then (A, f) is a crossed B-module, where B acts on A via β and the homomorphism $\alpha : A \times_B A \longrightarrow B$ defined by $\alpha(a, a') = f(a) = f(a')$ is a crossed B-module, where

$$
A \times_B A = \{(a, a') \mid f(a) = f(a')\}
$$

is a B-algebra:

$$
\alpha\big(b \cdot (a, a')\big) = f(b \cdot a) = b \cdot f(a) = b \cdot \alpha(a, a'),
$$

$$
\alpha(a, a') \cdot (a_1, a_1') = \big(f(a) \cdot a_1, f(a') \cdot a_1'\big) = (a, a')(a_1, a_1'),
$$

for all $(a, a'), (a_1, a_1') \in A \times_B A, b \in B$.

We can define $\partial' : A \times_B A \longrightarrow R$ by $\partial'(a, a') = \partial(a) = \partial(a')$, since $\beta\alpha = \beta f = \partial$. It is easily checked that $(A \times_B A, R, \partial')$ is a crossed module:

$$
\partial'\big(r \cdot (a, a')\big) = \partial(r \cdot a) = r \cdot \partial(a) = r \cdot \partial'(a, a'),
$$

$$
\begin{aligned}
\partial'(a, a') \cdot (a_1, a_1') &= \partial(a) \cdot (a_1, a_1') \\
&= \big(\partial(a) \cdot a_1, \partial(a) \cdot a_1'\big) \\
&= \big(\partial(a) \cdot a_1, \partial(a') \cdot a_1'\big) \\
&= (a a_1, a' a_1') \\
&= (a, a')(a_1, a_1'),
\end{aligned}
$$

for all $(a,a'),(a_1,a_1') \in A \times_B A, r \in R$.

Below we will show that $\{A \times_B A, \partial'\}$ is a braided crossed R-module with the braiding map

$$\{-,-\} : R \times R \longrightarrow A \times_B A$$

defined by $\{r,r'\} = (\{r,r'\}, \{r,r'\})$,

BCM1)

$$\partial'\{r,r'\} = \partial\{r,r'\} = rr',$$

BCM2)

$$\{\partial'(a,a'), \partial'(b,b')\} = (\{\partial a, \partial b\}, \{\partial a', \partial b'\})$$
$$= (ab, a'b')$$
$$= (a,a')(b,b'),$$

BCM3)

$$\{\partial'(a,a'), r\} = (\{\partial a, r\}, \{\partial a', r\})$$
$$= (r \cdot a, r \cdot a')$$
$$= r \cdot (a,a')$$
$$\{r, \partial'(a,a')\} = (\{r, \partial a\}, \{r, \partial a'\})$$
$$= (r \cdot a, r \cdot a')$$
$$= r \cdot (a,a'),$$

BCM4)

$$\{rr', r''\} - \{r, r'r''\} = (\{rr', r''\}, \{rr', r''\}) - (\{r, r'r''\}, \{r, r'r''\})$$
$$= (\{rr', r''\} - \{r, r'r''\}, \{rr', r''\} - \{r, r'r''\})$$
$$= (0,0),$$

for all $(a,a'),(b,b') \in A \times_B A, r,r',r'' \in R$.

The following diagram

$$
\begin{array}{ccccc}
A \times_B A & \underset{p_2}{\overset{p_1}{\rightrightarrows}} & A & \xrightarrow{f} & B \\
\partial' \downarrow & & \partial \downarrow & & \downarrow \beta \\
R & =\!=\!= & R & =\!=\!= & R
\end{array}
$$

commutes and the morphisms p_1 and p_2 above are morphisms of braided crossed R-modules. This construction satisfies universal property: Let $\{E, \delta\}$ be a braided

crossed R-module and p_1', $p_2' : \{E, \delta\} \longrightarrow \{A, \partial\}$ be any morphisms of braided crossed R-modules with $fp_1' = fp_2'$, then there exist a unique morphism

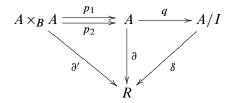$$h : \{E, \delta\} \longrightarrow \{A \times_B A, \partial'\}$$

given by $h(e) = \left(p_1'(e), p_2'(e)\right)$, for all $e \in E$, which makes the diagram
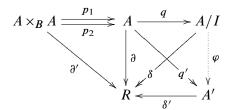

commutative.

Then (p_1, p_2) is the kernel pair of the morphism f.

Now we will show that the pair (p_1, p_2) has a coequaliser. Let I be an ideal of A generated by all the elements of the form $p_1(x) - p_2(x)$, for all $x = (a, a') \in A \times_B A$. We will define the braided crossed R-module $\delta : A/I \longrightarrow R$ by $\delta(a + I) = \partial(a)$, $\{r, r'\} = \{r, r'\} + I$ for $a \in A$, $r, r' \in R$. Since $I \subseteq Ker \partial$, δ is well defined. By the definition of $\{A/I, \delta\}$, the morphism $q : \{A, \partial\} \longrightarrow \{A/I, \delta\}$ is the induced projection and it is a morphism of braided crossed R-modules, i.e., the diagram

commutes. Suppose there exist a braided crossed R-module $\{A', \alpha'\}$ and a morphism of braided crossed R-modules $q' : \{A, \partial\} \longrightarrow \{A', \alpha'\}$ such that $q'p_1 = q'p_2$, then there exists a unique morphism $\varphi : \{A/I, \alpha\} \longrightarrow \{A', \alpha'\}$ defined by $\varphi(a + I) = q'(a)$, satisfying $\varphi q = q'$, i.e. q is the universal among all the morphisms q' :

$\{A,\partial\} \longrightarrow \{A',\alpha'\}$ for any braided crossed R-module $\{A',\alpha'\}$. So we get the following commutative diagram:
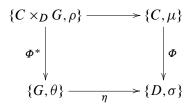
$$
\begin{array}{ccccc}
A \times_B A & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & A & \overset{q}{\longrightarrow} & A/I \\
& \partial' \searrow & \downarrow \partial & \delta \swarrow \quad q' \searrow & \downarrow \varphi \\
& & R & \overset{}{\underset{\delta'}{\longleftarrow}} & A'
\end{array}
$$

Then q is the coequaliser of the pair (p_1, p_2).

Therefore in BXMod/R, the kernel pair of every morphism exists and has a coequaliser. □

Proposition 5. *In* BXMod/R *every regular epimorphisms are stable under pullback.*

Proof. Let $\Phi : \{C,\mu\} \longrightarrow \{D,\sigma\}$ be a regular epimorphism in BXMod/R, which means that $C \longrightarrow D$ is a regular epimorphism of k-algebras. Let $\eta : \{G,\theta\} \longrightarrow \{D,\sigma\}$ be any morphism in BXMod/R. The pullback of Φ along η is
$\Phi^* : \{C \times_D G, \rho\} \longrightarrow \{G,\theta\}$ given by $\Phi^*(c,g) = g$
where $C \times_D G = \{(c,g) \mid \Phi(c) = \eta(g)\}$ and $\rho : C \times_D G \longrightarrow R$ is a braided crossed R-module defined by $\rho(c,g) = \mu(c) = \theta(g)$ with $\{r,r'\} = (\{r,r'\},\{r,r'\})$, for all $r,r' \in R$, $c \in C$, $g \in G$. Since in the category of k-algebras the regular epimorphisms are characterised as the surjective homomorphisms, these are closed in this way under pullback. Thus Φ^* is a surjective homomorphism.

$$
\begin{array}{ccc}
\{C \times_D G, \rho\} & \longrightarrow & \{C,\mu\} \\
\Phi^* \downarrow & & \downarrow \Phi \\
\{G,\theta\} & \overset{\eta}{\longrightarrow} & \{D,\sigma\}
\end{array}
$$

We claim that the surjective morphism Φ^* is a regular epimorphism, that is Φ^* is the coequaliser of a pair of morphisms. Define

$$E = \big\{(x,y) \in (C \times_D G) \times (C \times_D G) \mid \Phi^*(x) = \Phi^*(y)\big\}$$

and let $\{E,\alpha\} \overset{p}{\underset{q}{\rightrightarrows}} \{C \times_D G, \rho\}$ be the first and second projections. Since G is isomorphic to the quotient of $C \times_D G$, Φ^* is the coequaliser of p and q. Thus we get that if Φ is regular epimorphism, so is Φ^*, as required. □

Theorem 1. BXMod/R *is regular.*

Proof. The proof is a direct consequence of Propositions 3, 4 and 5. □

Now we recall the definition of equivalence relation from [2].

Definition 3. Let A be a category with finite limits. If A is an object, a subobject $(d^0, d^1) : E \longrightarrow A \times A$ is called an equivalence relation if it is

ER1. Reflexive: There is an arrow $r : A \longrightarrow E$ such that $d^0 r = d^1 r = id_A$;

ER2. Symmetric: There is an arrow $s : E \longrightarrow E$ such that $d^0 s = d^1$ and $d^1 s = d^0$;

ER3. Transitive: If

$$
\begin{array}{ccc}
T & \xrightarrow{\ q_1\ } & E \\
{\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle d^0} \\
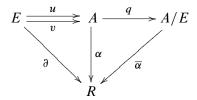E & \xrightarrow{\ d^1\ } & A
\end{array}
$$

is a pullback, there is an arrow $t : T \longrightarrow E$ such that $d^1 t = d^1 q_1$ and $d^0 t = d^0 q_2$.

Proposition 6. *Every equivalence relation*

$$
\{E, \partial\} \underset{v}{\overset{u}{\rightrightarrows}} \{A, \alpha\}
$$

in the category of braided crossed R-modules is effective.
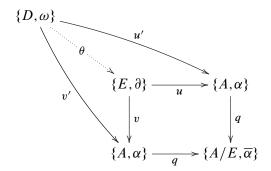
Proof. Let A/E be the set of all equivalence classes $[a]$ with respect to E, i.e., $[a] = \{b \in A \mid (a, b) \in E\}$. A/E has the structure of an R-algebra. $\overline{\alpha} : A/E \longrightarrow R$ induced by α is well defined since $\alpha u = \alpha v$. We can form the braided crossed R-module $\overline{\alpha}$ with braiding map $\{r, r'\} = [\{r, r'\}]$ and get the following diagram,

$$
\begin{array}{ccc}
E \underset{v}{\overset{u}{\rightrightarrows}} A & \xrightarrow{\ q\ } & A/E \\
{\scriptstyle \partial}\searrow \quad \downarrow{\scriptstyle \alpha} & \swarrow{\scriptstyle \overline{\alpha}} & \\
R & &
\end{array}
$$

of morphisms of braided crossed R-modules, where q is the projection onto A/E. By the definition of an equivalence relation on A, we have $(u(x), v(x)) \in E$, for $x \in E$. Since

$$
E \subseteq A \times_R A = \left\{ (a, a') \mid \alpha(a) = \alpha(a') \right\},
$$

we get $\alpha u(x) = \alpha v(x)$, thus $(0, u(x) - v(x)) \in E$, therefore $qu = qv$. Suppose that there exist a braided crossed R-module $\{D, \omega\}$ with $u', v' : \{D, \omega\} \longrightarrow \{A, \alpha\}$ such that $qu' = qv'$, so $[u'(d)] = [v'(d)]$, i.e. $(u'(d), v'(d)) \in E$, and therefore there

exists a unique morphism $\theta : \{D,\omega\} \longrightarrow \{E,\partial\}$, such that the diagram

$$
\begin{array}{ccc}
\{D,\omega\} & & \\
& \searrow^{\,u'} & \\
\theta \searrow & \{E,\partial\} \xrightarrow{\ u\ } \{A,\alpha\} & \\
v' \searrow & \downarrow^{v} \qquad\qquad \downarrow^{q} & \\
& \{A,\alpha\} \xrightarrow{\ q\ } \{A/E,\overline{\alpha}\} &
\end{array}
$$

commutes.

Thus (u,v) is the kernel pair of a morphism $q : \{A,\alpha\} \longrightarrow \{A/E,\overline{\alpha}\}$ in the category of braided crossed R-modules. $\qquad\square$

Theorem 2. *The category* BXMod/R *of braided crossed R-modules is a Barr exact category.*

Proof. The conditions of exact category for BXMod/R are satisfied by Theorem 1 and Proposition 6 , which completes the proof. $\qquad\square$

REFERENCES

[1] M. Barr, P. Grillet, and D. Van Osdol, *Exact categories and categories of sheaves*, ser. Lecture notes in mathematics. Springer-Verlag, 1971.

[2] M. Barr and C. Wells, "Toposes, triples and theories." *Repr. Theory Appl. Categ.*, vol. 2005, no. 12, pp. 1–288, 2005.

[3] R. Brown and N. Gilbert, "Algebraic models of 3-types and automorphism structures for crossed modules." *Proc. Lond. Math. Soc. (3)*, vol. 59, no. 1, pp. 51–73, 1989.

[4] M. Gerstenhaber, "On the deformation of rings and algebras," *Annals of Mathematics*, vol. 79, no. 1, pp. 59–103, 1964.

[5] S. Lichtenbaum and M. Schlessinger, "The cotangent complex of a morphism," *Transactions of the American Mathematical Society*, vol. 128, no. 1, pp. 41–70, 1967.

[6] S. MacLane and J. H. C. Whitehead, "On the 3-type of a complex," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 36, no. 1, pp. 41–48, 1950.

[7] M. Pedicchio and W. Tholen, *Categorical Foundations: Special Topics in Order, Topology, Algebra, and Sheaf Theory*, ser. Categorical Foundations: Special Topics in Order, Topology, Algebra, and Sheaf Theory. Cambridge University Press, 2004.

[8] T. Porter, "Some categorical results in the theory of crossed modules in commutative algebras," *Journal of Algebra*, vol. 109, no. 2, pp. 415 – 429, 1987.

[9] D. Quillen, *Higher algebraic K-theory: I.* Berlin, Heidelberg: Springer Berlin Heidelberg, 1973, pp. 85–147.

[10] E. Ulualan, "Değişmeli Cebirler Üzerinde Kuadratik Modüller," Ph.D. dissertation, ESOGU FBE, Turkey, 2004.

Authors' addresses

H. Gülsün Akay

Eskişehir Osmangazi University, Department of Mathematics and Computer Science, 26480 Eskişehir, Turkey

E-mail address: `hgulsun@ogu.edu.tr`

U. Ege Arslan

Eskişehir Osmangazi University, Department of Mathematics and Computer Science, 26480 Eskişehir, Turkey

E-mail address: `uege@ogu.edu.tr`