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ON K-CIRCULANT MATRICES INVOLVING THE FIBONACCI NUMBERS

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Abstract. Let k be a nonzero complex number. In this paper we consider a k-circulant matrix whose first row is $(F_1, F_2, \dots, F_n)$, where F_n is the n^{th} Fibonacci number, and investigate the eigenvalues and Euclidean (or Frobenius) norm of that matrix. Also, the upper and lower bounds for the spectral norm of the Hadamard inverse of that matrix are obtained.

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1. INTRODUCTION

In this paper, k is a nonzero complex number and $\mathbb{C}^{n\times n}$ denotes the set of all complex matrices of order n. Any n^{th} root of k and any primitive n^{th} root of unity are denoted by ψ and ω, respectively. Symbols λ_j, $j=\overline{0,n-1}$, $|C|$, $\|C\|_E$, $\|C\|_2$ and $C^{\circ -1}$ stand for the eigenvalues, the determinant, the Euclidean norm, the spectral norm and the Hadamard inverse of $C\in\mathbb{C}^{n\times n}$, respectively. Namely, for $C=\left[c_{i,j}\right]\in\mathbb{C}^{n\times n}$, $\|C\|_E=(\sum_{i,j=1}^{n}|c_{i,j}|^2)^{\frac{1}{2}}$, $\|C\|_2=\sqrt{\max_{1\le i\le n}\lambda_i(C^*C)}$, where C^* is the conjugate transpose of C, and $C^{\circ -1}=\left[c_{i,j}^{-1}\right]$.

Definition 1. A matrix C of order n with the first row $(c_0, c_1, c_2, \dots, c_{n-1})$ is called a *k-circulant matrix* if C has the following form:

$$C=\begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ kc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ kc_{n-2} & kc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ kc_2 & kc_3 & kc_4 & \cdots & c_0 & c_1 \\ kc_1 & kc_2 & kc_3 & \cdots & kc_{n-1} & c_0 \end{bmatrix}. \tag{1.1}$$

We shall write $C = circ_n\{_k(c_0, c_1, c_2, \ldots, c_{n-1})\}$ if a matrix C has the form (1.1). The designation for the order of a matrix can be omitted if the dimension of a matrix is known. Circulant matrices are k-circulant matrices for $k = 1$ and skew circulant matrices are k-circulant matrices for $k = -1$.

The Fibonacci numbers $\{F_n\}$ satisfy the following recursive relation:

$$F_n = F_{n-2} + F_{n-1}, \quad n \geq 2, \tag{1.2}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$.

Let α and β be the roots of the equation $x^2 - x - 1 = 0$ i.e.

$$\alpha = \frac{1+\sqrt{5}}{2},\ \beta = \frac{1-\sqrt{5}}{2},\ \alpha\beta = -1,\ \alpha + \beta = 1 \quad \text{and} \quad \alpha - \beta = \sqrt{5}. \tag{1.3}$$

Binet's formula for the Fibonacci numbers is:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right). \tag{1.4}$$

Let us mention that *the Lucas numbers* $\{L_n\}$ satisfy the same recursive relation (as the Fibonacci numbers) but with initial conditions $L_0 = 2$ and $L_1 = 1$ and

$$L_n = \alpha^n + \beta^n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \tag{1.5}$$

is *Binet's formula* for the Lucas numbers.

The following identities hold for the Fibonacci numbers:

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1 \quad \text{and} \quad \sum_{i=1}^{n} F_i^2 = F_n F_{n+1} \left[= \frac{L_{2n+1} - (-1)^n}{5} \right]. \tag{1.6}$$

More information about these numbers can be found in [2, 3, 8–10, 12, 14, 15].

In [4, 13] the authors investigated the determinants and inverses of circulant (of skew circulant) matrices whose first rows are $(F_1, F_2, \ldots, F_n)$ and $(L_1, L_2, \ldots, L_n)$. The paper [5] is devoted to k-circulant matrices with the Fibonacci and Lucas numbers, and an upper bound estimation of the spectral norm for such matrices was given in that paper.

The motivation for this paper is the paper [17] in which the authors investigated k-circulant matrices with the generalized r-Horadam numbers $\{H_{r,n}\}$ which are defined as follows:

$$H_{r,n+2} = f(r)H_{r,n+1} + g(r)H_{r,n}, \quad n \geq 0,$$

where $r \in \mathbb{R}^+$, $H_{r,0} = a$, $H_{r,1} = b$, $a, b \in \mathbb{R}$ and $f^2(r) + 4g(r) > 0$, and presented the upper and lower bounds for the spectral norms of such matrices.

Theorem 1 (Theorem 5. [17]). *Let* $H = circ\{_k(H_{r,0}, H_{r,1}, \ldots, H_{r,n-1})\}$.

a) If $|k| \geq 1$, *then*

$$\sqrt{\sum_{i=0}^{n-1} H_{r,i}^2} \leq \|H\|_2 \leq \sqrt{\left(a^2(1-|k|^2)+|k|^2\sum_{i=0}^{n-1} H_{r,i}^2\right)\left(1-a^2+\sum_{i=0}^{n-1} H_{r,i}^2\right)} \quad (1.7)$$

b) If $|k| < 1$, *then*

$$|k|\sqrt{\sum_{i=0}^{n-1} H_{r,i}^2} \leq \|H\|_2 \leq \sqrt{n\sum_{i=0}^{n-1} H_{r,i}^2}\,. \quad (1.8)$$

Also, in [17], the formulae for the eigenvalues and determinant of a k-circulant matrix with the generalized r-Horadam numbers were derived.

Theorem 2 (Theorem 7. [17]). *Let* $H = circ\{_k(H_{r,0}, H_{r,1}, \ldots, H_{r,n-1})\}$. *Then the eigenvalues of* H *are:*

$$\lambda_j = \frac{kH_{r,n} + (g(r)kH_{r,n-1} - b + af(r))\psi\omega^{-j} - H_{r,0}}{g(r)(\psi\omega^{-j})^2 + f(r)\psi\omega^{-j} - 1}, \; j = \overline{0, n-1}. \quad (1.9)$$

Theorem 3 (Theorem 8. [17]). *Let* $H = circ\{_k(H_{r,0}, H_{r,1}, \ldots, H_{r,n-1})\}$. *Then the determinant of* H *is:*

$$|H| = \frac{(H_{r,0} - kH_{r,n})^n - (g(r)kH_{r,n-1} - b + af(r))^n k}{(1-k\alpha^n)(1-k\beta^n)}, \quad (1.10)$$

where α *and* β *are the roots of the equation* $x^2 - f(r)x - g(r) = 0$.

Let us also mention that, in [16], the authors considered circulant matrices with the generalized r-Horadam numbers and obtained the determinants and inverses of such matrices.

In the present paper, we consider the matrix

$$F = circ\{_k(F_1, F_2, \ldots, F_n)\} \quad (1.11)$$

and improve the result in relation to the eigenvalues of (1.11) which can be obtained from (1.9) because the authors did not consider the case when the denominator is equal to zero. Also, we determine the Euclidean norm of (1.11) and derive the upper and lower bounds for the spectral norm of the Hadamard inverse of (1.11). The results are presented in the next section.

2. Results

Let us recall that ψ is any n^{th} root of k and ω is any primitive n^{th} root of unity. Also, throughout this section, α and β are the roots of the equation $x^2-x-1=0$. In order to obtain the eigenvalues of (1.11), we need the following lemma.

Lemma 1 (Lemma 4. [1])**.** *The eigenvalues of $C=circ\{_k(c_0,c_1,c_2,\dots,c_{n-1})\}$ are:*

$$\lambda_j=\sum_{i=0}^{n-1}c_i(\psi\omega^{-j})^i,\ j=\overline{0,n-1}. \tag{2.1}$$

Moreover, in this case:

$$c_i=\frac{1}{n}\sum_{j=0}^{n-1}\lambda_j(\psi\omega^{-j})^{-i},\ i=\overline{0,n-1}. \tag{2.2}$$

Theorem 4. *Let F be the matrix as in (1.11). Then the eigenvalues of F are given by the following formulae:*

1) If $\psi\omega^{-j}=\frac{1}{\alpha}$, then

$$\lambda_j=\frac{1}{\sqrt{5}}\left[n\alpha+\frac{1-(-1)^n\beta^{2n}}{\sqrt{5}}\right], \tag{2.3}$$

2) If $\psi\omega^{-j}=\frac{1}{\beta}$, then

$$\lambda_j=\frac{1}{\sqrt{5}}\left[\frac{1-(-1)^n\alpha^{2n}}{\sqrt{5}}-n\beta\right], \tag{2.4}$$

3) If $\psi\omega^{-j}\neq\frac{1}{\alpha}$ and $\psi\omega^{-j}\neq\frac{1}{\beta}$, then

$$\lambda_j=\frac{kF_{n+1}-1+kF_n\psi\omega^{-j}}{(\psi\omega^{-j})^2+\psi\omega^{-j}-1}. \tag{2.5}$$

Proof. Based on Lemma 1 and (1.4), it follows:

1) Suppose that $\psi\omega^{-j}=\frac{1}{\alpha}$. Then,

$$\begin{aligned}\lambda_j&=\sum_{i=0}^{n-1}F_{i+1}(\psi\omega^{-j})^i=\frac{1}{\sqrt{5}}\sum_{i=0}^{n-1}\left[\alpha^{i+1}-\beta^{i+1}\right](\frac{1}{\alpha})^i\\&=\frac{1}{\sqrt{5}}\left[\alpha\sum_{i=0}^{n-1}1-\beta\sum_{i=0}^{n-1}(\frac{\beta}{\alpha})^i\right]=\frac{1}{\sqrt{5}}\left[n\alpha-\beta\frac{1-(\frac{\beta}{\alpha})^n}{1-\frac{\beta}{\alpha}}\right]\\&=\frac{1}{\sqrt{5}}\left[n\alpha+\frac{1-(-1)^n\beta^{2n}}{\sqrt{5}}\right],\end{aligned}$$

2) Suppose that $\psi\omega^{-j} = \frac{1}{\beta}$. Then,

$$\begin{aligned}
\lambda_j &= \sum_{i=0}^{n-1} F_{i+1}(\psi\omega^{-j})^i = \frac{1}{\sqrt{5}}\sum_{i=0}^{n-1}\left[\alpha^{i+1}-\beta^{i+1}\right](\frac{1}{\beta})^i \\
&= \frac{1}{\sqrt{5}}\left[\alpha\sum_{i=0}^{n-1}(\frac{\alpha}{\beta})^i - \beta\sum_{i=0}^{n-1}1\right] = \frac{1}{\sqrt{5}}\left[\alpha\frac{1-(\frac{\alpha}{\beta})^n}{1-\frac{\alpha}{\beta}} - n\beta\right] \\
&= \frac{1}{\sqrt{5}}\left[\frac{1-(-1)^n\alpha^{2n}}{\sqrt{5}} - n\beta\right],
\end{aligned}$$

3) Suppose that $\psi\omega^{-j} \neq \frac{1}{\alpha}$ and $\psi\omega^{-j} \neq \frac{1}{\beta}$. Then, λ_j follows from (1.9). □

The previously obtained result will be illustrated by the following example.

Example 1. Let

$$F = circ\{_{9-4\sqrt{5}}(1,1,2,3,5,8)\}$$

i.e.

$$F = \begin{bmatrix}
1 & 1 & 2 & 3 & 5 & 8 \\
8(9-4\sqrt{5}) & 1 & 1 & 2 & 3 & 5 \\
5(9-4\sqrt{5}) & 8(9-4\sqrt{5}) & 1 & 1 & 2 & 3 \\
3(9-4\sqrt{5}) & 5(9-4\sqrt{5}) & 8(9-4\sqrt{5}) & 1 & 1 & 2 \\
2(9-4\sqrt{5}) & 3(9-4\sqrt{5}) & 5(9-4\sqrt{5}) & 8(9-4\sqrt{5}) & 1 & 1 \\
9-4\sqrt{5} & 2(9-4\sqrt{5}) & 3(9-4\sqrt{5}) & 5(9-4\sqrt{5}) & 8(9-4\sqrt{5}) & 1
\end{bmatrix}.$$

Since $n = 6$ and $k = 9-4\sqrt{5}$ i.e. $\psi = -\beta$ and $\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, based on Theorem 4, it follows that

★: $\psi\omega^0 = \frac{1}{\alpha}$, so λ_0 is obtained based on 1) of Theorem 4: $\lambda_0 = -29+15\sqrt{5}$;

★: $\psi\omega^{-j} \neq \frac{1}{\alpha}$ and $\psi\omega^{-j} \neq \frac{1}{\beta}$, for $j = \overline{1,5}$, so λ_j, for $j = \overline{1,5}$, are obtained based on 3) of Theorem 4: $\lambda_{1,5} = -\frac{1}{2}\left[51-23\sqrt{5} \pm i\sqrt{3}(29-13\sqrt{5})\right]$, $\lambda_{2,4} = 7-3\sqrt{5} \mp i\sqrt{3}(29-13\sqrt{5})$, $\lambda_3 = 72-32\sqrt{5}$

Bearing in mind that $|\ F\ | = \prod_{j=0}^{n-1}\lambda_j$, it follows that

$$|\ F\ | = -238\,300\,041\,216 - 106\,571\,018\,240\sqrt{5}.$$

Let us remark, in relation to the previous example, that the determinant of $F = circ\{_{9-4\sqrt{5}}(1,1,2,3,5,8)\}$ is not possible to obtain using the result of Theorem 3.

The next theorem is devoted to determining the Euclidean norm of (1.11). The following formula will be needed.

For all x,

$$\sum_{i=1}^{n-1} i x^i = \frac{x - nx^n + (n-1)x^{n+1}}{(1-x)^2} \,. \tag{2.6}$$

Theorem 5. *Let F be the matrix as in* (1.11)*. Then the Euclidean norm of F is:*

$$\|F\|_E = \sqrt{\frac{1}{5}\left\{n\left[L_{2n+1}-(-1)^n\right]+(|k|^2-1)\left[-L_{2n}+(n-1)L_{2n+1}+\frac{5}{2}+\frac{1-2n}{2}(-1)^n\right]\right\}}\,. \tag{2.7}$$

Proof. From the definition of the Euclidean norm, using (1.4), (1.5), (1.6) and (2.6), we obtain:

$$\begin{aligned}
&(\|F\|_E)^2 \\
&= \sum_{i,j=1}^{n} |f_{i,j}|^2 \\
&= nF_1^2 + \left[(n-1)+|k|^2\right]F_2^2 + \left[(n-2)+2|k|^2\right]F_3^2 + \cdots + \left[1+(n-1)|k|^2\right]F_n^2 \\
&= \sum_{i=0}^{n-1}(n-i)F_{i+1}^2 + |k|^2 \sum_{i=1}^{n-1} iF_{i+1}^2 \\
&= n\sum_{i=0}^{n-1} F_{i+1}^2 + (|k|^2-1)\sum_{i=1}^{n-1} iF_{i+1}^2 \\
&= \frac{1}{5}\left[n\left[L_{2n+1}-(-1)^n\right]+(|k|^2-1)\sum_{i=1}^{n-1} i\left[\alpha^{2i+2}-2(\alpha\beta)^{i+1}+\beta^{2i+2}\right]\right] \\
&= \frac{n}{5}\left[L_{2n+1}-(-1)^n\right]+\frac{|k|^2-1}{5}(\alpha^2\,\frac{\alpha^2-n\alpha^{2n}+(n-1)\alpha^{2n+2}}{\alpha^2} \\
&\quad +2\,\frac{-1-n(-1)^n+(n-1)(-1)^{n+1}}{4}+\beta^2\,\frac{\beta^2-n\beta^{2n}+(n-1)\beta^{2n+2}}{\beta^2}) \\
&= \frac{n}{5}\left[L_{2n+1}-(-1)^n\right]+\frac{|k|^2-1}{5}(-n\alpha^{2n}+(n-1)\alpha^{2n+2}+\frac{5}{2}-\frac{n}{2}(-1)^n \\
&\quad -\frac{n-1}{2}(-1)^n-n\beta^{2n}+(n-1)\beta^{2n+2}) \\
&= \frac{n}{5}\left[L_{2n+1}-(-1)^n\right]+\frac{|k|^2-1}{5}\left[-nL_{2n}+(n-1)L_{2n+2}+\frac{5}{2}+\frac{1-2n}{2}(-1)^n\right]
\end{aligned}$$

$$= \frac{n}{5}\left[L_{2n+1}-(-1)^n\right]+\frac{|k|^2-1}{5}\left[-L_{2n}+(n-1)L_{2n+1}+\frac{5}{2}+\frac{1-2n}{2}(-1)^n\right].$$

Therefore,

$$\|F\|_E=\sqrt{\tfrac{1}{5}\left\{n\left[L_{2n+1}-(-1)^n\right]+(|k|^2-1)\left[-L_{2n}+(n-1)L_{2n+1}+\tfrac{5}{2}+\tfrac{1-2n}{2}(-1)^n\right]\right\}}.$$

□

The upper and lower bounds for the spectral norm of the Hadamard inverse of (1.11) will be given by the following theorem. We use the well - known inequalities

$$\frac{\|C\|_E}{\sqrt{n}} \le \|C\|_2 \le \|C\|_E, \tag{2.8}$$

which hold for any matrix C of order n, and the following lemma.

Lemma 2 ([7]). *Let $M=\left[m_{i,j}\right]$ and $N=\left[n_{i,j}\right]$ be matrices of order $m\times n$. Then*

$$\|M\circ N\|_2 \le r_1(M)\circ c_1(N), \tag{2.9}$$

where $M\circ N=[m_{i,j}n_{i,j}]$ is the Hadamard product (or Schur product),

$$r_1(M)=\max_{1\le i\le m}\sqrt{\sum_{j=1}^{n}|m_{i,j}|^2} \quad and \quad c_1(N)=\max_{1\le j\le n}\sqrt{\sum_{i=1}^{m}|n_{i,j}|^2}.$$

More information about the Hadamard product can be found in [6, 11].

Theorem 6. *Let F be the matrix as in* (1.11).

1) If $|k|\ge 1$, then

$$\sqrt{\frac{5n}{L_{2n+1}-(-1)^n}} \le \|F^{\circ -1}\|_2 \le \sqrt{n(1+(n-1)|k|^2)}\,, \tag{2.10}$$

2) If $|k|<1$, then

$$|k|\sqrt{\frac{5n}{L_{2n+1}-(-1)^n}} \le \|F^{\circ -1}\|_2 \le n\,. \tag{2.11}$$

Proof. From the definition of the Euclidean norm, it follows that

$$\|F^{\circ -1}\|_E^2=\sum_{i=0}^{n-1}(n-i)\frac{1}{F_{i+1}^2}+|k|^2\sum_{i=1}^{n-1}i\,\frac{1}{F_{i+1}^2}. \tag{2.12}$$

1) If $|k| \geq 1$, then

$$\begin{aligned}
\|F^{\circ -1}\|_E^2 &\geq \sum_{i=0}^{n-1}(n-i)\frac{1}{F_{i+1}^2}+\sum_{i=1}^{n-1} i\,\frac{1}{F_{i+1}^2}=n\sum_{i=0}^{n-1}\frac{1}{F_{i+1}^2}\\
&=n\sum_{i=1}^{n}\frac{1}{F_i^2}\geq n\sum_{i=1}^{n}\frac{1}{F_n^2}=(\frac{n}{F_n})^2\geq \frac{n^2}{F_nF_{n+1}}\\
&=\frac{5n^2}{L_{2n+1}-(-1)^n}.
\end{aligned}$$

Therefore,

$$\frac{\|F^{\circ -1}\|_E}{\sqrt{n}}\geq \sqrt{\frac{5n}{L_{2n+1}-(-1)^n}}.$$

We conclude from (2.8) that

$$\|F^{\circ -1}\|_2\geq \sqrt{\frac{5n}{L_{2n+1}-(-1)^n}}.$$

Now, we shall obtain the upper bound for the spectral norm of $F^{\circ -1}$. Let R and S be the following matrices:

$$R=\begin{bmatrix}
\frac{1}{F_1} & \frac{1}{F_2} & \frac{1}{F_3} & \cdots & \frac{1}{F_n}\\
k & \frac{1}{F_1} & \frac{1}{F_2} & \cdots & \frac{1}{F_{n-1}}\\
k & k & \frac{1}{F_1} & \cdots & \frac{1}{F_{n-2}}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
k & k & k & \cdots & \frac{1}{F_1}
\end{bmatrix} \text{ and } S=\begin{bmatrix}
1 & 1 & 1 & \cdots & 1\\
\frac{1}{F_n} & 1 & 1 & \cdots & 1\\
\frac{1}{F_{n-1}} & \frac{1}{F_n} & 1 & \cdots & 1\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
\frac{1}{F_2} & \frac{1}{F_3} & \frac{1}{F_4} & \cdots & 1
\end{bmatrix}.$$

Then,

$$r_1(R)=\max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|\,r_{i,j}\,|^2}=\sqrt{1+(n-1)|k|^2}$$

and

$$c_1(S)=\max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|\,s_{i,j}\,|^2}=\sqrt{n}\,.$$

Since $F^{\circ -1}=R\circ S$, based on Lemma 2, we can write

$$\|F^{\circ -1}\|_2\leq r_1(R)\circ c_1(S)=\sqrt{n(1+(n-1)|k|^2)}\,.$$

2) If $|k|<1$, then

$$\begin{aligned}\|F^{\circ-1}\|_E^2 &\geq \sum_{i=0}^{n-1}(n-i)\,|k|^2\frac{1}{F_{i+1}^2}+\sum_{i=1}^{n-1}i\,|k|^2\frac{1}{F_{i+1}^2}=n|k|^2\sum_{i=0}^{n-1}\frac{1}{F_{i+1}^2}\\ &=n|k|^2\sum_{i=1}^{n}\frac{1}{F_i^2}\geq n|k|^2\sum_{i=1}^{n}\frac{1}{F_n^2}=|k|^2(\frac{n}{F_n})^2\geq |k|^2\frac{n^2}{F_nF_{n+1}}\\ &=|k|^2\frac{5n^2}{L_{2n+1}-(-1)^n}.\end{aligned}$$

Therefore,

$$\frac{\|F^{\circ-1}\|_E}{\sqrt{n}}\geq |k|\sqrt{\frac{5n}{L_{2n+1}-(-1)^n}}.$$

We conclude from (2.8) that

$$\|F^{\circ-1}\|_2\geq |k|\sqrt{\frac{5n}{L_{2n+1}-(-1)^n}}.$$

Now, we shall obtain the upper bound for the spectral norm of $F^{\circ-1}$. Let Q and W be the following matrices:

$$Q=\begin{bmatrix}\frac{1}{F_1} & 1 & 1 & \cdots & 1\\ \frac{k}{F_n} & \frac{1}{F_1} & 1 & \cdots & 1\\ \frac{k}{F_{n-1}} & \frac{k}{F_n} & \frac{1}{F_1} & \cdots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{k}{F_2} & \frac{k}{F_3} & \frac{k}{F_4} & \cdots & \frac{1}{F_1}\end{bmatrix}\text{ and } W=\begin{bmatrix}1 & \frac{1}{F_2} & \frac{1}{F_3} & \cdots & \frac{1}{F_n}\\ 1 & 1 & \frac{1}{F_2} & \cdots & \frac{1}{F_{n-1}}\\ 1 & 1 & 1 & \cdots & \frac{1}{F_{n-2}}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & 1 & \cdots & 1\end{bmatrix}.$$

Then,

$$r_1(Q)=\max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|\,q_{i,j}\,|^2}=\sqrt{n}$$

and

$$c_1(W)=\max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|\,w_{i,j}\,|^2}=\sqrt{n}\,.$$

Since $F^{\circ-1}=Q\circ W$, based on Lemma 2, we can write

$$\|F^{\circ-1}\|_2\leq r_1(Q)\circ c_1(W)=n\,.$$

□

3. Conclusion

In this paper, we investigated the eigenvalues, the Euclidean norm and the upper and lower bounds for the spectral norm of the Hadamard inverse of

$$F = circ\{_k(F_1, F_2, \ldots, F_n)\},$$

where F_n is the n^{th} Fibonacci number and k is a nonzero complex number.
From the fact that *the eigenvalues of an upper triangular matrix are the diagonal entries*, the eigenvalues of a semicirculant matrix (i.e. a k-circulant matrix for $k=0$) with the first row $(F_1, F_2, \ldots, F_n)$ are: $\lambda_j = 1$, $(j = \overline{0, n-1})$. The Euclidean norm of such (semicirculant) matrix can be obtained from (2.7) i.e. in (2.7) k can be equal to 0. Semicirculant matrices are not Hadamard invertible.

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References

[1] R. E. Cline, R. J. Plemmons, and G. Worm, "Generalized inverses of certain Toeplitz matrices," *Linear Algebra Appl.*, vol. 8, no. 1, pp. 25–33, 1974, doi: 10.1016/0024-3795(74)90004-4.

[2] J. H. Conway and R. K. Guy, *The Book of Numbers*. New York: Springer-Verlag New York, Inc., 1996. doi: 10.1007/978-1-4612-4072-3.

[3] B. Demirturk, "Fibonacci and Lucas sums by matrix methods," *Int. Math. Forum*, vol. 5, no. 3, pp. 99–107, 2010.

[4] Y. Gao, Z. Jiang, and Y. Gong, "On the determinants and inverses of skew circulant and skew left circulant matrices with Fibonacci and Lucas numbers," *WSEAS Trans. Math.*, vol. 12, no. 4, pp. 472–481, 2013.

[5] C. He, J. Ma, K. Zhang, and Z. Wang, "The upper bound estimation on the spectral norm of r-circulant matrices with the Fibonacci and Lucas numbers," *J. Inequal. Appl.*, vol. 2015, no. 72, 2015, doi: 10.1186/s13660-015-0596-5.

[6] R. A. Horn, "The Hadamard product," *Proc. Sympos. Appl. Math.*, vol. 40, pp. 87–169, 1990.

[7] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. Cambridge: Cambridge Univ. Press, 1991.

[8] R. Keskin and B. Demirturk, "Some new Fibonacci and Lucas identities by matrix methods," *Int. J. Math. Ed. Sci. Tech.*, vol. 41, no. 3, pp. 379–387, 2010, doi: 10.1080/00207390903236426.

[9] T. Koshy, *Fibonacci and Lucas Numbers with Applications*. New York: John Wiley and Sons, 2001. doi: 10.1002/9781118033067.

[10] M. Krzywkowski, "New proofs of some Fibonacci identities," *Int. Math. Forum*, vol. 5, no. 18, pp. 869–874, 2010.

[11] S. Liu and G. Trenkler, "Hadamard, Khatri-Rao, Kronecker and other matrix products," *Int. J. Inf. Syst. Sci.*, vol. 4, no. 1, pp. 160–177, 2008.

[12] R. S. Melham, "Sums of certain products of Fibonacci and Lucas numbers," *Fibonacci Quart.*, vol. 37, no. 3, pp. 248–251, 1999.

[13] S. Q. Shen, J. M. Cen, and Y. Hao, "On the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers," *Appl. Math. Comput.*, vol. 217, no. 23, pp. 9790–9797, 2011, doi: 10.1016/j.amc.2011.04.072.

[14] J. Silvester, "Fibonacci properties by matrix methods," *Math. Gaz.*, vol. 63, no. 425, pp. 188–191, 1979, doi: 10.2307/3617892.

[15] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications*. Ellis Horwood Limited, Chichester, England, 1989.

[16] Y. Yazlik and N. Taskara, "On the inverse of circulant matrix via generalized k-Horadam numbers," *Appl. Math. Comput.*, vol. 223, pp. 191–196, 2013, doi: 10.1016/j.amc.2013.07.078.

[17] Y. Yazlik and N. Taskara, "On the norms of an r-circulant matrix with the generalized k-Horadam numbers," *J. Inequal. Appl.*, vol. 2013, no. 394, 2013, doi: 10.1186/1029-242X-2013-394.

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