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INVERSE AND FACTORIZATION OF TRIANGULAR TOEPLITZ MATRICES

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Abstract. In this paper, we present a new approach for finding the inverse of some triangular Toeplitz matrices using the generalized Fibonacci polynomials and give a factorization of these matrices. We also give a new proof of Trudi's formula using our result.

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1. INTRODUCTION

Let L_n be the lower triangular Toeplitz matrix:

$$L_n = \begin{bmatrix} t_0 & 0 & \cdots & 0 \\ t_1 & t_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ t_{n-1} & \cdots & t_1 & t_0 \end{bmatrix}$$

where $(t_i)_{i=0,1,2,\ldots,n-1}$ are real and/or complex numbers. In the matrix theory, there is quite an interest in the theory and applications of triangular Toeplitz matrices. There are a number of studies focusing on the linear algebra of the triangular Toeplitz matrices. For example, in [2] the authors discussed the linear algebra of the Pascal matrix, in [8] the authors examined the linear algebra of the k-Fibonacci matrix and the symmetric k-Fibonacci matrix, in [6] the authors studied the Pell matrix. In [9], Lee *et al.* defined $n \times n$-Fibonacci matrix and obtained the inverse matrix of the Fibonacci matrix. The Fibonacci matrix $F_n = [f_{i,j}]_{i,j=1,2,\ldots,n}$ and the inverse matrix of F_n as follows:

$$F_n = [f_{i,j}] = \begin{cases} f_{i-j+1}, & \text{for } i-j+1 \geq 0, \\ 0, & \text{for } i-j+1 < 0, \end{cases}$$

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and

$$F_n^{-1} = [f_{i,j}^{\imath}] = \begin{cases} 1, & \text{for } i = j, \\ -1, & \text{for } i-2 \le j \le i-1, \\ 0, & \text{otherwise,} \end{cases}$$

where f_n is nth Fibonacci numbers.

The inverse of Toeplitz matrices was first studied by Trench [18] in 1964 and by Gohberg and Semencul [4] in 1972. In the last decades some papers related to computing the inverse of a nonsingular Toeplitz matrix and the lower triangular Toeplitz matrix were presented, etc. [1, 3, 5, 7, 11, 16, 17, 19, 21]. In [16] Merca derived the inverse of triangular Toeplitz matrix using symmetric functions.

In this paper, we obtain the inverse of n×n lower triangular Toeplitz matrix T_n as follows:

$$T_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -t_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -t_{n-1} & \cdots & -t_1 & 1 \end{bmatrix} \tag{1.1}$$

where $(t_i)_{i=1,2,\dots,n-1}$ are the elements of a ring.

To achieve this goal, we use the generalized Fibonacci polynomials which are general form of a large number of recurrent relation numbers and polynomials. MacHenry [12, 13] defined generalized Fibonacci polynomials $(F_{k,n}(t))$, where t_i $(1 \le i \le k)$ are constant coefficients of the core polynomial

$$P(x; t_1, t_2, \dots, t_k) = x^k - t_1 x^{k-1} - \cdots - t_k,$$

which is denoted by the vector $t = (t_1, t_2, \dots, t_k)$. $F_{k,n}(t)$ is defined inductively by

$$\begin{aligned} F_{k,n}(t) &= 0, \ n < 0; \\ F_{k,0}(t) &= 1, \\ F_{k,n+1}(t) &= t_1 F_{k,n}(t) + \cdots + t_k F_{k,n-k+1}(t). \end{aligned} \tag{1.2}$$

In addition, in [14] the authors obtained $F_{k,n}(t)$ $(n, k \in \mathbb{N}, n \ge 1)$ as,

$$F_{k,n}(t) = \sum_{b \vdash n} \binom{|b|}{b_1, \dots, b_n} t_1^{b_1} \dots t_n^{b_n}. \tag{1.3}$$

Throughout this paper, the notations $b \vdash n$ and $|b|$ are used instead of $\sum_{j=1}^{n} j b_j = n$ and $\sum_{j=1}^{n} b_j$, respectively.

Corollary 1 ([10]). *Let $F_{k,n}(t)$ be the generalized Fibonacci polynomials and $H_{-(k,n)}$ be the $n \times n$ lower Hessenberg matrix such that*

$$H_{-(k,n)} = \begin{bmatrix} t_1 & -1 & 0 & \cdots & 0 \\ t_2 & t_1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{k-1} & t_{k-2} & t_{k-3} & \cdots & -1 \\ t_k & t_{k-1} & t_{k-2} & \cdots & t_1 \end{bmatrix}.$$

Then,

$$\det H_{-(k,n)} = F_{k,n}(t).$$

2. THE INVERSE OF N×N LOWER TRIANGULAR TOEPLITZ MATRIX

In this section, we obtain the inverse of matrix T_n (1.1). This result was obtained in [16] as a result of the study on symmetry between complete symmetric functions and elementary symmetric functions. We present new approach for this using definition of generalized Fibonacci polynomials.

Theorem 1. *Let T_n be the $n{\times}n$ lower triangular Toeplitz matrix in (1.1) and $F_{k,n}(t)$ be the generalized Fibonacci polynomials defined in (1.2), then*

$$(T_n)^{-1} = [t_{i,j}] = \begin{cases} F_{k,i-j}(t), & \text{for } i-j>0, \\ 1, & \text{for } i-j=0, \\ 0, & \text{otherwise}. \end{cases}$$

Proof.

$$A_n = [a_{i,j}] = \begin{cases} F_{k,i-j}(t), & \text{for } i-j>0, \\ 1, & \text{for } i-j=0, \\ 0, & \text{otherwise}. \end{cases}$$

and R_i is ith row vector of A_n, C_i is ith column vector of T_n.

If we show the equation $A_nT_n = I = [e_{i,j}]$, the proof is completed. It is obvious that $e_{ij} = 0$ for $i<j$ and $e_{ii} = \langle R_i, C_i\rangle = 1$ from the definitions of A_n and T_n. Now we obtain e_{ij} for $i>j$;

$$e_{ij} = \left\langle R_i, C_j \right\rangle = F_{k,i-j+1}(t) - t_1 F_{k,i-j}(t) - t_2 F_{k,i-j}(t) - \ldots - t_k F_{k,0}(t).$$

From the definition of generalized Fibonacci polynomials and previous equation, we obtain $e_{ij} = 0$ for $i>j$, which ends the proof. □

Example 1. We obtain the inverse of matrix T_5 using Theorem 1;

$$(T_5)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -t_1 & 1 & 0 & 0 & 0 \\ -t_2 & -t_1 & 1 & 0 & 0 \\ -t_3 & -t_2 & -t_1 & 1 & 0 \\ -t_4 & -t_3 & -t_2 & -t_1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ F_{k,1}(t) & 1 & 0 & 0 & 0 \\ F_{k,2}(t) & F_{k,1}(t) & 1 & 0 & 0 \\ F_{k,3}(t) & F_{k,2}(t) & F_{k,1}(t) & 1 & 0 \\ F_{k,4}(t) & F_{k,3}(t) & F_{k,2}(t) & F_{k,1}(t) & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ t_1 & 1 & 0 & 0 & 0 \\ t_2+t_1^2 & t_1 & 1 & 0 & 0 \\ t_3+2t_1t_2+t_1^3 & t_2+t_1^2 & t_1 & 1 & 0 \\ t_4+2t_1t_3+t_2^2+t_1^4+3t_1^2t_2 & t_3+2t_1t_2+t_1^3 & t_2+t_1^2 & t_1 & 1 \end{bmatrix}.$$

Example 2. We obtain "$s_{8,3}$" for matrix $S_n = [s_{i,j}]_{n\times n}$ which is the inverse of matrix $R_n = [r_{ij}]_{n\times n} = \begin{cases} 1, & i=j, \\ 0, & i<j, \\ j-i, & i>j, \end{cases}$ using Theorem 1 and equation (1.3);

$$s_{8,3} = F_{k,5}(1,2,3,4,5) = \sum_{b\vdash 5} \binom{|b|}{b_1,\ldots,b_5} 1^{b_1}\ldots 5^{b_5} = 55.$$

Since $F_{k,n}(t)$ is the general form of the Fibonacci type numbers and polynomials, the results that we obtained are applicable for many polynomials and sequences, such as generalized order-k Fibonacci, Pell and Jacobsthal numbers, generalized bivariate Fibonacci p-polynomials, bivariate Jacobsthal p-polynomials, Chebyshev polynomial of the second kind and bivariate Pell p-polynomials etc.[see Table 1]

The generalized bivariate Fibonacci p-polynomials [20] are, for $n > p$,

$$F_{p,n}(x,y) = xF_{p,n-1}(x,y) + yF_{p,n-p-1}(x,y), \tag{2.1}$$

with boundary conditions for $n = 1,2,...,p$, $F_{p,0}(x,y) = 0$, $F_{p,n}(x,y) = x^{n-1}$.

Corollary 2. *Let $F_{p,n}(x,y)$ be the generalized bivariate Fibonacci p-polynomials defined in* (2.1) *and $T_{n,p}$ is $n \times n$ lower triangular Toeplitz matrix as*

$$T_{n,p} = [c_{i,j}] = \begin{cases} 1, & \text{for } i-j=0, \\ -x, & \text{for } i-j=1, \\ -y, & \text{for } i-j=p+1, \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$(T_{n,p})^{-1} = [a_{i,j}] = \begin{cases} F_{p,i-j}(x,y), & \text{for } i-j>0, \\ 1, & \text{for } i-j=0, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. It is obvious that $F_{k-1,n}(x,y) = F_{k,n}(t)$ for $t_1 = x, t_i = 0(2 \le i \le (k-1)), t_k = y$. So using Theorem 1, we obtain the required result. □

TABLE 1. [20] Cognate polynomial sequences.

x	y	p	$F_{p,n}(x,y)$
x	y	1	bivariate Fibonacci polynomials $F_n(x,y)$
x	1	p	Fibonacci p-polynomials $F_{p,n}(x)$
x	1	1	Fibonacci polynomials $f_n(x)$
1	1	p	Fibonacci p-numbers $F_p(n)$
1	1	1	Fibonacci numbers F_n
$2x$	y	p	bivariate Pell p-polynomials $F_{p,n}(2x,y)$
$2x$	y	1	bivariate Pell polynomials $F_n(2x,y)$
$2x$	1	p	Pell p-polynomials $P_{p,n}(x)$
$2x$	1	1	Pell polynomials $P_n(x)$
2	1	1	Pell numbers P_n
$2x$	-1	1	second kind Chebysev polynomials $U_{n-1}(x)$
x	$2y$	p	bivariate Jacobsthal p-polynomials $F_{p,n}(x,2y)$
x	$2y$	1	bivariate Jacobsthal polynomials $F_n(x,2y)$
1	$2y$	1	Jacobsthal Polynomials $J_n(y)$
1	2	1	Jacobsthal Numbers J_n

Corollary 3. *Let U_n be the $n \times n$ lower triangular Toeplitz matrix as*

$$U_n = [u_{i,j}] = \begin{cases} \sum_{m=0}^{\left\lfloor \frac{i-j-1}{2} \right\rfloor} \binom{i-j}{2m+1} x^{i-j-2m} (x^2-1)^m, & \text{for } i-j>0, \\ 1, & \text{for } i-j=0, \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$(U_n)^{-1} = \begin{cases} 1, & \text{for } i-j=0, i-j=2, \\ -2x, & \text{for } i-j=1, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. It is obvious in the Table 1 that $F_{1,n}(x,y) = U_{n-1}(x)$ and

$$U_n(x) = \sum_{m=0}^{\left\lfloor \frac{i-j}{2} \right\rfloor} \binom{i-j+1}{2m+1} x^{i-j-2m} (x^2-1)^m$$

from the definition of the second kind of Chebyshev polynomials. So using Corollary 2, we obtain the required result. □

3. FACTORIZATIONS OF LOWER TRIANGULAR TOEPLITZ MATRIX

The set of all square matrices of order n is denoted by P_n. A matrix $P \in P_n$ of the form

$$P = \begin{bmatrix} P_{11} & 0 & \cdots & 0 \\ 0 & P_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & P_{kk} \end{bmatrix}$$

in which $P_{ii} \in P_{n_i}$, $i \in \{1,2,...,k\}$, and $\sum_{k=1}^{n} n_i = n$, is called block diagonal. Notationally, such a matrix is often indicated as $P = P_{11} \oplus P_{22} \oplus \cdots \oplus P_{kk}$; this is called the direct sum of the matrices $P_{11}, P_{22}, \cdots, P_{kk}$.

In [9], Lee *et al.* gave the factorization of Fibonacci matrix. Now we consider the factorization of lower triangular Toeplitz matrix. We define the matrices C_n and $\widetilde{T_n}$ by

$$C_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -F_{k,1}(t) & & & \\ \vdots & & I_{n+2} & \\ -F_{k,n+2}(t) & & & \end{bmatrix} \text{ and } \widetilde{T_n} = [1] \oplus T_n. \tag{3.1}$$

Theorem 2. *Let T_n be the $n \times n$ lower triangular Toeplitz matrix in (1.1) and matrices $C_n, \widetilde{T_n}$ defined in (3.1). Then $(\widetilde{T_{k-1}})(C_{k-3}) = T_n$ for $k \geq 3$.*

Proof. For $k=3$, we have $(\widetilde{T_2})(C_0)=T_3$. Letting $k>3$ and applying the definition of generalized Fibonacci polynomials, the proof complete. □

Theorem 3. *Let T_n be the $n\times n$ lower triangular Toeplitz matrix in* (1.1)*, the matrix C_n defined in* (3.1) *and I_n be the $n\times n$ identity matrix. Then $T_n=(I_{n-2}\oplus(C_{-1}))\ldots(I_1\oplus(C_{n-4}))(C_{n-3})$.*

Proof. The proof is an immediate consequence of Theorem 2. □

Example 3. We obtain the factorization of the matrix T_5 using Theorem 3;

$$T_5=(I_3\oplus C_{-1})(I_2\oplus C_0)(I_1\oplus C_1)C_2,$$

i.e.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -t_1 & 1 & 0 & 0 & 0 \\ -t_2 & -t_1 & 1 & 0 & 0 \\ -t_3 & -t_2 & -t_1 & 1 & 0 \\ -t_4 & -t_3 & -t_2 & -t_1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -t_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -t_1 & 1 & 0 \\ 0 & 0 & -(t_1^2+t_2) & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -t_1 & 1 & 0 & 0 \\ 0 & -(t_1^2+t_2) & 0 & 1 & 0 \\ 0 & -(t_3+2t_1t_2+t_1^3) & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -t_1 & 1 & 0 & 0 & 0 \\ -(t_1^2+t_2) & 0 & 1 & 0 & 0 \\ -(t_3+2t_1t_2+t_1^3) & 0 & 0 & 1 & 0 \\ -(t_4+2t_1t_3+t_2^2+t_1^4+3t_1^2t_2) & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Lemma 1. *Let C_n be the $(n+3)\times(n+3)$ Hessenberg matrix in 3.1 , then*

$$(C_n)^{-1}=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ F_{k,1}(t) & & & \\ \vdots & & I_{n+2} & \\ F_{k,n+2}(t) & & & \end{bmatrix}$$

Proof. The proof is obvious from the matrix product. □

Corollary 4. *Let T_n be the $n{\times}n$ lower triangular Toeplitz matrix in (1.1), the matrix C_n defined in (3.1) and I_n be the $n\times n$ identity matrix. Then* $(T_n)^{-1} = (C_{n-3})^{-1}(I_1\oplus(C_{n-4}))^{-1}\dots(I_{n-2}\oplus(C_{-1}))^{-1}$.

Proof. The proof follows by a simple calculation using the previous lemma and the equation $(I_k\oplus(C_{n-k-3}))^{-1} = I_k\oplus(C_{n-k-3})^{-1}$. □

4. A NEW PROOF OF TRUDI'S FORMULA

Merca [15] gave a proof of the Trudi's formula. We give a different proof of this identity using our results.

Theorem 4 (Trudi's formula [15]). *Let m be a positive integer. Then*

$$\det\begin{bmatrix} a_1 & a_0 & \cdots & & 0\\ a_2 & a_1 & \cdots & & \\ \vdots & \ddots & \ddots & & \vdots\\ a_{m-1} & a_{m-2} & \cdots & a_1 & a_0\\ a_m & a_{m-1} & \cdots & a_2 & a_1\end{bmatrix} = \sum_{(k_1,k_2,\dots,k_m)}\binom{k_1+\cdots+k_m}{k_1,\cdots,k_m}(-a_0)^{m-k_1-\cdots-k_m}a_1^{k_1}a_2^{k_2}\cdots a_m^{k_m}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+mt_m = m$.

Proof. Using the properties of determinants of Hessenberg matrices, we can write

$$\det\begin{bmatrix} a_1 & a_0 & 0 & 0\\ a_2 & a_1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & a_0\\ a_m & \cdots & a_2 & a_1\end{bmatrix}$$

$$= \det\begin{bmatrix} a_1 & -1 & 0 & \cdots & 0\\ -a_0a_2 & a_1 & -1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ (-1)^{m-2}a_0^{m-2}a_{m-1} & (-1)^{m-3}a_0^{m-3}a_{m-2} & \ddots & a_1 & -1\\ (-1)^{m-1}a_0^{m-1}a_m & (-1)^{m-2}a_0^{m-2}a_{m-1} & & -a_0a_2 & a_1\end{bmatrix}.$$

And if we take

$$t_1 = a_1,\ t_2 = -a_0a_2,\dots,t_m = (-1)^{m-1}a_0^{m-1}a_m$$

in equation 1.3 and Corollary 1, we obtain

$$\det\begin{bmatrix} a_1 & -1 & 0 & \cdots & 0 \\ -a_0a_2 & a_1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ (-1)^{m-2}a_0^{m-2}a_{m-1} & (-1)^{m-3}a_0^{m-3}a_{m-2} & \ddots & a_1 & -1 \\ (-1)^{m-1}a_0^{m-1}a_m & (-1)^{m-2}a_0^{m-2}a_{m-1} & & -a_0a_2 & a_1 \end{bmatrix}$$
$$= \sum_{(k_1,k_2,\ldots,k_m)} \binom{k_1+\cdots+k_m}{k_1,\cdots,k_m} a_1^{k_1}(-a_0a_2)^{k_2}\cdots((-a_0)^{m-1}a_m)^{k_m}$$

Finally, when we make the necessary calculations, equation

$$\sum_{(k_1,k_2,\ldots,k_m)} \binom{k_1+\cdots+k_m}{k_1,\cdots,k_m} a_1^{k_1}(-a_0a_2)^{k_2}\cdots((-a_0)^{m-1}a_m)^{k_m}$$
$$= \sum_{(k_1,k_2,\ldots,k_m)} \binom{k_1+\cdots+k_m}{k_1,\cdots,k_m} (-a_0)^{m-k_1-\cdots-k_m} a_1^{k_1}a_2^{k_2}\cdots a_m^{k_m}$$

is obtained. □

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