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EXISTENCE OF TWO SYMMETRIC SOLUTIONS FOR NEUMANN PROBLEMS

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Abstract. In this paper, we investigate the existence of at least two distinct cylindrically symmetric weak solutions for some elliptic problems involving a p-Laplace operator, subject to Neumann boundary conditions in a strip-like domain of the Euclidean space.

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1. Introduction

Let $\mathcal{O} \subset \mathbb{R}^m$ be a bounded domain with smooth boundary and $\Omega := \mathcal{O} \times \mathbb{R}^n$ be a strip-like domain. Define the space of cylindrically symmetric functions by

$$W_c^{1,p}(\Omega) := \{u \in W^{1,p}(\Omega) : u(x,\cdot) \text{ is radially symmetric for all } x \in \mathcal{O}\}.$$

In this space, Molica Bisci and Rădulescu in [7, Theorem 2.1] studied the existence of at least three cylindrically symmetric solutions for the following elliptic Neumann problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda \alpha(x,y) f(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ν denotes the outward unit normal to $\partial\Omega$, $p > m+n$ is a real number, λ is a positive real parameter and $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. Moreover, $\alpha \in L^1(\Omega)$ is a non-negative cylindrically symmetric function and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

In this paper, our goal is to obtain the existence of at least two distinct cylindrically symmetric weak solutions for problem (1.1) under suitable conditions on α and f.

We denote by c_p the best embedding constant of $W_c^{1,p}(\Omega)$ into $L^\infty(\Omega)$, i.e.,

$$c_p := \sup_{u \in W^{1,p}(\Omega)} \frac{\|u\|_{L^\infty(\Omega)}}{\|u\|_{W^{1,p}(\Omega)}}, \tag{1.2}$$

where

$$\|u\|_{L^\infty} := \text{esssup}_{(x,y)\in\Omega}|u(x,y)|;$$

see [4, Theorem 2.2]. Further, Let $\alpha \in L^1(\Omega)$ is a non-negative cylindrically symmetric function such that

$$\alpha_0 := \inf_{(x,y)\in\Omega}\alpha(x,y) > 0,$$

and $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following condition:

(f_1) $$|f(t)| \leq a_1 + a_2|t|^{s-1}, \qquad \forall t \in \mathbb{R},$$

for some non-negative constants a_1, a_2 and $s > p$. We put $F(\xi) := \int_0^\xi f(t)dt$, for every $\xi \in \mathbb{R}$. Moreover, we introduce the functional $I_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ associated with problem (1.1),

$$I_\lambda(u) := \frac{1}{p}\left(\int_\Omega |\nabla u(x,y)|^p dxdy + \int_\Omega |u(x,y)|^p dxdy\right) - \lambda \int_\Omega \alpha(x,y)F(u(x,y))dxdy.$$

Fixing the real parameter λ, a function $u \in W^{1,p}(\Omega)$ is said to be a weak solution of (1.1) if for all $v \in W^{1,p}(\Omega)$,

$$\int_\Omega |\nabla u(x,y)|^{p-2}\nabla u(x,y)\cdot\nabla v(x,y)dxdy + \int_\Omega |u(x,y)|^{p-2}u(x,y)v(x,y)dxdy$$
$$= \lambda \int_\Omega \alpha(x,y)f(u(x,y))v(x,y)dxdy.$$

Hence, the critical points of I_λ are exactly the weak solutions of problem (1.1).

Definition 1. A Gâteaux differentiable function I satisfies the Palais-Smale condition (in short (PS)-condition) if any sequence $\{u_n\}$ such that

(a) $\{I_\lambda(u_n)\}$ is bounded,
(b) $\|I'_\lambda(u_n)\|_{X^*} \to 0, \quad$ as $n \to \infty$,

has a convergent subsequence.

We shall prove our results applying the following critical point theorem, which is a more precise version of Ricceri's variational principle [12, Theorem 2.5]. We point out that Ricceri's variational principle generalizes the celebrated three critical point theorem of Pucci and Serrin [9, 10] and is an useful result that gives alternatives for the multiplicity of critical points of certain functions depending on a parameter.

Theorem 1 (see [2, Theorem 3.2])**.** *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u) < +\infty$ and assume that, for each*

$$\lambda \in \left]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}\right[,$$

the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies (PS)*-condition and it is unbounded from below. Then, for each* $\lambda \in \left]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}\right[$, *the functional I_λ admits two distinct critical points.*

For completeness, we refer the interested reader to the recent papers [3, 6] where Ricceri's variational principle has been developed on studying nonlinear Neumann problems. See also [1, 5].

2. Main results

In this section we establish the main abstract result of this paper. We recall that c_p is the constant of the continuous embedding $W_c^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$; see (1.2).

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying condition* (f_1). *Moreover, assume that*

(f_2) *there exist two constants $\eta > p$ and $L > 0$ such that*

$$0 < \eta F(t) \le t f(t), \qquad |t| \ge L.$$

Then, for each $\lambda \in]0, \lambda^\star[$, problem (1.1) *admits at least two distinct cylindrically symmetric weak solutions, where*

$$\lambda^\star := \frac{s}{\left(s a_1 c_p p^{1/p} + a_2 c_p^s p^{s/p}\right)\|\alpha\|_{L^1}}.$$

Proof. Our aim is to apply Theorem 1 to problem (1.1) in the case $r = 1$ to the Banach space $X := W_c^{1,p}(\Omega)$ endowed with the norm

$$\|u\|_{W^{1,p}} := \left(\int_\Omega |\nabla u(x,y)|^p dxdy + \int_\Omega |u(x,y)|^p dxdy\right)^{1/p}.$$

For every $u \in X$ we set

$$\Phi(u) := \frac{\|u\|_{W^{1,p}}^p}{p}, \qquad \Psi(u) := \int_\Omega \alpha(x,y) F(u(x,y)) dxdy.$$

Clearly Φ and Ψ are continuously Gâteaux differentiable and

$$\Phi'(u)(v) :=$$
$$\int_\Omega |\nabla u(x,y)|^{p-2} \nabla u(x,y) \cdot \nabla v(x,y) dxdy + \int_\Omega |u(x,y)|^{p-2} u(x,y) v(x,y) dxdy,$$

and

$$\Psi'(u)(v) := \int_\Omega \alpha(x,y) f(u(x,y)) v(x,y) dxdy,$$

for every $v \in X$. Moreover, Φ' admits a continuous inverse on X^* and Ψ' is a compact operator.

Now we prove that $I_\lambda := \Phi - \lambda\Psi$ satisfies (PS)-condition for every $\lambda > 0$. Namely, we will prove that any sequence $\{u_n\} \subset X$ satisfying

$$m := \sup_n I_\lambda(u_n) < +\infty, \qquad \lim_{n\to+\infty} \|I'_\lambda(u_n)\|_{X^*} = 0,$$

contains a convergent subsequence. From above, we can actually assume that

$$|\frac{1}{\eta}\langle I'_\lambda(u_n), u_n\rangle| \leq \|u_n\|_{W_{1,p}}.$$

For n large enough, we have

$$\begin{aligned} m \geq I_\lambda(u_n) = \frac{1}{p}\left(\int_\Omega |\nabla u_n(x,y)|^p dxdy + \int_\Omega |u_n(x,y)|^p dxdy\right) \\ -\lambda \int_\Omega \alpha(x,y) F(u_n(x,y)) dxdy, \end{aligned}$$

then

$$\begin{aligned} I_\lambda(u_n) &\geq \frac{1}{p}\left(\int_\Omega |\nabla u_n(x,y)|^p dxdy + \int_\Omega |u_n(x,y)|^p dxdy\right) \\ &-\frac{\lambda}{\eta}\int_\Omega \alpha(x,y) f(u_n(x,y)) u_n(x,y) dxdy \\ &= \left(\frac{1}{p} - \frac{1}{\eta}\right)\left(\int_\Omega |\nabla u_n(x,y)|^p dxdy + \int_\Omega |u_n(x,y)|^p dxdy\right) \\ &+\frac{1}{\eta}\Big(\int_\Omega |\nabla u_n(x,y)|^p dxdy + \int_\Omega |u_n(x,y)|^p dxdy \\ &-\lambda \int_\Omega \alpha(x,y) f(u_n(x,y)) u_n(x,y) dxdy\Big) \\ &= \left(\frac{1}{p} - \frac{1}{\eta}\right)\|u_n\|^p_{W^{1,p}} + \frac{1}{\eta}\langle I'_\lambda(u_n), u_n\rangle. \end{aligned}$$

Thus,

$$m + \|u_n\|_{W^{1,p}} \geq I_\lambda(u_n) - \frac{1}{\eta}\langle I'_\lambda(u_n), u_n\rangle \geq \left(\frac{1}{p} - \frac{1}{\eta}\right)\|u_n\|^p_{W^{1,p}}.$$

Consequently, $\{\|u_n\|\}$ is bounded. By the Eberlian-Smulyan theorem, without loss of generality, we assume that $u_n \rightharpoonup u$. Then $\Psi'(u_n) \to \Psi'(u)$ because of compactness. Since $I'_\lambda(u_n) = \Phi'(u_n) - \lambda\Psi'(u_n) \to 0$, then $\Phi'(u_n) \to \lambda\Psi'(u)$. Since Φ' has a continuous inverse, then $u_n \to u$ and so I_λ satisfies (PS)-condition.

From (f_2), there is a positive constant C such that

$$F(t) \geq C|t|^\eta \tag{2.1}$$

for all $|t| > L$. In fact, setting $b := \min_{|\xi|=L} F(\xi)$ and

$$\varphi_t(\beta) := F(\beta t), \quad \forall \beta > 0, \tag{2.2}$$

by (f_2), for every $|t| > L$ one has

$$0 < \eta\varphi_t(\beta) = \eta F(\beta t) \le \beta t \cdot f(\beta t) = \beta\varphi_t'(\beta), \quad \forall \beta > \frac{L}{|t|}.$$

Therefore,

$$\int_{L/|t|}^{1} \frac{\varphi_t'(\beta)}{\varphi_t(\beta)} d\beta \ge \int_{L/|t|}^{1} \frac{\eta}{\beta} d\beta.$$

Then

$$\varphi_t(1) \ge \varphi_t\Big(\frac{L}{|t|}\Big)\frac{|t|^\eta}{L^\eta}.$$

Taking into account of (2.2), we obtain

$$F(t) \ge F\Big(\frac{L}{|t|}t\Big)\frac{|t|^\eta}{L^\eta} \ge b\frac{|t|^\eta}{L^\eta} \ge C|t|^\eta,$$

where $C > 0$ is a constant. Thus, (2.1) is proved.

Fixed $u_0 \in X\backslash\{0\}$, for each $t > 1$ one has

$$I_\lambda(tu_0) \le \frac{1}{p}t^p\|u_0\|_{W^{1,p}}^p - \lambda\alpha_0 C t^\eta \int_\Omega |u_0(x,y)|^\eta dxdy.$$

Since $\eta > p$, this condition guarantees that I_λ is unbounded from below. Fixed $\lambda \in]0,\lambda^\star[$, from definition of Φ it follows that

$$\|u\|_{W^{1,p}} < p^{1/p}, \tag{2.3}$$

for each $u \in X$ such that $u \in \Phi^{-1}(]-\infty,1[)$. Moreover, (f_1), the compact embedding $X \hookrightarrow L^\infty(\Omega)$ and (2.3) imply that, for each $u \in \Phi^{-1}(]-\infty,1[)$, we have

$$\begin{aligned}\Psi(u) &\le \int_\Omega \alpha(x,y)(a_1|u(x,y)| + \frac{a_2}{s}|u(x,y)|^s)dxdy\\ &\le (a_1\|u\|_{L^\infty} + \frac{a_2}{s}\|u\|_{L^\infty}^s)\|\alpha\|_{L^1}\\ &\le (a_1c_p\|u\|_{W^{1,p}} + \frac{a_2c_p^s}{s}\|u\|_{W^{1,p}}^s)\|\alpha\|_{L^1}\\ &< (a_1c_p p^{1/p} + \frac{a_2}{s}c_p^s p^{s/p})\|\alpha\|_{L^1},\end{aligned}$$

and so,

$$\sup_{u\in\Phi^{-1}(]-\infty,1[)} \Psi(u) \le (a_1c_p p^{1/p} + \frac{a_2}{s}c_p^s p^{s/p})\|\alpha\|_{L^1} = \frac{1}{\lambda^*} < \frac{1}{\lambda} \tag{2.4}$$

From (2.4) one has

$$\lambda \in]0,\lambda^\star[\subseteq \Bigg]0, \frac{1}{\sup_{u\in\Phi^{-1}(]-\infty,1[)}\Psi(u)}\Bigg[.$$

Hence, Theorem 1.2 assures the existence of at least two distinct critical points for problem (1.1). Also, it is proved in [7, proof of Theorem 2.1] that I_λ is an invariant functional with respect to the action of the compact group of linear isometries of $\mathbb{R}^n$. Thus, we can apply the principle of symmetric criticality (see [8]) to the smooth and isometric invariant functional I_λ and deduce that problem (1.1) admits at least two distinct cylindrically symmetric weak solutions. The proof is complete. □

Remark 1. We observe that, if f is non-negative and $f(0) \neq 0$, then Theorem 2 ensures the existence of two positive cylindrically symmetric weak solutions for problem (1.1) (see, e.g., [11, Theorem 11.1]).

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