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CONVERGENCE OF CESÁRO MEANS WITH VARYING PARAMETERS OF WALSH-FOURIER SERIES

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Abstract. In 2007 Akhobadze [1] (see also [2]) introduced the notion of Cesàro means of Fourier series with variable parameters. In the present paper we prove the almost everywhere convergence of the the Cesàro (C,α_n) means of integrable functions $\sigma_n^{\alpha_n} f \to f$, where $\mathbb{N}_{\alpha,K} \ni n \to \infty$ for $f \in L^1(I)$, where I is the Walsh group for every sequence $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$. This theorem for constant sequences α that is, $\alpha \equiv \alpha_1$ was proved by Fine [3].

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1. INTRODUCTION AND MAIN RESULTS

We follow the standard notions of dyadic analysis introduced by the mathematicians F. Schipp, P. Simon, W. R. Wade (see e.g. [9]) and others. Denote by $\mathbb{N} := \{0, 1, ...\}$, $\mathbb{P} := \mathbb{N} \setminus \{0\}$, the set of natural numbers, the set of positive integers and $I := [0, 1)$ the unit interval. Denote by $\lambda(B) = |B|$ the Lebesgue measure of the set $B (B \subset I)$. Denote by $L^p(I)$ the usual Lebesgue spaces and $\|.\|_p$ the corresponding norms $(1 \le p \le \infty)$. Set

$$\mathcal{J} := \left\{ \left[\frac{p}{2^n}, \frac{p+1}{2^n} \right) : p, n \in \mathbb{N} \right\}$$

the set of dyadic intervals and for given $x \in I$ and let $I_n(x)$ denote the interval $I_n(x) \in \mathcal{J}$ of length 2^{-n} which contains $x\,(n \in \mathbb{N})$. Also use the notion $I_n := I_n(0)\,(n \in \mathbb{N})$. Let

$$x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)}$$

be the dyadic expansion of $x \in I$, where $x_n = 0$ or 1 and if x is a dyadic rational number ($x \in \{\frac{p}{2^n} : p, n \in \mathbb{N}\}$) we choose the expansion which terminates in 0's. The

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notion of the Hardy space $H(I)$ is introduced in the following way [9]. A function $a \in L^{\infty}(I)$ is called an atom, if either $a = 1$ or a has the following properties: $\operatorname{supp} a \subseteq I_a, \|a\|_{\infty} \le |I_a|^{-1}, \int_I a = 0$, for some $I_a \in \mathcal{J}$. We say that the function f belongs to H, if f can be represented as $f = \sum_{i=0}^{\infty} \lambda_i a_i$, where a_i's are atoms and for the coefficients (λ_i) the inequality $\sum_{i=0}^{\infty} |\lambda_i| < \infty$ is true. It is known that H is a Banach space with respect to the norm

$$\|f\|_H := \inf \sum_{i=0}^{\infty} |\lambda_i|,$$

where the infimum is taken over all decompositions $f = \sum_{i=0}^{\infty} \lambda_i a_i \in H$.

Set the definition of the nth ($n \in \mathbb{N}$) Walsh-Paley function at point $x \in I$ as:

$$\omega_n(x) := \prod_{j=0}^{\infty} (-1)^{x_j n_j},$$

where $\mathbb{N} \ni n = \sum_{n=0}^{\infty} n_j 2^j$ $(n_j \in \{0,1\} \, (j \in \mathbb{N}))$. It is known (see [8] or [10]) that the system $(\omega_n, n \in \mathbb{N})$ is the character system of $(I, +)$, where the group operation $+$ is the so-called dyadic or logical addition on I. That is, for any $x, y \in I$

$$x + y := \sum_{n=0}^{\infty} |x_n - y_n| 2^{-(n+1)}.$$

Denote by

$$\hat{f}(n) := \int_I f \omega_n d\lambda, \quad D_n := \sum_{k=0}^{n-1} \omega_k, \quad K_n^1 := \frac{1}{n+1} \sum_{k=0}^{n} D_k$$

the Fourier coefficients, the Dirichlet and the Fejér or $(C,1)$ kernels, respectively. It is also known that the Fejér or $(C,1)$ means of f is

$$\sigma_n^1 f(y) := \frac{1}{n+1} \sum_{k=0}^{n} S_k f(y) = \int_I f(x) K_n^1(y+x) d\lambda(x)$$

$$= \frac{1}{n+1} \sum_{k=0}^{n} \int_I f(x) D_k(y+x) d\lambda(x), \quad (n \in \mathbb{N}, y \in I).$$

It is known [9] that for $n \in \mathbb{N}, x \in I$ it holds

$$D_{2^n}(x) = \begin{cases} 2^n, \text{ if } x \in I_n \\ 0 \ , \text{ if } x \notin I_n \end{cases}$$

and also that

$$D_n(x) = \omega_n(x) \sum_{k=1}^{\infty} D_{2^k}(x) n_k (-1)^{x_k},$$

where $n=\sum_{i=1}^{\infty} n_i 2^i$, $n_i=\{0,1\}\,(i\in\mathbb{N})$.

Denote by $K_n^{\alpha_n}$ the kernel of the summability method (C,α_n) and call it the (C,α_n) kernel or the Cesàro kernel for $\alpha_n\in\mathbb{R}\setminus\{-1,-2,\dots\}$

$$K_n^{\alpha_n}=\frac{1}{A_n^{\alpha_n}}\sum_{k=0}^{n}A_{n-k}^{\alpha_n-1}D_k,$$

where

$$A_k^{\alpha_n}=\frac{(\alpha_n+1)(\alpha_n+2)...(\alpha_n+n)}{k!}.$$

It is known [12] that $A_n^{\alpha_n}=\sum_{k=0}^{n}A_k^{\alpha_n-1}$, $A_k^{\alpha_n}-A_{k+1}^{\alpha_n}=-\frac{\alpha_n A_k^{\alpha_n}}{k+1}$. The (C,α_n) Cesàro means of integrable function f is

$$\sigma_n^{\alpha_n}f(y):=\frac{1}{A_n^{\alpha_n}}\sum_{k=0}^{n}A_{n-k}^{\alpha_n-1}S_kf(y)=\int_I f(x)K_n^{\alpha_n}(y+x)d\lambda(x).$$

In [3] Fine proved the almost everywhere convergence $\sigma_n^{\alpha_n}f\longrightarrow f$ for all integrable function f with constant sequence $\alpha_n=\alpha_1>0$. On the rate of convergence of Cesàro means in this constant case see the paper of Fridli [4]. For the two-dimensional situation see the paper of Goginava [7].

Comment 1. With respect to other locally constant orthonormal sysytems for instance it was a question of Taibleson [8] open for a long time, that does the Fejér-Lebesgue theorem, that is the a.e. convergence $\sigma_n^1 f\longrightarrow f$ hold for all integrable function f with respect to the character system of the group of 2-adic integers. In 1997 Gát answered [1] this question in the affirmative. Zheng and Gát generalized this result [9,2] for more general orthonormal systems.

Set two variable function $P(n,\alpha):=\sum_{i=0}^{\infty}n_i 2^{i\alpha}$ for $n\in\mathbb{N},\alpha\in\mathbb{R}$. For instance $P(n,1)=n$. Also set for sequences $\alpha=(\alpha_n)$ and positive reals K the subset of natural numbers

$$\mathbb{N}_{\alpha,K}:=\left\{n\in\mathbb{N}:\frac{P(n,\alpha_n)}{n^{\alpha_n}}\le K\right\}.$$

We can easily remark that for a sequence α such that $1>\alpha_n\ge\alpha_0>0$ we have $\mathbb{N}_{\alpha,K}=\mathbb{N}$ for some K depending only on α_0. We also remark that $2^n\in\mathbb{N}_{\alpha,K}$ for every $\alpha=(\alpha_n)$, $0<\alpha_n<1$ and $K\ge 1$.

In this paper C denotes an absolute constant and C_K another one which may depend only on K. The introduction of (C,α_n) means due to Akhobadze investigated [1] the behavior of the L^1-norm convergence of $\sigma_n^{\alpha_n}f\to f$ for the trigonometric system. In this paper it is also supposed that $1>\alpha_n>0$ for all n.

The main aim of this paper is to prove :

Theorem 1. *Suppose that $1>\alpha_n>0$. Let $f\in L^1(I)$. Then we have the almost everywhere convergence $\sigma_n^{\alpha_n}f\to f$ provided that $\mathbb{N}_{\alpha,K}\ni n\to\infty$.*

The method we use to prove Theorem 1 is to investigate the maximal operator $\sigma_*^\alpha f := \sup_{n\in\mathbb{N}_{\alpha,K}} |\sigma_n^{\alpha_n} f|$. We also prove that this operator is a kind of type (H, L) and of type (L^p, L^p) for all $1 < p \le \infty$. That is,

Theorem 2. *Suppose that* $1 > \alpha_n > 0$. *Let* $|f| \in H(I)$. *Then we have*

$$\|\sigma_*^\alpha f\|_1 \le C_K \||f|\|_H.$$

Moreover, the operator σ_*^α *is of type* (L^p, L^p) *for all* $1 < p \le \infty$. *That is,*

$$\|\sigma_*^\alpha f\|_p \le C_{K,p}\|f\|_p$$

for all $1 < p \le \infty$.

For the sequence $\alpha_n = 1$ Theorem 2 is due to Fujii [5]. For the more general but constant sequence $\alpha_n = \alpha_1$ see Weisz [11].

Basically, in order to prove Theorem 1 we verify that the maximal operator $\sigma_*^\alpha f$ $(\alpha = (\alpha_n))$ is of weak type (L^1, L^1). The way we get this, the investigation of kernel functions, and its maximal function on the unit interval I by making a hole around zero and some quasi locality issues (for the notion of quasi-locality see [9]). To have the proof of Theorem 2 is the standard way. We need several Lemmas in the next section.

2. Proofs

Lemma 1. *For* $j, n \in \mathbb{N}$, $j < 2^n$ *we have*

$$D_{2^n - j}(x) = D_{2^n}(x) - \omega_{2^n-1}(x) D_j(x).$$

Proof.

$$D_{2^n}(x) = \sum_{k=0}^{2^n-1} \omega_k(x) = \sum_{k=0}^{2^n-j-1} \omega_k(x) + \sum_{k=2^n-j}^{2^n-1} \omega_k(x) = D_{2^n-j} + \sum_{k=2^n-j}^{2^n-1} \omega_k(x).$$

We have to prove :

$$\sum_{k=2^n-j}^{2^n-1} \omega_k(x) = \omega_{2^n-1}(x) D_j(x).$$

For $k < j, k = k_{n-1}2^{n-1} + \ldots + k_1 2^1 + k_0$ we have

$$\begin{aligned}
&\omega_{2^n-1}(x)\omega_k \\
&= \omega_{2^{n-1}+\ldots+2^1+2^0}(x)\omega_{k_{n-1}2^{n-1}+\ldots+k_0}(x) \\
&= \omega_{(1+k_{n-1}(mod\,2))2^{n-1}+\ldots+(1+k_0(mod\,2))2^0}(x) \\
&= \omega_{(1-k_{n-1}(mod\,2))2^{n-1}+\ldots+(1-k_0(mod\,2))2^0}(x) \\
&= \omega_{2^{n-1}+2^{n-2}+\ldots+2^0-(k_{n-1}2^{n-1}+\ldots+k_0)}(x) = \omega_{2^n-1-k}(x).
\end{aligned}$$

Thus,

$$\omega_{2^n-1}(x)D_j(x)=\omega_{2^n-1}(x)\sum_{k=0}^{j-1}\omega_k(x)=\sum_{k=0}^{j-1}\omega_{2^n-1-k}(x)=\sum_{k=2^n-j}^{2^n-1}\omega_k(x).$$

This completes the proof of Lemma 1. □

Introduce the following notations: for $a,n,j\in\mathbb{N}$ let $n_{(j)}:=\sum_{i=0}^{j-1}n_i2^i$, that is, $n_{(0)}=0$, $n_{(1)}=n_0$ and for $2^B\le n<2^{B+1}$, let $|n|:=B$, $n=n_{(B+1)}$. Moreover, introduce the following functions and operators for $n\in\mathbb{N}$ and $1>\alpha_n>0$

$$\begin{aligned}
T_n^{\alpha_a} &:= \frac{1}{A_n^{\alpha_a}}\sum_{j=0}^{2^{|n|}-1}A_{n-j}^{\alpha_a-1}D_j,\\
\tilde{T}_n^{\alpha_a} &:= \frac{1}{A_n^{\alpha_a}}D_{2^B}\sum_{j=0}^{2^B-1}A_{n_{(B)}+j}^{\alpha_a-1}\\
&\qquad+(1-\alpha_a)\sum_{j=0}^{2^B-1}A_{n_{(B)}+j}^{\alpha_a-1}\frac{j+1}{n_{(B)}+j+1}\left|K_j^1\right|+A_n^{\alpha_a-1}2^B\left|K_{2^B-1}^1\right|,\\
t_n^{\alpha_a}f(y) &:= \int_I f(x)T_n^{\alpha_a}(y+x)d\lambda(x),\\
\tilde{t}_n^{\alpha_a}f(y) &:= \int_I f(x)\tilde{T}_n^{\alpha_a}(y+x)d\lambda(x).
\end{aligned}$$

Now, we need to prove the next Lemma which means that maximal operator $\sup_{n,a}|\tilde{t}_n^{\alpha_a}|$ is quasi-local. This lemma together with the next one are the most important tools in the proof of the main results of this paper.

Lemma 2. *Let $1>\alpha_a>0$, $f\in L^1(I)$ such that supp $f\subset I_k(u)$, $\int_{I_k(u)}fd\lambda=0$ for some dyadic interval $I_k(u)$. Then we have*

$$\int_{I\setminus I_k(u)}\sup_{n,a\in\mathbb{N}}|\tilde{t}_n^{\alpha_a}f|d\lambda\le C\|f\|_1.$$

Moreover, $\left|T_n^{\alpha_a}\right|\le\tilde{T}_n^{\alpha_a}$.

Proof. It is easy to have that for $n<2^k$ and $x\in I_k(u)$ we have $\tilde{T}_n^{\alpha_a}(y+x)=\tilde{T}_n^{\alpha_a}(y+u)$ and

$$\begin{aligned}
&\int_{I_k(u)}f(x)\tilde{T}_n^{\alpha_a}(y+x)d\lambda(x)\\
&=\tilde{T}_n^{\alpha_a}(y+u)\int_{I_k(u)}f(x)d\lambda(x)=0.
\end{aligned}$$

Therefore,

$$\int_{I\setminus I_k(u)} \sup_{n,a\in\mathbb{N}} \tilde{t}_n^{\alpha_a} f d\lambda = \int_{I\setminus I_k(u)} \sup_{n\geq 2^k, a\in\mathbb{N}} \tilde{t}_n^{\alpha_a} f d\lambda.$$

Recall that $B = |n|$. Then

$$\begin{aligned} &A_n^{\alpha_a} T_n^{\alpha_a} \\ &= \sum_{j=0}^{2^B-1} A_{2^B+n_{(B)}-j}^{\alpha_a-1} D_j \\ &= \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_{2^B-j} \end{aligned}$$

By Lemma 1 we have

$$\begin{aligned} &A_n^{\alpha_a} T_n^{\alpha_a} \\ &= D_{2^B} \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} - \omega_{2^B-1} \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_j. \end{aligned}$$

It is easy to have that $\frac{1}{A_n^{\alpha_a}} D_{2^B}(z) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} = 0$, for any $z \in I \setminus I_k$. This holds because $D_{2^B}(z) = 0$ for $B = |n| \geq k$ and $z \in I \setminus I_k$. By the help of the Abel transform we get:

$$
\begin{aligned}
&\left|\sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_j\right| \\
&= \left|\sum_{j=0}^{2^B-1} (A_{n_{(B)}+j}^{\alpha_a-1} - A_{n_{(B)}+j+1}^{\alpha_a-1}) \sum_{i=0}^{j} D_i + A_{n_{(B)}+2^B}^{\alpha_a-1} \sum_{i=0}^{2^B-1} D_i\right| \\
&= \left|(1-\alpha_a) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} K_j^1 + A_n^{\alpha_a-1} 2^B K_{2^B-1}^1\right| \\
&= \left|(1-\alpha_a) \sum_{j=0}^{2^k-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} K_j^1 + (1-\alpha_a) \sum_{j=2^k}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} K_j^1\right. \\
&\left. + A_n^{\alpha_a-1} 2^B K_{2^B-1}^1\right| \\
&\le (1-\alpha_a) \sum_{j=0}^{2^k-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} \left|K_j^1\right| \\
&+ (1-\alpha_a) \sum_{j=2^k}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} \left|K_j^1\right| + A_n^{\alpha_a-1} 2^B \left|K_{2^B-1}^1\right| \\
&=: I + II + III.
\end{aligned}
$$

These equalities above immediately proves inequality $\left|T_n^{\alpha_a}\right| \le \tilde{T}_n^{\alpha_a}$.

Since for any $j < 2^k$ we have that the Fejér kernel $K_j^1(y+x)$ depends (with respect to x) only on coordinates $x_0, \ldots, x_{k-1}$, then $\int_{I_k(u)} f(x)|K_j^1(y+x)| d\lambda(x) = |K_j^1(y+u)| \int_{I_k(u)} f(x) d\lambda(x) = 0$ gives $\int_{I_k(u)} f(x) I(y+x) d\lambda(x) = 0$.

On the other hand,

$$
\begin{aligned}
II &= (1-\alpha_a) \sum_{j=2^k}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} |K_j^1| \\
&\le \sup_{j \ge 2^k} |K_j^1| (1-\alpha_a) \sum_{j=0}^{n} A_j^{\alpha_a-1} = A_n^{\alpha_a} (1-\alpha_a) \sup_{j \ge 2^k} |K_j^1|.
\end{aligned}
$$

This by Lemma 3 in [6] gives

$$
\int_{I \setminus I_k} \sup_{n \ge 2^k, a \in \mathbb{N}} \frac{1}{A_n^{\alpha_a}} II \, d\lambda \le \int_{I \setminus I_k} \sup_{j \ge 2^k} |K_j^1| \, d\lambda \le C.
$$

The situation with III is similar. Namely,

$$\frac{A_n^{\alpha_a-1}n}{A_n^{\alpha_a}}=\frac{\alpha_a\cdot n}{(\alpha_a+n)}\le\alpha_a<1.$$

So, just as in the case of II we apply Lemma 3 in [6]

$$\int_{I\setminus I_k}\sup_{n\ge 2^k,a\in\mathbb{N}}\frac{1}{A_n^{\alpha_a}}III\,d\lambda\le\int_{I\setminus I_k}\sup_{n\ge 2^k,a\in\mathbb{N}}|K^1_{2^{|n|}-1}|\,d\lambda\le C.$$

Therefore, substituting $z=x+y\in I\setminus I_k$ (since $x\in I_k(u)$ and $y\in I\setminus I_k(u)$)

$$\begin{aligned}
&\int_{I\setminus I_k(u)}\sup_{n\ge 2^k,a\in\mathbb{N}}\tilde{t}_n^{\alpha_a}f\,d\lambda\\
&=\int_{I\setminus I_k(u)}\sup_{n\ge 2^k,a\in\mathbb{N}}\left|\int_{I_k(u)}f(x)\tilde{T}_n^{\alpha_a}(y+x)\,d\lambda(x)\right|d\lambda(y)\\
&\le\int_{I\setminus I_k(u)}\int_{I_k(u)}|f(x)|\sup_{n\ge 2^k,a\in\mathbb{N}}\frac{1}{A_n^{\alpha_a}}\left(II(y+x)+III(y+x)\right)d\lambda(x)\\
&=\int_{I_k(u)}|f(x)|\int_{I\setminus I_k}\sup_{n\ge 2^k,a\in\mathbb{N}}\frac{1}{A_n^{\alpha_a}}\left(II(z)+III(z)\right)d\lambda(z)\,d\lambda(x)\\
&\le C\int_{I_k(u)}|f(x)|\,d\lambda(x).
\end{aligned}$$

This completes the proof of Lemma 2. □

A straightforward corollary of this lemma is:

Corollary 1. *Let* $1>\alpha_a>0$. *Then we have* $\|T_n^{\alpha_a}\|_1\le\|\tilde{T}_n^{\alpha_a}\|_1\le C$, $\|t_n^{\alpha_a}f\|_1$, $\|\tilde{t}_n^{\alpha_a}f\|_1\le C\|f\|_1$ *and* $\|t_n^{\alpha_a}g\|_\infty,\|\tilde{t}_n^{\alpha_a}g\|_\infty\le C\|g\|_\infty$ *for all natural numbers* a,n, *where* C *is some absolute constant and* $f\in L^1, g\in L^\infty$. *That is, operators* $\tilde{t}_n^{\alpha_a},t_n^{\alpha_a}$ *is of type* (L^1,L^1) *and* (L^∞,L^∞) *(uniformly in* n*).*

Proof. The proof is a straightforward consequence of Lemma 2 and an easy estimation below. Let $B=|n|$. Then

$$\begin{aligned}
&\left\|A_n^{\alpha_a}\tilde{T}_n^{\alpha_a}\right\|_1\le\|D_{2^B}\|_1\sum_{j=0}^{2^B-1}A^{\alpha_a-1}_{n_{(B)}+j}\\
&+(1-\alpha_a)\sum_{j=0}^{2^B-1}A^{\alpha_a-1}_{n_{(B)}+j}\frac{j+1}{n_{(B)}+j+1}\|K_j^1\|_1+A_n^{\alpha_a-1}2^B\|K^1_{2^B-1}\|_1.
\end{aligned}$$

Then by $\|D_{2^B}\|_1=1,\|K_j^1\|_1\le C$ we complete the proof of Corollary 1.

□

Lemma 3. *Let n, N be any natural numbers and $0 < \alpha < 1$. Then we have*

$$\frac{A_n^\alpha}{A_N^\alpha} \le 2\left(\frac{n+1}{N}\right)^\alpha .$$

Proof. By definition we have

$$\frac{A_n^\alpha}{A_N^\alpha} = \left(1-\frac{\alpha}{n+1+\alpha}\right)\cdots\left(1-\frac{\alpha}{N+\alpha}\right) \le \left(1-\frac{\alpha}{n+2}\right)\cdots\left(1-\frac{\alpha}{N+1}\right).$$

It is well-known that

$$\left(1-\frac{\alpha}{i(n+1)+1}\right)\cdots\left(1-\frac{\alpha}{(i+1)(n+1)}\right) \le \left(1-\frac{\alpha}{(i+1)(n+1)}\right)^{n+1} \le \left(e^{-1}\right)^{\frac{\alpha}{i+1}}$$

for every $n \in \mathbb{N}$. This gives

$$\begin{aligned}
&\left(1-\frac{\alpha}{n+2}\right)\cdots\left(1-\frac{\alpha}{N+1}\right) \le \left(e^{-1}\right)^{\alpha \sum_{i=2}^{\left\lfloor \frac{N}{n+1}\right\rfloor} \frac{1}{i}} \\
&\le \left(e^{-1}\right)^{\alpha \log_e \left\lfloor \frac{N}{n+1}\right\rfloor -1+c} \\
&\le 2\left(e^{-1}\right)^{\alpha \log_e \left(\frac{N}{n+1}\right)} = 2\left(\frac{n+1}{N}\right)^\alpha ,
\end{aligned}$$

where $c \approx 0.5772$ is the Euler-Mascheroni constant. This completes the proof of Lemma 3. □

Recall that the two variable function $P(n,\alpha) = \sum_{i=0}^{\infty} n_i 2^{i\alpha}$ for $n \in \mathbb{N}, \alpha \in \mathbb{R}$ and $K \in \mathbb{R}$ determines the set of natural numbers

$$\mathbb{N}_{\alpha,K} = \left\{ n \in \mathbb{N} : \frac{P(n,\alpha_n)}{n^{\alpha_n}} \le K \right\}.$$

Let $n = 2^{h_s} + \cdots + 2^{h_0}$, where $h_s > \cdots > h_0 \ge 0$ are integers. That is, $|n| = h_s$. Let $n^{(j)} := 2^{h_j} + \cdots + 2^{h_0}$. This means $n = n^{(s)}$. Define the following kernel function and operators

$$\tilde{K}_n^{\alpha_n} := \tilde{T}_{n^{(s)}}^{\alpha_n} + \sum_{l=0}^{s} \left(\frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} D_{2^{h_l}} + \frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} \tilde{T}_{n^{l-1}}^{\alpha_n} \right)$$

and

$$\tilde{\sigma}_n^{\alpha_n} f := f * \tilde{K}_n^{\alpha_n}, \quad \tilde{\sigma}_*^{\alpha} f := \sup_{n \in \mathbb{N}_{\alpha,K}} |f * \tilde{K}_n^{\alpha}|.$$

In the sequel we prove that maximal operator $\tilde{\sigma}_*^{\alpha} f$ is quasi-local. This is the very base of the proof of the main results of this paper. That is, Theorem 1 and Theorem 2.

Lemma 4. *Let $1 > \alpha_n > 0$, $f \in L^1(I)$ such that supp $f \subset I_k(u)$, $\int_{I_k(u)} f d\lambda = 0$ for some dyadic interval $I_k(u)$. Then we have*

$$\int_{I \setminus I_k(u)} \tilde{\sigma}_*^{\alpha} f d\lambda \leq C_K \|f\|_1,$$

where constant C_K can depend only on K.

Proof. Recall that $n = 2^{h_s} + \cdots + 2^{h_0}$, where $h_s > \cdots > h_0 \geq 0$ are integers. That is, $|n| = h_s$. Let $n^{(j)} := 2^{h_j} + \cdots + 2^{h_0}$. This means $n = n^{(s)}$. Use also the notation

$$\begin{aligned}
&\tilde{K}_{n^{(s)}}^{\alpha_n} \\
&= \tilde{T}_{n^{(s)}}^{\alpha_n} + \sum_{l=0}^{s} \left(\frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} D_{2^{h_l}} + \frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} \tilde{T}_{n^{l-1}}^{\alpha_n} \right) \\
&=: G_1 + G_2 + G_3.
\end{aligned}$$

Since $n^{(l-1)} < 2^{h_{(l-1)}+1}$, then by Lemma 3 we have

$$\frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} \leq 2 \left(\frac{n^{(l-1)} + 1}{n^{(s)}} \right)^{\alpha_n} \leq 2 \frac{2^{\alpha_n (h_{l-1}+1)}}{2^{\alpha_n h_s}} \leq C \frac{2^{h_{l-1} \alpha_n}}{n^{\alpha_n}}.$$

That is, by the above written we also have

$$\begin{aligned}
&\int_{I \setminus I_k(u)} \sup_{n \in \mathbb{N}} \left| \int_{I_k(u)} f(x) G_2(y+x) d\lambda(x) \right| d\lambda(y) \\
&\int_{I \setminus I_k(u)} \sup_{n \in \mathbb{N}} \sum_{l=0}^{s} \frac{2^{h_{l-1}\alpha_n}}{n^{\alpha_n}} \left| \int_{I_k(u)} f(x) D_{2^h}(y+x) d\lambda(x) \right| d\lambda(y) = 0
\end{aligned}$$

since $f * D_{2^h} = 0$ for $h \leq k$ because of the $\mathcal{A}_k$ measurability of D_{2^h} and $\int f = 0$. Besides, for $h > k$ $D_{2^h}(y+x) = 0$ $(y + x \notin I_k)$.

As a result of these estimations above and by the help of Lemma 2, that is the quasi-locality of operator $\tilde{t}_*^{\alpha} = \sup_{n,a \in \mathbb{N}} |\tilde{t}_n^{\alpha_a}|$ we conclude

$$\begin{aligned}
&\int_{I \setminus I_k(u)} \sup_{n \in \mathbb{N}} \left| \int_{I_k(u)} f(x) (G_1(y+x) + G_3(y+x)) d\lambda(x) \right| d\lambda(y) \\
&\leq C_K \int_{I \setminus I_k(u)} \sup_{n,a \in \mathbb{N}} \left| \int_{I_k(u)} f(x) \tilde{T}_n^{\alpha_a}(y+x) d\lambda(x) \right| d\lambda(y) \\
&\leq C_K \|f\|_1.
\end{aligned}$$

This completes the proof of Lemma 4. □

Lemma 5. *The operator $\tilde{\sigma}_*^{\alpha}$ is of type (L^∞, L^∞) $(\tilde{\sigma}_*^{\alpha} f := \sup_{n \in \mathbb{N}_{\alpha,K}} |\tilde{\sigma}_n^{\alpha_n} f|)$.*

Proof. By the help of the method of Lemma 4 and by Corollary 1 we have

$$\|\tilde{K}_n^{\alpha_n}\|_1 = \left\|\tilde{K}_{n^{(s)}}^{\alpha_n}\right\|_1 \leq \left\|\tilde{T}_{n^{(s)}}^{\alpha_n}\right\|_1 + \sum_{l=0}^{s}\left(\frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}}\|D_{2^{h_l}}\|_1 + \frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}}\|\tilde{T}_{n^{l-1}}^{\alpha_n}\|_1\right)$$
$$\leq C + C\sum_{l=0}^{s}\frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} \leq C_K$$

because $n \in \mathbb{N}_{\alpha,K}$. Hence $\tilde{\sigma}_*^{\alpha}$ is of type (L^{∞}, L^{∞}) (with constant C_K). This completes the proof of Lemma 5. □

Now, we can prove the main tool in order to have Theorem 1 for operator $\sigma_*^{\alpha} f := \sup_{n\in\mathbb{N}_{\alpha,K}}\left|\sigma_n^{\alpha_n} f\right|$.

Lemma 6. *The operators $\tilde{\sigma}_*^{\alpha}$ and σ_*^{α} are of weak type (L^1, L^1).*

Proof. First, we prove Lemma 6 for operator $\tilde{\sigma}_*^{\alpha}$. We apply the Calderon-Zygmund decomposition lemma [9]. That is, let $f \in L^1$ and $\|f\|_1 < \delta$. Then there is a decomposition:

$$f = f_0 + \sum_{j=1}^{\infty} f_j$$

such that $\|f_0\|_{\infty} \leq C\delta$, $\|f_0\|_1 \leq C\|f\|_1$ and $I^j = I_{k_j}(u^j)$ are disjoint dyadic intervals for which

$$\operatorname{supp} f_j \subset I^j\,,\ \int_{I^j} f_j\, d\lambda = 0\,,\ |F| \leq \frac{C\|f_1\|}{\delta}$$

$(u^j \in I\,,\ k_j \in N\,,\ j \in P)$, where $F = \cup_{i=1}^{\infty} I^j$. By the σ-sublinearity of the maximal operator with an appropriate constant C_K we have

$$\lambda(\tilde{\sigma}_*^{\alpha} f > 2C_K\delta) \leq \lambda(\tilde{\sigma}_*^{\alpha} f_0 > C_K\delta) + \lambda(\tilde{\sigma}_*^{\alpha}(\sum_{i=1}^{\infty} f_i) > C_K\delta) := I + II\,.$$

Since by Lemma 5 $\|\tilde{\sigma}_*^{\alpha} f_0\|_{\infty} \leq C_K\|f_0\|_{\infty} \leq C_K\delta$ then we have $I = 0$. So,

$$\lambda(\tilde{\sigma}_*^{\alpha}(\sum_{i=1}^{\infty} f_i) > C_K\delta) \leq |F| + \lambda(\bar{F} \cap \{\tilde{\sigma}_*^{\alpha}(\sum_{i=1}^{\infty} f_i) > C_K\delta\})$$
$$\leq \frac{C_K\|f\|_1}{\delta} + \frac{C_K}{\delta}\sum_{i=1}^{\infty}\int_{I\setminus I^j}\tilde{\sigma}_*^{\alpha} f_j\, d\lambda =: \frac{C_K\|f\|_1}{\delta} + \frac{C_K}{\delta}\sum_{i=1}^{\infty} III_j\,,$$

where

$$\begin{aligned} III_j &:= \int_{I\setminus I^j} \tilde{\sigma}_*^\alpha f_j \, d\lambda \\ &\le \int_{I\setminus I_{k_j}(u^j)} \sup_{n\in N_{\alpha,K}} \left| \int_{I_{k_j}(u^j)} f_j(x) \tilde{K}_n^{\alpha_n}(y+x) d\lambda(x) \right| d\lambda(y). \end{aligned}$$

The forthcoming estimation of III_j is given by the help Lemma 4

$$III_j \le C_K \|f_j\|_1.$$

That is, operator $\tilde{\sigma}_*^\alpha$ is of weak type (L^1, L^1). Next, we prove the estimation

$$|K_n^{\alpha_n}| \le \tilde{K}_n^{\alpha_n}. \tag{1}$$

To prove (1) recall again that $n = 2^{h_s} + \cdots + 2^{h_0}$, where $h_s > \cdots > h_0 \ge 0$ are integers. Since $n = 2^{h_s} + n^{(s-1)}$, then we have

$$\begin{aligned} \sum_{j=2^{h_s}}^{2^{h_s}+n^{(s-1)}} A_{n^{(s-1)}+2^{h_s}-j}^{\alpha_n-1} D_j &= \sum_{k=0}^{n^{(s-1)}} A_{n^{(s-1)}-k}^{\alpha_n-1} D_{2^{h_s}+k} \\ &= D_{2^{h_s}} \sum_{k=0}^{n^{(s-1)}} A_{n^{(s-1)}-k}^{\alpha_n-1} + \omega_{2^{h_s}} \sum_{k=0}^{n^{(s-1)}} A_{n^{(s-1)}-k}^{\alpha_n-1} D_k \\ &= D_{2^{h_s}} A_{n^{(s-1)}}^{\alpha_n} + \omega_{2^{h_s}} A_{n^{(s-1)}}^{\alpha_n} K_{n^{(s-1)}}^{\alpha_n}. \end{aligned}$$

So, by the help of the equalities above we get

$$K_{n^{(s)}}^{\alpha_n} = T_{n^{(s)}}^{\alpha_n} + \frac{A_{n^{(s-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} \left(D_{2^{h_s}} + \omega_{2^{h_s}} K_{n^{(s-1)}}^{\alpha_n} \right).$$

Apply this last formula recursively and Lemma 2. (Note that $n^{(-1)} = 0, T_0^{\alpha_n} = K_0^{\alpha_n} = 0, A_0^{\alpha_n} = 1$.)

$$\begin{aligned} |K_n^{\alpha_n}| = |K_{n^{(s)}}^{\alpha_n}| &\le |T_{n^{(s)}}^{\alpha_n}| + \sum_{l=0}^{s} \left(\prod_{j=l}^{s} \frac{A_{n^{(j-1)}}^{\alpha_n}}{A_{n^{(j)}}^{\alpha_n}} D_{2^{h_l}} + \prod_{j=l}^{s} \frac{A_{n^{(j-1)}}^{\alpha_n}}{A_{n^{(j)}}^{\alpha_n}} |T_{n^{l-1}}^{\alpha_n}| \right) \\ &= |T_{n^{(s)}}^{\alpha_n}| + \sum_{l=0}^{s} \left(\frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} D_{2^{h_l}} + \frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} |T_{n^{l-1}}^{\alpha_n}| \right) \\ &\le \tilde{K}_{n^{(s)}}^{\alpha_n} = \tilde{K}_n^{\alpha_n}. \end{aligned}$$

This completes the proof of inequality (1). This inequality gives that the operator σ_*^α is also of weak type (L^1, L^1) since

$$\lambda(\sigma_*^\alpha f > 2C_K\delta) \leq \lambda(\tilde{\sigma}_*^\alpha |f| > 2C_K\delta) \leq C_K \frac{\||f|\|_1}{\delta} = C_K \frac{\|f\|_1}{\delta}.$$

This completes the proof of Lemma 6. □

Proof of Theorem 1. Let $P \in \mathbf{P}$ be a polynomial where $P(x) = \sum_{i=0}^{2^k-1} c_i\omega_i$. Then for all natural number $n \geq 2^k$, $n \in \mathbb{N}_{\alpha,K}$ we have that $S_n P \equiv P$. Consequently, the statement $\sigma_n^{\alpha_n} P \longrightarrow P$ holds everywhere (of course not only for restricted $n \in \mathbb{N}_{\alpha,K}$). Now, let $\epsilon, \delta > 0$, $f \in L^1$. Let $P \in \mathbf{P}$ be a polynomial such that $\|f - P\|_1 < \delta$. Then

$$\begin{aligned}
&\lambda(\overline{\lim_{n\in\mathbb{N}_{\alpha,K}}} |\sigma_n^{\alpha_n} f - f| > \epsilon) \\
&\leq \lambda(\overline{\lim_{n\in\mathbb{N}_{\alpha,K}}} |\sigma_n^{\alpha_n}(f - P)| > \frac{\epsilon}{3}) + \lambda(\overline{\lim_{n\in\mathbb{N}_{\alpha,K}}} |\sigma_n^{\alpha_n} P - P| > \frac{\epsilon}{3}) \\
&\qquad + \lambda(\overline{\lim_{n\in\mathbb{N}_{\alpha,K}}} |P - f| > \frac{\epsilon}{3}) \\
&\leq \lambda(\sup_{n\in\mathbb{N}_{\alpha,K}} |\sigma_n^{\alpha_n}(f - P)| > \frac{\epsilon}{3}) + 0 + \frac{3}{\epsilon}\|P - f\|_1 \leq C_K \|P - f\|_1 \frac{3}{\epsilon} \leq \frac{C_K}{\epsilon}\delta
\end{aligned}$$

because σ_*^α is of weak type (L^1, L^1) (with any fixed $K > 0$). This holds for all $\delta > 0$. That is, for an arbitrary $\epsilon > 0$ we have

$$\lambda(\overline{\lim_{n\in\mathbb{N}_{\alpha,K}}} |\sigma_n^{\alpha_n} f - f| > \epsilon) = 0$$

and consequently we also have

$$\lambda(\overline{\lim_{n\in\mathbb{N}_{\alpha,K}}} |\sigma_n^{\alpha_n} f - f| > 0) = 0.$$

This finally gives

$$\overline{\lim_{n\in\mathbb{N}_{\alpha,K}}} |\sigma_n^{\alpha_n} f - f| = 0 \ \ a.e,$$

$$\sigma_n^{\alpha_n} f \longrightarrow f \ \ a.e. \quad (n \in \mathbb{N}_{\alpha,K}).$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. Inequality (1), Lemma 5 and Lemma 6 by the interpolation theorem of Marcinkiewicz [9] give that the operator σ_*^α is of type (L^p, L^p) for all $1 < p \leq \infty$. In the sequel we prove that operator $\tilde{\sigma}_*^\alpha f = \sup_{n\in\mathbb{N}_{\alpha,K}} |f * \tilde{K}_n^\alpha|$ is of type (H, L).

Let a be an atom ($a \neq 1$ can be supposed), $\operatorname{supp} a \subset I_k(x)$, $\|a\|_\infty \leq 2^k$ for some $k \in N$ and $x \in I$. Then, $n < 2^k$, $n \in \mathbb{N}_{\alpha,K}$ implies $a * \tilde{K}_n^\alpha = 0$ because $\tilde{K}_n^\alpha$ is $\mathcal{A}_k$ measurable for $n < 2^k$ and $\int_{I_k(x)} a(t)d\lambda(t) = 0$. That is,

$$\tilde{\sigma}_*^\alpha a = \sup_{\mathbb{N}_{\alpha,K} \ni n \geq 2^k} |\tilde{\sigma}_n^{\alpha_n} f|.$$

By the help Lemma 4 we have

$$\int_{I\setminus I_k(x)} \tilde{\sigma}_*^{\alpha} a \, d\lambda = \int_{I\setminus I_k(x)} \sup_{\mathbb{N}_{\alpha,K}\ni n\geq 2^k} \left| \int_{I_k(x)} a(y) \tilde{K}_n^{\alpha_n}(z+y) d\lambda(y) \right| d\lambda(z)$$
$$\leq C_K \int_{I_k(x)} |a(y)| d\lambda(y)$$
$$\leq C_K \|a\|_1$$
$$\leq C_K.$$

Since the operator $\tilde{\sigma}_*^{\alpha}$ is of type (L^2, L^2) (i.e $\|\tilde{\sigma}_*^{\alpha} f\|_2 \leq C_K \|f\|_2$ for all $f \in L^2(I)$), then we have

$$\|\tilde{\sigma}_*^{\alpha} a\|_1 = \int_{I\setminus I_k(x)} \tilde{\sigma}_*^{\alpha} a + \int_{I_k(x)} \tilde{\sigma}_*^{\alpha} a$$
$$\leq C_K + |I_k(x)|^{\frac{1}{2}} \|\tilde{\sigma}_*^{\alpha} a\|_2$$
$$\leq C_K + C_K 2^{\frac{-k}{2}} \|a\|_2$$
$$\leq C_K + C_K 2^{\frac{-k}{2}} 2^{\frac{k}{2}}$$
$$\leq C_K.$$

That is $\|\tilde{\sigma}_*^{\alpha} a\|_1 \leq C_K$ and consequently the σ-sublinearity of $\tilde{\sigma}_*^{\alpha}$ gives

$$\|\tilde{\sigma}_*^{\alpha} f\|_1 \leq \sum_{i=0}^{\infty} |\lambda_i| \|\tilde{\sigma}_*^{\alpha} a_i\|_1$$
$$\leq C_K \sum_{i=0}^{\infty} |\lambda_i|$$
$$\leq C_K \|f\|_H$$

for all $\sum_{i=0}^{\infty} \lambda_i a_i \in H$. That is, the operator $\tilde{\sigma}_*^{\alpha}$ is of type (H, L). This by inequality (1) and then by $\|\sigma_*^{\alpha} f\|_1 \leq \|\tilde{\sigma}_*^{\alpha} |f|\|_1 \leq C_K \||f|\|_H$ completes the proof of Theorem 2. $\square$

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