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# UNIFORM NUMERICAL APPROXIMATION FOR PARAMETER DEPENDENT SINGULARLY PERTURBED PROBLEM WITH INTEGRAL BOUNDARY CONDITION

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*Abstract.* In this paper, a parameter-uniform numerical method for a parameterized singularly perturbed ordinary differential equation containing integral boundary condition is studied. Asymptotic estimates on the solution and its derivatives are derived. A numerical algorithm based on upwind finite difference operator and an appropriate piecewise uniform mesh is constructed. Parameter-uniform error estimate for the numerical solution is established. Numerical results are presented, which illustrate the theoretical results.

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## 1. INTRODUCTION

In this paper, we consider the following parameterized singular perturbation problem with integral boundary condition arising in many scientific applications [16, 23](see also references therein):

$$\varepsilon u' + f(t,u,\lambda) = 0,\ t \in \Omega = (0,T],\ T > 0, \tag{1.1}$$

$$u(0) + \int_0^T c(s)u(s)ds = A, \tag{1.2}$$

$$u(T) = B, \tag{1.3}$$

where $\varepsilon \in (0,1]$ is the perturbation parameter, $\lambda$ is known as the control parameter, $A$ and $B$ are given constants. The functions $c(t) \geq 0$ and $f(t,u,\lambda)$ are assumed to be sufficiently continuously differentiable for our purpose in $\overline{\Omega} = \Omega \cup \{t = 0\}$ and $\Omega \times \mathbb{R}^2$ respectively and moreover

$$0 < \alpha \leq \frac{\partial f}{\partial u} \leq a^* < \infty,$$

$$0 < m_1 \leq \left| \frac{\partial f}{\partial \lambda} \right| \leq M_1 < \infty.$$

By a solution of (1.1)-(1.3) we mean $\{u(t), \lambda\} \in C^1[0,T] \times \mathbb{R}$ for which problem (1.1)-(1.3) is satisfied.

Singularly perturbed differential equations are typically characterized by a small parameter $\varepsilon$ multiplying some or all of the highest order terms in the differential equation as normally boundary layers occur in their solutions. These equations play an important role in today's advanced scientific computations. Many mathematical models starting from fluid dynamics to the problems in mathematical biology are modelled by singularly perturbed problems. Typical examples include high Reynold's number flow in the fluid dynamics, heat transport problem etc. For more details on singular perturbation, one can refer to the books [10, 12, 19, 21] and the references therein. The numerical analysis of singular perturbation cases has always been far from trivial because of the boundary layer behavior of the solution. Such problem undergo rapid changes within very thin layers near the boundary or inside the problem domain [19, 21]. It is well known that standard numerical methods for solving such problems are unstable and fail to give accurate results when the perturbation parameter is small. Therefore, it is important to develop suitable numerical methods to these problems, whose accuracy does not depend on the parameter value, i.e. methods that are convergence $\varepsilon-$uniformly. For the various approaches on the numerical solution of differential equations with steep gradients and continuous solutions we may refer to the studies [8, 10–12]. Parameterized boundary value problems have been considered by many researchers for many years. Such problems arise in physical chemistry and physics, describing the exothermic and isothermal chemical reactions, the steady-state temperature distributions, the oscillation of a mass attached by two springs lead to a differential equation with a parameter [18, 22]. An overview of some existence and uniqueness results and applications of parameterized equations may be obtained, for example, in [13, 16, 18, 22](see, also references therein). In [18, 22], the authors have also been considered some approximating aspects of this kind of problems. But in the above-mentioned papers, algorithms are only concerned with the regular cases (i.e., when the boundary layers are absent). In recent years, many researchers presented the numerical methods for the singular perturbation cases of parameterized problems. Uniform convergent finite-difference schemes for solving parameterized singularly perturbed two-point boundary value problems have been considered in [2, 3, 9, 17, 24, 25](see, also references therein). In [2, 3, 17] authors used boundary layer technique for solving analogous problem. A methodology based on the homotopy analysis technique to approximate the analytic solution was investigated in[24, 25]. Also it is well known that nonlinear differential equations with integral boundary conditions have been used in description of many phenomena in the applied sciences, e.g., heat conduction, chemical engineering, underground water flow and so on [6, 15, 20]. Therefore, boundary value problems involving integral boundary conditions have been studied by many authors [1, 4, 5, 7, 11, 14, 16, 23](see, also references therein). Some approximating aspects of this kind of problems in the regular cases, i.e, in absence of layers, were investigated in [4, 11, 14, 16, 23]. In recent years, many researchers considered the singularly perturbed case for these problems. In [1, 5, 7] authors develop a finite difference scheme on Shishkin mesh for problem with integral boundary conditions and proved that the method is nearly first order convergent except for a logarithmic factor. A hybrid scheme, which is second order convergent on Shishkin mesh was discussed in [7]. For the numerical methods, concerning to second order singularly perturbed differential equations with integral boundary conditions can be seen e.g., [5]. In this paper, as far as we know the numerical solution of the singularly perturbed boundary value problem containing both control parameter and integral condition is first being considered. For the numerical solution of such problems, requires specific approach in constructing of the appropriate difference scheme and examining the error analysis. The scheme is constructed by the method of integral identities with the use of appropriate quadrature rules with the remainder terms in integral form. We show that the proposed scheme is uniformly convergent in the discrete maximum norm accuracy of $O(N^{-1}\ln N)$ on Shishkin meshes. First, the asymptotic estimates for the continuous solution are given in Section 2, which are needed in later sections for the analysis of appropriate numerical solution. In Section 3, we describe the finite discretization and give the difference scheme on a piecewise uniform grid. In Section 4, the convergence analysis is carried out. Finally, in Section 5 presents some numerical results to confirm the theoretical analysis. Henceforth, $C$ and $c$ denote the generic positive constants independent of both the perturbation parameter $\varepsilon$ and mesh parameter $N$. Such subscripted constants are also independent of $\varepsilon$ and mesh parameter, but whose values are fixed.

## 2. Asymptotic behavior of the exact solution

In this section, we give a priori estimates for the solution and its derivatives of the problem (1.1)-(1.3), which indicate the asymptotic behavior of the solution and its first derivative in respect to perturbation parameter. These estimates are unimprovable in terms of the view of behavior in $\varepsilon$ and will be used in order to analyse the numerical solution. We also denote $\|g\|_\infty = \max_{[0,T]}|g(t)|$ for any $g \in C\,[0,T]$.

**Lemma 2.1** *The solution $\{u(t),\lambda\}$ of the problem (1.1)-(1.3) satisfies the following bounds:*

$$|\lambda| \le c_0, \tag{2.1}$$

$$\|u\|_\infty \le c_1, \tag{2.2}$$

*where*

$$c_0 = m_1^{-1}\left\{\frac{\alpha\,|A|}{e^{\alpha T}-1} + \frac{|B|\,a^*(1-\|c\|_\infty T)}{m_1\left(e^{a^*T}-1\right)} + \|F\|_\infty\right\},$$

$$c_1 = |u(0)| + \alpha^{-1}(\|F\|_\infty + |\lambda| M_1) = |u(0)| + \alpha^{-1}(\|F\|_\infty + c_0 M_1),$$

$$\left|u'(t)\right| \leq C\left(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha t}{\varepsilon}}\right), \; t \in [0,T], \tag{2.3}$$

*provided* $a \in C^1[0,T]$ *and* $\left|\frac{\partial f}{\partial t}\right| \leq C$ *for* $t \in [0,T]$ *and* $|u| \leq c_1, |\lambda| \leq c_0$.

*Proof.* The quasilinear equation (1.1) can be written as

$$\varepsilon u' + a(t)u = F(t) + \lambda b(t), \quad t \in [0,T], \tag{2.4}$$

where

$$a(t) = \frac{\partial f}{\partial u}(t, \tilde{u}, \tilde{\lambda}),$$

$$b(t) = -\frac{\partial f}{\partial \lambda}(t, \tilde{u}, \tilde{\lambda}),$$

$\widetilde{u} = \gamma u$, $\widetilde{\lambda} = \gamma\lambda$ $(0 < \gamma < 1)$—intermediate values.

Integrating (2.4), (1.3) we have

$$u(t) = Be^{\frac{1}{\varepsilon}\int_t^T a(\xi)d\xi} - \frac{1}{\varepsilon}\int_t^T F(\xi)e^{\frac{1}{\varepsilon}\int_t^\xi a(\eta)d\eta}d\xi + \frac{\lambda}{\varepsilon}\int_t^T b(\xi)e^{\frac{1}{\varepsilon}\int_t^\xi a(\eta)d\eta}d\xi,$$

from which, after using the integral boundary condition (1.2), it follows that,

$$Be^{\frac{1}{\varepsilon}\int_0^T a(\xi)d\xi} - \frac{1}{\varepsilon}\int_0^T F(\xi)e^{\frac{1}{\varepsilon}\int_0^\xi a(\eta)d\eta}d\xi + \frac{\lambda}{\varepsilon}\int_0^T b(\xi)e^{\frac{1}{\varepsilon}\int_0^\xi a(\eta)d\eta}d\xi$$
$$+B\int_0^T c(s)e^{\frac{1}{\varepsilon}\int_s^T a(\xi)d\xi}ds - \frac{1}{\varepsilon}\int_0^T c(s)[\int_s^T F(\xi)e^{\frac{1}{\varepsilon}\int_s^\xi a(\eta)d\eta}d\xi]ds$$
$$+\frac{\lambda}{\varepsilon}\int_0^T c(s)[\int_s^T b(\xi)e^{\frac{1}{\varepsilon}\int_s^\xi a(\eta)d\eta}d\xi]ds = A$$

and

$$\lambda = \frac{A}{\frac{1}{\varepsilon}\int_0^T b(\xi)e^{\frac{1}{\varepsilon}\int_0^\xi a(\eta)d\eta}d\xi + \frac{1}{\varepsilon}\int_0^T b(\xi)[\int_s^T c(s)e^{\frac{1}{\varepsilon}\int_s^\xi a(\eta)d\eta}ds]d\xi}$$
$$-\frac{B(e^{\frac{1}{\varepsilon}\int_0^T a(\xi)d\xi} + \int_0^T c(s)e^{\frac{1}{\varepsilon}\int_s^T a(\xi)d\xi}ds)}{\frac{1}{\varepsilon}\int_0^T b(\xi)e^{\frac{1}{\varepsilon}\int_0^\xi a(\eta)d\eta}d\xi + \frac{1}{\varepsilon}\int_0^T b(\xi)[\int_s^T c(s)e^{\frac{1}{\varepsilon}\int_s^\xi a(\eta)d\eta}ds]d\xi}$$
$$+\frac{\frac{1}{\varepsilon}\int_0^T F(\xi)e^{\frac{1}{\varepsilon}\int_0^\xi a(\eta)d\eta}d\xi + \frac{1}{\varepsilon}\int_0^T F(\xi)[\int_s^T c(s)e^{\frac{1}{\varepsilon}\int_s^\xi a(\eta)d\eta}ds]d\xi}{\frac{1}{\varepsilon}\int_0^T b(\xi)e^{\frac{1}{\varepsilon}\int_0^\xi a(\eta)d\eta}d\xi + \frac{1}{\varepsilon}\int_0^T b(\xi)[\int_s^T c(s)e^{\frac{1}{\varepsilon}\int_s^\xi a(\eta)d\eta}ds]d\xi}. \tag{2.5}$$

As $c(t) \geq 0$, then after applying the mean value theorem for integrals, we deduce that,

$$\left| \frac{\frac{1}{\varepsilon}\int_0^T F(\xi)e^{\frac{1}{\varepsilon}\int_0^\xi a(\eta)d\eta}d\xi + \frac{1}{\varepsilon}\int_0^T F(\xi)[\int_s^T c(s)e^{\frac{1}{\varepsilon}\int_s^\xi a(\eta)d\eta}ds]d\xi}{\frac{1}{\varepsilon}\int_0^T b(\xi)e^{\frac{1}{\varepsilon}\int_0^\xi a(\eta)d\eta}d\xi + \frac{1}{\varepsilon}\int_0^T b(\xi)[\int_s^T c(s)e^{\frac{1}{\varepsilon}\int_s^\xi a(\eta)d\eta}ds]d\xi} \right| \leq m_1^{-1}\|F\|_\infty \tag{2.6}$$

and

$$\left| \frac{B(e^{\frac{1}{\varepsilon}\int_0^T a(\xi)d\xi} + \int_0^T c(s)e^{\frac{1}{\varepsilon}\int_s^T a(\xi)d\xi}ds)}{\frac{1}{\varepsilon}\int_0^T b(\xi)e^{\frac{1}{\varepsilon}\int_0^\xi a(\eta)d\eta}d\xi + \frac{1}{\varepsilon}\int_0^T b(\xi)[\int_s^T c(s)e^{\frac{1}{\varepsilon}\int_s^\xi a(\eta)d\eta}ds]d\xi} \right| \tag{2.7}$$

$$\leq \frac{|B|(1+\|c\|_\infty T)}{m_1\varepsilon^{-1}\int_0^T e^{\frac{1}{\varepsilon}\int_\xi^T a(\eta)d\eta}d\xi}$$

$$\leq \frac{|B|(1+\|c\|_\infty T)}{m_1(a*)^{-1}(1-e^{-\frac{a^*T}{\varepsilon}})} \leq \frac{|B|(1+\|c\|_\infty T)}{m_1(a^*)^{-1}(1-e^{-a^*T})}, (\varepsilon \leq 1).$$

Also, for the first term in right side of (2.5) for $\varepsilon \leq 1$ values, we get

$$\left| \frac{A}{\frac{1}{\varepsilon}\int_0^T b(\xi)e^{\frac{1}{\varepsilon}\int_0^\xi a(\eta)d\eta}d\xi + \frac{1}{\varepsilon}\int_0^T b(\xi)[\int_s^T c(s)e^{\frac{1}{\varepsilon}\int_s^\xi a(\eta)d\eta}ds]d\xi} \right|$$

$$\leq \frac{|A|}{\frac{1}{\varepsilon}\int_0^T b(\xi)e^{\frac{1}{\varepsilon}\int_0^\xi a(\eta)d\eta}d\xi}$$

$$\leq \frac{|A|}{m_1\alpha^{-1}(e^{\frac{\alpha T}{\varepsilon}}-1)} \leq \frac{|A|}{m_1\alpha^{-1}(e^{\alpha T}-1)} \leq \frac{\alpha|A|}{m_1(e^{\alpha T}-1)}. \tag{2.8}$$

The relation (2.5), by taking into consideration here (2.6)-(2.8), immediately leads to (2.1).

Now, integrating (2.4), we have

$$u(t) = u(0)e^{-\frac{1}{\varepsilon}\int\limits_0^t a(\eta)d\eta} + \frac{1}{\varepsilon}\int\limits_0^t \Phi(\xi)e^{-\frac{1}{\varepsilon}\int\limits_\xi^t a(\eta)d\eta}d\xi; \;\; \Phi(s) = F(s) - \lambda b(s),$$

from which, by setting the integral boundary condition (1.2), we get

$$u(0) = \frac{A - \frac{1}{\varepsilon}\int\limits_0^T c(s)[\int\limits_0^s \Phi(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^s a(\eta)d\eta}d\xi]ds}{1 + \int\limits_0^T c(s)e^{-\frac{1}{\varepsilon}\int_0^s a(\xi)d\xi}ds}.$$

Since $c(t)$ is nonnegative, then

$$|u(0)| = \left| \frac{A - \frac{1}{\varepsilon}\int_0^T c(s)[\int_0^s \Phi(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^s a(\eta)d\eta}d\xi]ds}{1+\int_0^T c(s)e^{-\frac{1}{\varepsilon}\int_0^s a(\xi)d\xi}ds} \right|$$

$$\le |A| + \frac{1}{\varepsilon}\int_0^T c(s)[\int_0^s |\Phi(\xi)|e^{-\frac{1}{\varepsilon}\int_\xi^s a(\eta)d\eta}d\xi]ds$$

$$\le |A| + \frac{1}{\varepsilon}\|c\|_\infty(\|F\|_\infty + M_1c_0)\alpha^{-1}\varepsilon\int_0^T (1-e^{-\frac{\alpha s}{\varepsilon}})ds$$

$$\le |A| + \alpha^{-1}\|c\|_\infty T(\|F\|_\infty + M_1c_0). \tag{2.9}$$

Next, by virtue of maximum principle we have

$$\|u\|_\infty \le |u(0)| + \alpha^{-1}\|F - b\lambda\|_\infty$$
$$\le |u(0)| + \alpha^{-1}(\|F\|_\infty + |\lambda|M_1),$$

which, after taking into account (2.1) and (2.9) leads to (2.2).
To prove (2.3), first we estimate $u'(0)$:

$$|u'(0)| \le \frac{|F(0) - a(0)u(0) - b(0)\lambda)|}{\varepsilon} \le \frac{C}{\varepsilon}.$$

Differentiating, now the equation (2.4), we have

$$\varepsilon v'' + p(t)v' = g(t),$$

with

$$v = u',\ p(t) = \frac{\partial f}{\partial u}(t,u(t),\lambda) \text{ and } g(t) = \frac{\partial f}{\partial t}(t,u(t),\lambda).$$

So

$$v(t) = v(0)e^{-\frac{1}{\varepsilon}\int_0^t a(s)ds} + \frac{1}{\varepsilon}\int_0^t g(s)e^{-\frac{1}{\varepsilon}\int_s^t a(\xi)d\xi}ds.$$

Since $p(t) \ge \alpha > 0$ and $|g(t)| \le C$, for $v(t)$ we then obtain

$$|v(t)| \le \frac{C}{\varepsilon}e^{\frac{-\alpha t}{\varepsilon}} + \frac{C}{\varepsilon}\int_0^t e^{\frac{-\alpha(t-s)}{\varepsilon}}ds$$

$$\le \frac{C}{\varepsilon}e^{\frac{-\alpha t}{\varepsilon}} + C(1-e^{\frac{-\alpha t}{\varepsilon}})$$

which implies validity of (2.3). □

## 3. Discrete problem

Let $\omega_N$ be any non-uniform mesh on $\Omega$ :

$$\omega_N = \{0 < t_1 < t_2 < ... < t_{N-1} < t_N = T\}$$

and $\bar{\omega}_N = \omega_N \cup \{t = 0\}$. For each $i \geq 1$, we set the step size $h_i = t_i - t_{i-1}$. To simplify the notation we set $g_i = g(t_i)$ for any function $g(t)$, while $g_i^N$ denotes an approximation of $g(t)$ at $t_i$.
For any mesh function $\{w_i\}$ defined on $\overline{\omega}_N$ we use

$$w_{\bar{t},i} = (w_i - w_{i-1})/h_i,$$
$$\|w\|_\infty \equiv \|w\|_{\infty,\bar{\omega}_N} := \max_{0\leq i\leq N} |w_i|.$$

To obtain approximation for (1.1) we integrate (1.1) over $(t_{i-1}, t_i)$ :

$$\varepsilon u_{\bar{t},i} + h_i^{-1} \int\limits_{t_{i-1}}^{t_i} f(t, u(t), \lambda) dt = 0, \; 1 \leq i \leq N,$$

which yields the relation

$$\varepsilon u_{\bar{t},i} + f(t_i, u_i, \lambda) + R_i = 0, \; 1 \leq i \leq N, \tag{3.1}$$

with local truncation error

$$R_i = -h_i^{-1} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) \frac{d}{dt} f\,(t, u(t), \lambda)\, dt. \tag{3.2}$$

To define an approximation for the boundary condition (1.2), here we use the composite right-side rectangle rule:

$$u(0) + \int_0^T c(s)u(s)ds = u_0 + \sum_{i=1}^N h_i c_i u_i + r$$

with remainder term

$$r = -\sum_{i=1}^N \int_{t_{i-1}}^{t_i} (t - t_{i-1}) \frac{d}{dt} (c(t)u(t))\, dt. \tag{3.3}$$

Consequently

$$u_0 + \sum_{i=1}^N h_i c_i u_i + r = A. \tag{3.4}$$

Neglecting $R_i$ and $r$ in (3.1) and (3.4), we propose the following difference scheme for approximating (1.1)-(1.3):

$$\varepsilon u_{\bar{t},i}^N + f(t_i, u_i^N, \lambda^N) = 0, \; 1 \leq i \leq N, \tag{3.5}$$

$$u_0^N+\sum_{i=1}^{N}h_i c_i u_i^N=A, \tag{3.6}$$

$$u_N^N=B. \tag{3.7}$$

The difference scheme (3.5)-(3.7), in order to be $\varepsilon-$uniform convergent, we will use the Shishkin mesh. For an even number $N$, the piecewise uniform mesh takes $N/2$ points in the interval $[0,\sigma]$ and also $N/2$ points in the interval $[\sigma,T]$, where the transition point $\sigma$, which separates the fine and coarse portions of the mesh, is obtained by taking

$$\sigma=\min\left\{\frac{T}{2},\alpha^{-1}\varepsilon\ln\varepsilon\right\}.$$

In practice one usually has $\sigma\le T$, so the mesh is fine on $[0,\sigma]$ and coarse on $[\sigma,T]$. Hence, if we denote by $h^{(1)}$ and $h^{(2)}$ the step size in $[0,\sigma]$ and $[\sigma,T]$, respectively, we have

$$h^{(1)}=2\sigma N^{-1},\quad h^{(2)}=2(T-\sigma)N^{-1},$$

$$h^{(1)}\le TN^{-1},TN^{-1}\le h^{(2)}<2TN^{-1},\quad h^{(1)}+h^{(2)}=2TN^{-1},$$

so

$$\bar{\omega}_N=\begin{cases} t_i=ih^{(1)}, & \text{for } i=0,1,...,N/2;\ h^{(1)}=2\sigma/N,\\ t_i=\sigma+(i-N/2)h^{(2)}, & \text{for } i=N/2+1,...,N;\ h^{(2)}=2(T-\sigma)/N.\end{cases}$$

In the rest of the paper we only consider this mesh.

## 4. Uniform error estimates

To investigate the convergence of the method, note that the error functions $z_i^N=u_i^N-u_i$, $0\le i\le N$, $\mu^N=\lambda^N-\lambda$ are the solution of the discrete problem

$$\varepsilon z_{\bar{t},i}^N+f(t_i,u_i^N,\lambda^N)-f(t_i,u_i,\lambda)=R_i,\ 1\le i\le N, \tag{4.1}$$

$$z_0^N+\sum_{i=1}^{N}h_i c_i z_i^N-r=0, \tag{4.2}$$

$$z_N^N=0. \tag{4.3}$$

where the truncation errors $R_i$ and $r$ are given by (3.2) and (3.3), respectively.

**Lemma 4.1.** The solution of the first order difference equation

$$y_i=q_i y_{i-1}+\varphi_i\ ,\ 1\le i\le N$$

can be expressed in the following forms:

$$y_i=y_0 Q_i+\sum_{k=1}^{i}\varphi_k Q_{i-k} \tag{4.4}$$

or

$$y_i = y_N Q_{N-i}^{-1} - \sum_{k=i+1}^{N} \varphi_k Q_{k-i}^{-1} \tag{4.5}$$

where

$$Q_{i-k} = \begin{cases} 1, & k = i, \\ \prod_{\ell=k+1}^{i} q_\ell, & 1 \le k \le i-1. \end{cases}$$

The relations (4.4) and (4.5) can be easily verified by induction in $i$.

**Lemma 4.2.** Under the above assumptions of Section 1 and Lemma 2.1, for the error functions $R$ and $r$, the following estimates hold:

$$\|R\|_{\infty,\omega_N} \le CN^{-1}\ln N, \tag{4.6}$$

$$|r| \le CN^{-1}\ln N. \tag{4.7}$$

*Proof.* From explicit expression (3.2) for $R_i$, on an arbitrary mesh we have

$$|R_i| \le h_i^{-1}\int_{t_{i-1}}^{t_i} (t-t_{i-1})\left|\frac{\partial f}{\partial t}(t,u(t),\lambda) + \frac{\partial f}{\partial u}(t,u(t),\lambda)\,u'(t)\right| dt$$

$$\le C h_i^{-1}\int_{t_{i-1}}^{t_i} (t-t_{i-1})\left(1+\left|u'(t)\right|\right) dt,\ 1 \le i \le N.$$

This inequality together with (2.3) enables us to write

$$|R_i| \le C\left\{h_i^{-1} + h_i^{-1}\varepsilon^{-1}\int_{t_{i-1}}^{t_i} (t-t_{i-1})\,e^{-\alpha t/\varepsilon}dt\right\},\ 1 \le i \le N, \tag{4.8}$$

in which

$$h_i = \begin{cases} h^{(1)}, & 1 \le i \le N/2, \\ h^{(2)}, & N/2+1 \le i \le N. \end{cases}$$

We consider first the case $\sigma = T/2$ and so $T/2 \le \alpha^{-1}\varepsilon \ln N$ and $h^{(1)} = h^{(2)} = TN^{-1}$. Hereby, since

$$h_i^{-1}\varepsilon^{-1}\int_{t_{i-1}}^{t_i} (t-t_{i-1})\,e^{-\alpha t/\varepsilon}dt \le \varepsilon^{-1}h^{(1)} \le \frac{2\ln N}{\alpha T}\frac{T}{N} = 2\alpha^{-1}N^{-1}\ln N,$$

it follows from (4.8) that

$$|R_i| \le CN^{-1}\ln N,\ 1 \le i \le N. \tag{4.9}$$

We now consider the case $\sigma = \alpha^{-1}\varepsilon \ln N$ and estimate $R_i$ on $[0,\sigma]$ and $[\sigma,T]$ separately. In the layer region $[0,\sigma]$, inequality (4.8) reduces to

$$|R_i| \le C\left(1+\varepsilon^{-1}\right)h^{(1)} = C\left(1+\varepsilon^{-1}\right)\frac{\alpha^{-1}\varepsilon\ln N}{N/2},\ 1 \le i \le N/2.$$

Hence

$$|R_i| \le CN^{-1}\ln N,\ 1\le i\le N/2. \tag{4.10}$$

It remains to estimate $R_i$ for $N/2+1\le i\le N$. In this case we are able to write (4.8) as

$$|R_i| \le C\left\{h^{(2)}+\alpha^{-1}\left(e^{-\frac{\alpha t_{i-1}}{\varepsilon}}-e^{-\frac{\alpha t_i}{\varepsilon}}\right)\right\},\ N/2+1\le i\le N. \tag{4.11}$$

Since $t_i=\alpha^{-1}\varepsilon\ln N+(i-N/2)h^{(2)}$ it follows that:

$$e^{-\frac{\alpha t_{i-1}}{\varepsilon}}-e^{-\frac{\alpha t_i}{\varepsilon}}=\frac{1}{N}e^{-\frac{\alpha(i-1-\frac{N}{2})h^{(2)}}{\varepsilon}}\left(1-e^{-\frac{\alpha h^{(2)}}{\varepsilon}}\right)<N^{-1}$$

and this together with (4.11) to give the bound

$$|R_i|\le CN^{-1}. \tag{4.12}$$

The inequalities (4.9) , (4.10) and (4.12) finish the proof of (4.6).
Finally, we estimate the remainder term $r$. From the explicit expression (3.3) we obtain

$$|r|\le\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}c(t)|(t-t_{i-1})||u'(t)|dt,\ 1\le i\le N,$$

This inequality together with (2.3) enable use to write

$$|r|\le\|c\|_\infty C\sum_{i=1}^{N}h_i\int_{t_{i-1}}^{t_i}\left(1+\frac{1}{\varepsilon}e^{-\frac{\alpha t}{\varepsilon}}\right)dt,\ 1\le i\le N. \tag{4.13}$$

From (4.13), the validity of (4.7) follows:

$$\begin{aligned}|r|\le C\sum_{i=1}^{N/2}h^{(1)}\int_{t_{i-1}}^{t_i}\left(1+\frac{1}{\varepsilon}e^{-\frac{\alpha t}{\varepsilon}}\right)dt+C\sum_{i=N/2+1}^{N}h^{(2)}\int_{t_{i-1}}^{t_i}\left(1+\frac{1}{\varepsilon}e^{-\frac{\alpha t}{\varepsilon}}\right)dt,\\ \le Ch^{(1)}\int_0^T\left(1+\frac{1}{\varepsilon}e^{-\frac{\alpha t}{\varepsilon}}\right)dt+Ch^{(2)}\int_0^T\left(1+\frac{1}{\varepsilon}e^{-\frac{\alpha t}{\varepsilon}}\right)dt,\\ \le C\left(h^{(1)}+h^{(2)}\right)\le CN^{-1}\ln N.\end{aligned}$$

□

**Lemma 4.3.** For the solution of (4.1)-(4.3), the following estimates hold

$$\left|\mu^N\right|\le C|r|, \tag{4.14}$$

$$\left|z_0^N\right|\le|r|+\|c\|_\infty TB_N\left(\left|\mu^N\right|M_1+\|R\|_\infty\right), \tag{4.15}$$

$$\left|z_i^N\right|\le\left|z_0^N\right|+\alpha^{-1}(M_1\left|\mu^N\right|+\|R\|_\infty),\ 1\le i\le N-1, \tag{4.16}$$

where

$$B_N = \sum_{\ell=1}^{N} \frac{h_\ell}{\varepsilon + a_\ell h_\ell} Q_{N-\ell},$$

$$Q_{N-\ell} = \begin{cases} 1, & \text{for } \ell = N, \\ \prod_{s=\ell+1}^{N} \frac{\varepsilon}{\varepsilon + a_s h_s}, & \text{for } 1 \le \ell \le N-1. \end{cases}$$

*Proof.* The equation (4.1) can be rewritten as

$$\varepsilon z_{\bar{t},i}^{N} + a_i z_i^N = b_i \mu^N + R_i, \quad 1 \le i \le N-1, \tag{4.17}$$

with

$$a_i = \frac{\partial f}{\partial u}\left(t_i, u_i + \gamma z_i^N, \lambda + \gamma \mu^N\right),$$

$$b_i = -\frac{\partial f}{\partial \lambda}\left(t_i, u_i + \gamma z_i^N, \lambda + \gamma \mu^N\right), \ 0 < \gamma < 1.$$

From (4.17) we have

$$z_i^N = \frac{\varepsilon}{\varepsilon + a_i h_i} z_{i-1}^N + \mu^N \frac{h_i b_i}{\varepsilon + a_i h_i} + \frac{h_i R_i}{\varepsilon + a_i h_i}. \tag{4.18}$$

Solving the first-order difference equation with respect to $z_i^N$ by using (4.5) and setting the boundary condition (4.3), we get

$$z_i^N = -\mu^N \sum_{k=i+1}^{N} \frac{h_k b_k}{\varepsilon + a_k h_k} Q_{k-i}^{-1} - \sum_{k=i+1}^{N} \frac{h_k R_k}{\varepsilon + a_k h_k} Q_{k-i}^{-1}. \tag{4.19}$$

Taking into consideration in (4.19) the integral boundary condition (4.2), we have

$$\mu^N = \frac{r}{\sum_{k=1}^{N} \frac{h_k b_k}{\varepsilon + a_k h_k} Q_k^{-1} + \sum_{k=1}^{N} h_k c_k \sum_{s=k+1}^{N} \frac{h_s b_s}{\varepsilon + a_s h_s} Q_{s-k}^{-1}}$$

$$+ \frac{\sum_{k=1}^{N} \frac{h_k R_k}{\varepsilon + a_k h_k} Q_k^{-1} + \sum_{k=1}^{N} h_k c_k \sum_{s=k+1}^{N} \frac{h_s R_s}{\varepsilon + a_s h_s} Q_{s-k}^{-1}}{\sum_{k=1}^{N} \frac{h_k b_k}{\varepsilon + a_k h_k} Q_k^{-1} + \sum_{k=1}^{N} h_k c_k \sum_{s=k+1}^{N} \frac{h_s b_s}{\varepsilon + a_s h_s} Q_{s-k}^{-1}}. \tag{4.20}$$

Now, we estimate separately the terms on the right-hand side of equality (4.20). For the first term, we have

$$\left| \frac{r}{\sum_{k=1}^{N} \frac{h_k b_k}{\varepsilon + a_k h_k} Q_k^{-1} + \sum_{k=1}^{N} h_k c_k \sum_{s=k+1}^{N} \frac{h_s b_s}{\varepsilon + a_s h_s} Q_{s-k}^{-1}} \right|$$

$$\le \frac{|r|}{m_1 \sum_{k=1}^{N} \frac{h_k}{\varepsilon + a_k h_k} Q_k^{-1}} \le \frac{a^* |r|}{m_1 \rho_* \sum_{k=1}^{N} (1+\rho_*)^{k-1}} = \frac{a^* |r|}{m_1 \left[(1+\rho_*)^N - 1\right]},$$

here $\rho_k = a_k h_k/\varepsilon$ and $\rho_* = \min \rho_k$. Therefore, it is not hard to see that

$$\left| \frac{r}{\sum_{k=1}^{N} \frac{h_k b_k}{\varepsilon + a_k h_k} Q_k^{-1} + \sum_{k=1}^{N} h_k c_k \sum_{s=k+1}^{N} \frac{h_s b_s}{\varepsilon + a_s h_s} Q_{s-k}^{-1}} \right| \le C\,|r| \qquad (4.21)$$

Next, evidently

$$\left| \frac{\sum_{k=1}^{N} \frac{h_k R_k}{\varepsilon + a_k h_k} Q_k^{-1} + \sum_{k=1}^{N} h_k c_k \sum_{s=k+1}^{N} \frac{h_s R_s}{\varepsilon + a_s h_s} Q_{s-k}^{-1}}{\sum_{k=1}^{N} \frac{h_k b_k}{\varepsilon + a_k h_k} Q_k^{-1} + \sum_{k=1}^{N} h_k c_k \sum_{s=k+1}^{N} \frac{h_s b_s}{\varepsilon + a_s h_s} Q_{s-k}^{-1}} \right| \le m_1^{-1} \|R\|_\infty . \quad (4.22)$$

After taking into consideration (4.21) and (4.22) in (4.20), we arrive at (4.14). Now, we need to estimate $z_0$. From (4.18), by using (4.4) we have

$$z_i^N = z_0^N Q_i + \mu^N \sum_{k=1}^{i} \frac{h_k b_k}{\varepsilon + a_k h_k} Q_{i-k} + \sum_{k=1}^{i} \frac{h_k R_k}{\varepsilon + a_k h_k} Q_{i-k}.$$

From here, by virtue of (4.2) it follows that

$$z_0^N = \frac{r - \sum_{k=1}^{N} h_k c_k \left( \mu^N \sum_{\ell=1}^{k} \frac{h_\ell b_\ell}{\varepsilon + a_\ell h_\ell} Q_{k-\ell} + \sum_{\ell=1}^{k} \frac{h_\ell R_\ell}{\varepsilon + a_\ell h_k} Q_{k-\ell} \right)}{1 + \sum_{k=1}^{N} h_k c_k Q_k}.$$

Thereby

$$\left| z_0^N \right| \le |r| + \|c\|_\infty T \left\{ \left| \mu^N \right| \sum_{\ell=1}^{N} \frac{h_\ell |b_\ell|}{\varepsilon + a_\ell h_k} Q_{N-\ell} + \sum_{\ell=1}^{N} \frac{h_\ell |R_\ell|}{\varepsilon + a_\ell h_k} Q_{k-\ell} \right\},$$

$$\le |r| + \|c\|_\infty T \left\{ (M_1 \left| \mu^N \right| + \|R\|_\infty \right\} \sum_{\ell=1}^{N} \frac{h_\ell}{\varepsilon + a_\ell h_\ell} Q_{N-\ell},$$

which implies validity of (4.15).

Finally, an applying the maximum principle for the difference operator $L^N z_i^N := \varepsilon z_{\bar{t},i}^N + a_i z_i^N$, $1 \le i \le N$, to Eq. (4.17) immediately leads to (4.16). □

Combining the two previous lemmas gives us the following convergence result.

**Theorem 4.1.** Let $\{u(t), \lambda\}$ and $\{u_i^N, \lambda^N\}$ be the exact solution and discrete solution on $\bar{\omega}_N$ respectively. Then the following estimates hold

$$\left| \lambda - \lambda^N \right| \le C N^{-1} \ln N,$$

$$\left\| u - u^N \right\|_{\infty, \varpi_N} \le C N^{-1} \ln N.$$

*Proof.* This follows immediately by combining the previous lemmas. □

## 5. Algorithm and numerical results

Here, we consider a test problem to show the applicability and efficiency of the method described in this paper.

a) We solve the nonlinear problem (3.5)-(3.7) using the following quasilinearization technique:

$$\lambda^{(n)} = \lambda^{(n-1)} - \frac{\left(B - u_{N-1}^{(n-1)} - u_{i-1}^{(n)}\right)\rho_N^{-1} + f\left(T, B, \lambda^{(n-1)}\right)}{\partial f/\partial\lambda\left(T, B, \lambda^{(n)}\right)},$$

$$u_0^{(n)} = A - c_N h_N B - \sum_{i=1}^{N-1} h_i b_i u_i^{(n-1)},$$

$$u_i^{(n)} = u_i^{(n-1)} - \frac{\left(u_i^{(n-1)} - u_{i-1}^{(n)}\right)\rho_i^{-1} + f\left(t_i, u_i^{(n-1)}, \lambda^{(n)}\right)}{\partial f/\partial u\left(t_i, u_i^{(n-1)}, \lambda^{(n)}\right) + \rho_i^{-1}}, \; n = 1, 2, ...$$

where, $\rho_i = h_i/\varepsilon$; $\lambda^{(0)}$ and $u_i^{(0)}(1 \le i \le N-1)$ are the initial iterations given.

b) Consider the test problem:

$$\varepsilon u' + 2u - e^{-u} + t^2 + \lambda + \tanh(\lambda + t) = 0, \;\; 0 < t < 1,$$

$$u(0) + \frac{1}{4}\int_0^1 e^{-s} u(s)ds = 1,$$

$$u(1) = 0.$$

The exact solution of our test problem is not available. Therefore we use the double mesh principle to estimate the errors and to compute the experimental rates of convergence. The error estimates obtained in this way are denoted by

$$e_u^{\varepsilon,N} = \max_{\omega_N}\left|u^{\varepsilon,N} - \tilde{u}^{\varepsilon,2N}\right|, \; e_\lambda^{\varepsilon,N} = \left|\lambda^{\varepsilon,N} - \tilde{\lambda}^{\varepsilon,2N}\right|,$$

where $\left\{\tilde{u}^{\varepsilon,2N}, \tilde{\lambda}^{\varepsilon,2N}\right\}$ is the approximate solution on the mesh

$$\widetilde{\omega}_{2N} = \left\{t_{i/2} : i = 0, 1, ..., 2N\right\}$$

with $t_{i+1/2} = (t_i + t_{i+1})/2$ for $i = 0, 1, ..., N-1$. The corresponding rates of convergence are calculated by

$$p_u^{\varepsilon,N} = \ln(e_u^{\varepsilon,N}/e_u^{\varepsilon,2N})/\ln 2$$

for $u$, and

$$p_\lambda^{\varepsilon,N} = \ln(e_\lambda^{\varepsilon,N}/e_\lambda^{\varepsilon,2N})/\ln 2$$

for $\lambda$. The $\varepsilon-$uniform errors $p_u^N$, $p_\lambda^N$ are estimated from

$$e_u^N = \max_\varepsilon e_u^{\varepsilon,N}, \; e_\lambda^N = \max_\varepsilon e_\lambda^{\varepsilon,N}.$$

The corresponding $\varepsilon-$uniform convergence rates are

$$p_u^N = \ln(e_u^N / e_u^{2N}) / \ln 2, \quad p_\lambda^N = \ln(e_\lambda^N / e_\lambda^{2N}) / \ln 2.$$

In the computations in this section we take $\alpha = 2$. The initial guess in the iteration process is taken as $u_i^{(0)} = 1 - t_i^2$, $\lambda^{(0)} = -0.4$ and the stopping criterion is

$$\max_i \left| u_i^{(n)} - u_i^{(n-1)} \right| \le 10^{-5}, \quad \left| \lambda^{(n)} - \lambda^{(n-1)} \right| \le 10^{-5}.$$

The values of $\varepsilon$ and $N$ for which we solve the test problem are $\varepsilon = 2^{-i}$, $i = 2, 4, ..., 16$; $N = 64, 128, 256, 512, 1024$. Some results of numerical experiment are displayed in Tables 1 and 2. The numerical results are the clear illustration of the error estimates.

TABLE 1. Errors $e_u^{\varepsilon,N}$ computed $\varepsilon-$uniform errors $e_u^N$ and convergence rates $p_u^{\varepsilon,N}$ on $\omega_N$.

| $\varepsilon$ | $N = 64$ | $N = 128$ | $N = 256$ | $N = 512$ | $N = 1024$ |
|---|---|---|---|---|---|
| $2^{-2}$ | 0.00047526 | 0.00025293 | 0.00133677 | 0.00069196 | 0.00035325 |
| | 0.91 | 0.92 | 0.95 | 0.97 | |
| $2^{-4}$ | 0.00477384 | 0.00254057 | 0.00133344 | 0.00069070 | 0.00035261 |
| | 0.91 | 0.93 | 0.95 | 0.97 | |
| $2^{-6}$ | 0.00357052 | 0.00203655 | 0.00115359 | 0.00064445 | 0.00035261 |
| | 0.81 | 0.82 | 0.84 | 0.87 | |
| $2^{-8}$ | 0.00354583 | 0.00203655 | 0.00115359 | 0.00064445 | 0.00035261 |
| | 0.80 | 0.82 | 0.84 | 0.87 | |
| $2^{-10}$ | 0.00349702 | 0.00203655 | 0.00115359 | 0.00064445 | 0.00035261 |
| | 0.78 | 0.82 | 0.84 | 0.87 | |
| $2^{-12}$ | 0.00349702 | 0.00203655 | 0.00115359 | 0.00064445 | 0.00035261 |
| | 0.78 | 0.82 | 0.84 | 0.87 | |
| $2^{-14}$ | 0.00349702 | 0.00203655 | 0.00115359 | 0.00064445 | 0.00035261 |
| | 0.78 | 0.82 | 0.84 | 0.87 | |
| $2^{-16}$ | 0.00349702 | 0.00203655 | 0.00115359 | 0.00064445 | 0.00035261 |
| | 0.78 | 0.82 | 0.84 | 0.87 | |
| $e_u^{\varepsilon,N}$ | 0.00349702 | 0.00203655 | 0.00133677 | 0.00069196 | 0.00035261 |
| $p_u^{\varepsilon,N}$ | 0.78 | 0.82 | 0.84 | 0.97 | |

TABLE 2. Errors $e_{\lambda}^{\varepsilon,N}$ computed $\varepsilon-$uniform errors $e_{\lambda}^{N}$ and convergence rates $p_{\lambda}^{\varepsilon,N}$ on $\omega_N$.

| $\varepsilon$ | $N=64$ | $N=128$ | $N=256$ | $N=512$ | $N=1024$ |
|---|---|---|---|---|---|
| $2^{-2}$ | 0.03255711 | 0.01831433 | 0.01002063 | 0. 00533283 | 0.00276045 |
| | 0.83 | 0.87 | 0.91 | 0.95 | |
| $2^{-4}$ | 0.03134362 | 0.01838045 | 0.01048388 | 0.00581630 | 0.00313856 |
| | 0.77 | 0.81 | 0.85 | 0.89 | |
| $2^{-6}$ | 0.03134362 | 0.01838045 | 0.01048388 | 0.00581630 | 0.00313856 |
| | 0.77 | 0.81 | 0.85 | 0.89 | |
| $2^{-8}$ | 0.03134362 | 0.01838045 | 0.01048388 | 0.00581630 | 0.00313856 |
| | 0.77 | 0.81 | 0.85 | 0.89 | |
| $2^{-10}$ | 0.03134362 | 0.01838045 | 0.01048388 | 0.00581630 | 0.00313856 |
| | 0.77 | 0.81 | 0.85 | 0.89 | |
| $2^{-12}$ | 0.03134362 | 0.01838045 | 0.01048388 | 0.00581630 | 0.00313856 |
| | 0.77 | 0.81 | 0.85 | 0.89 | |
| $2^{-14}$ | 0.03134362 | 0.01838045 | 0.01048388 | 0.00581630 | 0.00313856 |
| | 0.77 | 0.81 | 0.85 | 0.89 | |
| $2^{-16}$ | 0.03134362 | 0.01838045 | 0.01048388 | 0.00581630 | 0.00313856 |
| | 0.77 | 0.81 | 0.85 | 0.89 | |
| $e_{\lambda}^{\varepsilon,N}$ | 0.03134362 | 0.01838045 | 0.01048388 | 0.00581630 | 0.00313856 |
| $p_{\lambda}^{\varepsilon,N}$ | 0.77 | 0.81 | 0.85 | 0.89 | |

## 6. CONCLUSION

A parameterized singular perturbation problem with integral boundary condition is considered. The difference scheme is constructed by the method of integral identities with the use of interpolating quadrature rules with the weight and remainder terms in integral form. The numerical method presented here comprises a backward difference operator on a non-uniform mesh for the equation and composite rectangle rule for the integral condition. It is shown that the method displays uniform convergence with respect to the perturbation parameter. Numerical results confirm our theoretical analysis. The main lines for the analysis of the uniform convergence carried out here can be used for the study of more complicated nonlinear singularly perturbed analogous type problems.

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