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# On a decomposition formula in commutative group rings

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# ON A DECOMPOSITION FORMULA IN COMMUTATIVE GROUP RINGS

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Abstract. A necessary and sufficient condition is obtained when the $p$-component of the group of all normalized units can be decomposed into two special subgroups provided that the group basis has a standard proper decomposition. The established formula is useful for proving the niceness of certain subgroups in modular group rings and improves our identical claim in Hokkaido Math. J. (2000) as well.

We also point out a confusion due to Mollov in his reviewer's report (Zbl. Math. 2001) concerning our paper published in Rend. Sem. Mat. Univ. Padova (1999).

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Suppose $R$ is a commutative ring with 1 of prime characteristic $p$ and suppose $G$ is an abelian group with $p$-torsion part $G_p$. Throughout the text, $S(RG)$ shall designate the normed Sylow $p$-subgroup in the group ring $RG$. For $H$ a subgroup of $G$, the symbol $I_p(RG; H)$ will denote the nil-radical of the relative augmentation ideal $I(RG; H)$ of the group ring $RG$ with respect to $H$.

In [2], we have found a convenient direct decomposition of normalized $p$-units of the group $A \times B$ where $A$ and $B$ are arbitrary abelian groups (see, e. g., [3, 4]).

The aim of this paper is to inspect here the validity of the same formula when the product of the subgroups generating the basis is however not direct. The ratio deduced is next used for proving the niceness in commutative modular group rings.

We come now to the central statement motivating the present study.

**Proposition 1** (Decomposition)**.** *Let $G = AB$ with $A \leq B$ and $B \leq G$. Then $S(RG) = S(RA)(1 + I_p(RG; B))$ if and only if $G_p = A_pB_p$.*

Proof. *Necessity.* For given $g_p \in G_p$ we write $g_p(r_1a_1 + r_2a_2 + \ldots + r_ta_t) = f_1b_1 + f_2g_2 + \ldots + f_tg_t$. The canonical forms yield $g_pa_1 = b_1$, $g_pa_2 = g_2$, $\ldots$, $g_pa_t = g_t$. Further, we will distinguish two basic cases:

*Case* 1: $b_1 \in B_p$. Hence it is plain that $g_p = a_1^{-1}b_1 \in A_pB_p$ and everything is proved.

*Case* 2: $b_1 \in B \setminus B_p$. Without loss of generality let $a_2 \in A_p$ and $g_2 \in G_p$; such elements exist because $r_1a_1 + r_2a_2 + \ldots + r_ta_t \in S(RA)$ and $f_1b_1 + f_2g_2 + \ldots + f_tg_t \in 1 + I_p(RG; B) \subseteq S(RG)$. Next, we distinguish four independent subcases (all other remaining possibilities have similar proofs):

2.1. $g_3 \in b_1G_p$, $g_3 = g_2b$ $(b \in B)$, $g_2 \notin B$; $f_1 + f_3$ lies in the nil-ideal of $R$, $f_2 + f_3 = 0$.

In this situation we extract $b_1b^{-1} \in B_p$ and $g_pa_1a_2a_3^{-1} = b_1b^{-1} \in B$, i. e., in other words $g_p \in A_pB_p$.

2.2. $g_3 \in g_2B_p$, $g_2 \notin B$, $g_4 \in b_1G_p$, $g_5 = g_4b' \in G_p$ $(b' \in B)$; $f_2 + f_3 = 0$, $f_1 + f_4$ belongs to the nil-ideal of $R$, $f_4 + f_5 = 0$.

Under these circumstances $b'b_1 \in B_p$, hence $g_pa_1a_5a_4^{-1} = b_1b' \in B_p$, i. e., equivalently $g_p \in A_pB_p$.

2.3. $g_3 = g_2b$ $(b \in B)$, $g_5 = g_4b'$ $(b' \in B)$, $g_4 \in b_1G_p$, $g_3 \in g_5G_p$; $f_2 + f_3 = 0$, $f_4 + f_5 = 0$, $f_3 + f_5$ lies in the nil-radical of $R$.

In that case $b_1^{-1}bb'^{-1} \in B_p$ and $g_p^{-1}a_1^{-1}a_2^{-1}a_3a_4a_5^{-1} = b_1^{-1}b'^{-1}b$ which substantiates our claim.

2.4. $g_3 = g_2b$ $(b \in B)$, $g_5 = g_4b'$ $(b' \in B)$, $g_5 \in b_1G_p$, $g_3 \in g_4G_p$; $f_2 + f_3 = 0$, $f_4 + f_5 = 0$, $f_3 + f_4$ belongs to the nil-radical of $R$.

Because this case is analogous to the last preceding one, by similar computations $b'bb_1^{-1} \in B_p$ and $g_p^{-1}a_1^{-1}a_2^{-1}a_3a_4^{-1}a_5 = b'bb_1^{-1}$.

Finally, in all cases, we deduce $g_p \in A_pB_p$ whence $G_p = A_pB_p$, as stated.

*Sufficiency.* Choosing $x \in S(RG)$, we write $x = f_1a_1b_1 + \ldots + f_ta_tb_t$. Furthermore in the canonical record there is a group member from $G_p$, say, e. g., $a_tb_t$; thus $a_tb_t = a_{tp}b_{tp}$ for some $a_{tp} \in A_p$ and $b_{tp} \in B_p$. Moreover, with no loss of generality, we may presume that the following relations are fulfilled (the remaining cases are analogous): for some $k \in \mathbf{N}$

$$a_1^{p^k}b_1^{p^k} = a_2^{p^k}b_2^{p^k} = \ldots = a_s^{p^k}b_s^{p^k} \neq 1 \qquad (s \in \mathbf{N}),$$

$(f_1 + f_2 + \ldots + f_s)^{p^k} = 0$; $a_{s+1}^{p^k}b_{s+1}^{p^k} = \ldots = a_t^{p^k}b_t^{p^k} = 1$, $(f_{s+1} + \ldots + f_t)^{p^k} = 1$. By the above listed assumptions $a_1a_2^{-1}a_{2p} = b_2b_1^{-1}b_{2p}$ for $a_{2p} \in A_p$, $b_{2p} \in B_p$, etc.; similarly $a_1a_s^{-1}a_{sp} = b_sb_1^{-1}b_{sp}$ for $a_{sp} \in A_p$ and $b_{sp} \in B_p$. Besides this, $a_{s+1}b_{s+1} = a_{s+1p}b_{s+1p}$ with $a_{s+1p} \in A_p$ and $b_{s+1p} \in B_p$, etc., $a_tb_t = a_{tp}b_{tp}$ with $a_{tp} \in A_p$ and $b_{tp} \in B_p$.

After this, we examine the sum $f_1a_1 + \ldots + f_sa_s = f_1a_1 + f_2a_1a_{2p} + \ldots + f_sa_1a_{sp} + f_2a_2 - f_2a_2b_2b_1^{-1}b_{2p} + \ldots + f_sa_s - f_sa_sb_sb_1^{-1}b_{sp} = f_1a_1 + f_2a_1a_{2p} + \ldots + f_sa_1a_{sp} + f_2a_2(1 - b_2b_1^{-1}b_{2p}) + \ldots + f_sa_s(1 - b_sb_1^{-1}b_{sp})$. It is a simple matter to see that $f_1a_1 + f_2a_1a_{2p} + \ldots + f_sa_1a_{sp}$ is a nil-element from $I(RA; A)$, i. e., it lies in $I_p(RA; A)$.

On the other hand, $f_{s+1}a_{s+1}b_{s+1} + \ldots + f_ta_tb_t = f_{s+1}a_{s+1p} + f_{s+1}a_{s+1p}(b_{s+p} - 1) + \ldots + f_ta_{tp} + f_ta_{tp}(b_{tp} - 1)$. Certainly, $f_{s+1}a_{s+1p} + \ldots + f_ta_{tp}$ is a $p$-element, i. e., it belongs to $S(RA)$. As a final step, we detect that $y = f_1a_1 + f_2a_1a_{2p} + \ldots + f_sa_1a_{sp} + f_{s+1}a_{s+1p} + \ldots + f_ta_{tp} \in S(RA)$. Thus, $x = y + f_1a_1(b_1 - 1) + \ldots + f_sa_s(b_s - 1) + f_2a_2(1 - b_2b_1^{-1}b_{2p}) + \ldots + f_sa_s(1 - b_sb_1^{-1}b_{sp}) + f_{s+1}a_{s+1p}(b_{s+1p} - 1) + \ldots + f_ta_{tp}(b_{tp} - 1) \in S(RA) + I(RG; B)$

whence we infer at once that $x \in S(RA)(1 + I_p(RG; B))$, as desired. The proof is now complete. □

*Comment.* At this point we shall give a detailed analysis of the arguments used in the proof of the necessity of Proposition 2.4 from [1]; the notions and notations are as in the considered assertion.

In fact, we have claimed that the non-existence of zero divisors in the field does imply that in the right hand-side of the canonical record of $x$ (see, e. g., [1, p. 54, line 5] and [4, p. 8, lines 1, 2, 3 (-)]) there is an element from $MB$ with nonzero coefficient in $K$, which is correct, because in the right hand-side all possible group relations if they eventually exist between the group elements of the two sums lead us to that; the existing of no zero divisors guarantees that all group members in these relations are of equal worth. If such relations do not exist, everything is done.

Such a claim is true even without this assumption on zero divisors by which we have just shown above or by exploiting the trick that if $0 \neq r \in R$, $\alpha_1, \dots, \alpha_n \in R$, $r\alpha_2 = r\alpha_3 = \dots = r\alpha_n = 0$ and $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$ then $r = r\alpha_1 \neq 0$, i. e., the condition on zero divisors can be removed.

Moreover, the use of the necessity in Proposition 2.4 from [1] in the verification of Proposition 2.6 of [1] for infinitely many factors is correct, of course, since the infinite case follows immediately from the finite one by means of a standard transfinite induction. That is why this argumentation was omitted in the text of the manuscript of [1].

*Remark* 1. The technique shown above leads us to the following intersection identity:

$$[G_p S(RA)] \cap (1 + I_p(RG; B)) = B_p(1 + I_p(RA; A \cap B)).$$

**Claim.** *For any two abelian groups A and B, the following conditions are equivalent:*

(1) $A^{p^n} \cap B^{p^n} = (A \cap B)^{p^n}, \forall n \in \mathbf{N}$;
(2) $(AB)[p^n] = A[p^n]B[p^n], \forall n \in \mathbf{N}$;
(3) $(AB)_p = A_p B_p$ *and for each* $x = ab$ *when* $a \in A_p$ *and* $b \in B_p$ *the condition* $x^{p^m} = 1$ *holds* $\iff$ $a^{p^m} = 1$ *and* $b^{p^m} = 1$ *for any positive integer m.*

Proof. Evidently (2) $\Leftrightarrow$ (3). Now we consider the implication (1) $\Rightarrow$ (2). For this purpose, given $x \in (AB)[p^n]$, hence $x = ab$ with $a \in A$, $b \in B$ and $(ab)^{p^n} = 1$. Thus, $a^{p^n} = b^{-p^n} = c^{p^n}$ for $c \in A \cap B$ and we find $x = ac^{-1}cb \in A[p^n]B[p^n]$.

We treat now (2) $\Rightarrow$ (1). For this goal, choose $x \in A^{p^n} \cap B^{p^n}$. So, $x = a^{p^n} = b^{p^n}$ whenever $a \in A$ and $b \in B$. Henceforth $ab^{-1} \in (AB)[p^n] = A[p^n]B[p^n]$, i. e., $ab^{-1} = a_n b_n$ where $a_n \in A[p^n]$ and $b_n \in B[p^n]$. Observing that $aa_n^{-1} = b_n b \in A \cap B$, we see that $x = (aa_n^{-1})^{p^n} \in (A \cap B)^{p^n}$. The proof is complete . □

A direct consequence is the following.

**Corollary.** *Let* $G = AB$ *with* (1) *fulfilled. Then* $S(RG) = S(RA)(1 + I_p(RG; B))$.

Proof. Although we can copy our original proof from the foregoing main Proposition along with the Claim, we will give another confirmation of the attainment.

Indeed, given $x \in S(RG)$, hence $x = r_1a_1b_1 + \ldots + r_ta_tb_t = r_1a_1(b_1-1) + r_1a_1 + \ldots + r_ta_t(b_t-1) + r_ta_t = r_1a_1 + \ldots + r_ta_t + r_1a_1(b_1-1) + \ldots + r_ta_t(b_t-1)$. Therefore there exists $k \in \mathbf{N}$ so that $1 - (r_1a_1 + \ldots + r_ta_t)^{p^k} = [r_1a_1(b_1-1) + \ldots + r_ta_t(b_t-1)]^{p^k} \in R^{p^k}A^{p^k} \cap I(R^{p^k}G^{p^k}; B^{p^k}) = I(R^{p^k}A^{p^k}; A^{p^k} \cap B^{p^k}) = I(R^{p^k}A^{p^k}; (A \cap B)^{p^k}) = I^{p^k}(RA; A \cap B)$. Consequently there is $v \in I(RA; A \cap B)$ such that $r_1a_1 + \ldots + r_ta_t + v \in S(RA)$. Finally, $x = (r_1a_1 + \ldots r_ta_t + v)(1 + (r_1a_1 + \ldots + r_ta_t + v)^{-1}(r_1a_1(b_1-1) + \ldots + r_ta_t(b_t-1) - v)) \in S(RA)(1 + I_p(RG; B))$, as promised. The proof is complete. □

*Critical Remark.* In his reviewer's report [6], Mollov has claimed that our method of proof of [1, Proposition 2.4] contains an error. But his conclusion is obviously wrong by what we have already quoted above.

Now we are ready to study the niceness in group rings. Conforming with the classical definition for nice subgroups of $p$-primary groups (see, e. g., [5]), we shall say that the subgroup $N$ of the arbitrary abelian group $G$ is nice only when $\cap_{\alpha<\tau}(NG^{p^\alpha}) = NG^{p^\tau}$ for each prime $p$ and for each limit ordinal number $\tau$. If this equality is valid only for one single prime $p$, $N$ is called $p$-nice in $G$.

Owing to the above decomposition formula, we are now in a position to obtain the following

**Proposition 2** (Niceness). *Suppose $N$ is $p$-balanced, that is, $p$-nice and $p$-isotype subgroup of $G$. Then $1 + I_p(RG; N)$ is nice in $S(RG)$ provided $R$ is perfect.*

Proof. Consuming the definition for niceness of $p$-torsion groups, it is enough to show that for every limit ordinal $\tau$ one has $\bigcap_{\alpha<\tau}[S^{p^\alpha}(RG)(1 + I_p(RG; N))] = S^{p^\tau}(RG)(1 + I_p(RG; N))$. For this aim, take an arbitrary element $x$ from the intersection. Thus we may write $x \in (r_{1\alpha}g_{1\alpha} + \ldots + r_{t\alpha}g_{t\alpha})(1 + I_p(RG; N))$ and $x \in (f_1g_{1\beta} + \ldots + f_tg_{t\beta})(1 + I_p(RG; N))$, where $r_{1\alpha}, \ldots, r_{t\alpha}; f_1, \ldots, f_t \in R$ and $g_{1\alpha}, \ldots, g_{t\alpha} \in G^{p^\alpha}$; $g_{1\beta}, \ldots, g_{t\beta} \in G^{p^\beta}$ for an arbitrary ordinal $\beta$ with the property $\alpha < \beta < \tau$. That is why we can write $r_{1\alpha}g_{1\alpha} + \ldots + r_{t\alpha}g_{t\alpha} = (f_1g_{1\beta} + \ldots + f_tg_{t\beta})(e_1a_1 + \ldots + e_ta_t)$, where $1 \neq e_1a_1 + \ldots + e_ta_t \in 1 + I_p(RG; N)$. We note that $r_{1\alpha} + \ldots + r_{t\alpha} = f_1 + \ldots + f_t = e_1 + \ldots + e_t = 1$ and that we may presume $a_1 \in N$. Moreover, let us assume that in $R$ there exist zero divisors; otherwise if there are no zero divisors, the conclusions are simpler or similar to those presented above. Thus, suppose $f_1e_1 \neq 0$ and $f_1e_2 = \ldots = f_1e_t = 0$. Certainly, if $f_1e_1 = 0$, $f_1 = 0$, which is false. Besides, $f_2e_2 \neq 0$, $f_2e_3 \neq 0$ and $f_2e_1 = f_2e_4 = \ldots = f_2e_t = 0$; $f_3e_2 \neq 0$, $f_3e_3 \neq 0$ and $f_3e_1 = f_3e_4 = \ldots = f_3e_t = 0$, etc., due to the symmetry $f_{k-1}e_{k-1} \neq 0$, $f_{k-1}e_k \neq 0$ and $f_{k-1}e_1 = \ldots = f_{k-1}e_{k-2} = f_{k-1}e_{k+1} = \ldots = f_{k-1}e_t = 0$; $f_ke_{k-1} \neq 0$, $f_ke_k \neq 0$ and $f_ke_1 = \ldots = f_ke_{k-2} = f_{k-1}e_{k+1} = \ldots = f_{k-1}e_t = 0$ for some positive integer $k < t$. For the remaining cases, $f_{k+1}e_{k+1} \neq 0$ and $f_{k+1}e_1 = \ldots = f_{k+1}e_k = f_{k+1}e_{k+2} = \ldots = f_{k+1}e_t = 0, \ldots, f_te_t \neq 0$ and $f_te_1 = \ldots = f_te_{t-1} = 0$. We note that $f_1e_1 = f_1, \ldots, f_te_t = f_t$.

The existence of the following relations is natural: $a_3 \in a_2N$ and $e_2 + e_3 = 0$, $a_5 \in a_4N$ and $e_4 + e_5 = 0$, etc., $a_k \in a_{k-1}N$, $e_{k-1} + e_k = 0$. Also, $e_1 + e_{k+1} + \ldots + e_t = 1$ and $a_{k+1}, a_{k+2}, \ldots, a_t \in N$ or $e_1 = 1$, $e_{k+1} + \ldots + e_t = 0$ and $a_{k+1} \in a_{k+2}N \in \ldots \in a_tN$.

So, we obtain $r_{1\alpha}g_{1\alpha} + \ldots + r_{t\alpha}g_{t\alpha} = f_1e_1g_{1\beta}a_1 + f_2e_2g_{2\beta}a_2 - f_2e_2g_{2\beta}a_3 + f_3e_2g_{3\beta}a_2 - f_3e_2g_{3\beta}a_3 + \ldots + f_{k-1}e_1g_{k-1\beta}a_{k-1} - f_{k-1}e_{k-1}g_{k-1\beta}a_k + f_ke_{k-1}g_{k\beta}a_{k-1} - f_ke_{k-1}g_{k\beta}a_k + f_{k+1}e_{k+1}g_{k+1\beta}a_{k+1} + \ldots + f_te_tg_{t\beta}a_t$.

For an extra difficulty, suppose also that $g_{1\beta}a_1$ does not lie in the support of $x$; e. g., $g_{1\beta}a_1 = g_{k+1\beta}a_{k+1}$ and $f_1e_1 + f_{k+1}e_{k+1} = 0$. But then $g_{k+2\beta}a_{k+2} \in g_{k+2\beta}a_{k+1}N = g_{k+2\beta}g^{-1}_{k+1\beta}g_{1\beta}a_1N \in G^{p^\beta}N$ and analogously $g_{t\beta}a_t \in G^{p^\beta}N$. If $g_{k+2\beta}a_{k+2} = \ldots = g_{t\beta}a_t \in G^{p^\beta}N$, then the coefficient in $R$ of this element, equal to $f_{k+2}e_{k+2} + \ldots + f_te_t$, is nonzero. Otherwise if $f_{k+2}e_{k+2} + \ldots + f_te_t = (f_{k+2} + \ldots + f_t)(e_{k+2} + \ldots + e_t) = 0$, we derive $(f_1 + f_{k+1} + f_{k+2} + \ldots + f_t)(e_1 + e_{k+1} + e_{k+2} + \ldots + e_t) = 0$. Because $e_1 + e_{k+1} + e_{k+2} + \ldots + e_t = 1$, we have $f_1 + f_{k+1} + f_{k+2} + \ldots + f_t = 0$, hence $f_2 + f_3 + \ldots + f_k = 1$. Furthermore, since $e_2 + e_3 + \ldots + e_k = 0$, we get $1 = 1.1 = (f_1 + \ldots + f_t)(e_1 + \ldots + e_t) = (f_2 + f_3 + \ldots + f_k)(e_1 + e_{k+1} + e_{k+2} + \ldots + e_t) = 0$, which is a contradiction.

On the other hand, if $g_{k+2\beta}a_{k+2} = g_{2\beta}a_2$ and $f_{k+2}e_{k+2} + f_2e_2 = 0$, we deduce $g_{2\beta}a_3 \in g_{2\beta}a_2N \in G^{p^\beta}N$. In all that follows, in the right hand-side of the main equality does exist an element from $G^{p^\beta}N$. Since $\{\beta < \tau \geq \omega\}$ is an infinite set, and the support is finite, we can assume that all dependences are of the form presented.

And so, independently from the additional relations between the elements from the right hand-side if they eventually exist, we observe that $g_{1\alpha} \in \cap_{\beta<\tau}(G^{p^\beta}N) = G^{p^\tau}N$, ..., $g_{s\alpha} \in G^{p^\tau}N$ and $g_{s+1\,\alpha} \in g_{s+2\,\alpha}N$, ..., $g_{t-1\,\alpha} \in g_{t\alpha}N$ together with $r_{s+1\alpha} + r_{s+2\,\alpha} = 0, \ldots, r_{t-1\alpha} + r_{t\alpha} = 0$; $s \in N$. Because $x$ is a $p$-torsion element, let $g_{s+1\alpha} \in G_p$, $g_{s+3\,\alpha} \in G_p$, ..., $g_{t-1\alpha} \in G_p$ with $r_{s+1\alpha} + r_{s+3\,\alpha} + \ldots + r_{t-1\alpha} - 1 \in \operatorname{rad}(R)$, the nil-radical of $R$. Moreover, since $1 = r_{1\alpha} + r_{2\alpha} + \ldots + r_{s\alpha} \notin \operatorname{rad}(R)$, the ratios $g_{1\alpha} \in g_{2\,\alpha}G_p \in \ldots \in g_{s\alpha}G_p$ are impossible. Thereby it is a real matter to presume that $g_{1\alpha} \in g_{s+2\,\alpha}G_p$. This, along with $g_{1\alpha} \in G^{p^\tau}N$ and $G_p \ni g_{s+1\,\alpha} \in g_{s+2\,\alpha}N$, leads us to $g_{1\alpha} \in (G^{p^\tau}N) \cap (G_pN) = N(G^{p^\tau}N)_p$. But the $p$-isotypity of $N$ in $G$ means $(G^{p^\tau}N)_p = G_p^{p^\tau}N_p$ whence $g_{1\alpha} \in G_p^{p^\tau}N$. By symmetry $g_{2\,\alpha}, \ldots, g_{s\alpha} \in G_p^{p^\tau}N$ eventually when $r_{2\alpha} \notin \operatorname{rad}(R)$, ..., $r_{s\alpha} \notin \operatorname{rad}(R)$. Bearing in mind that $r_{s+1\,\alpha}g_{s+1\,\alpha} + \ldots + r_{t\alpha}g_{t\alpha} \in I(RG; N)$, that $r_{1\alpha}g_{1\alpha} + r_{2\alpha}g_{2\alpha} + \ldots + r_{s\alpha}g_{s\alpha} = r_{1\alpha}(g_{1\alpha} - 1) + r_{2\alpha}(g_{2\alpha} - 1) + \ldots + r_{s\alpha}(g_{s\alpha} - 1) + 1 = r_{1\alpha}(b_{p\tau}u_1 - 1) + r_{2\alpha}(b_{2\tau}u_2 - 1) + \ldots + r_{s\alpha}(b_{s\tau}u_s - 1) + 1 = 1 + r_{1\alpha}(b_{p\tau} - 1) + r_{1\alpha}b_{p\tau}(u_1 - 1) + \ldots + r_{s\alpha}(b_{s\tau} - 1) + r_{s\alpha}b_{s\tau}(u_s - 1)$ where $b_{p\tau} \in G_p^{p^\tau}$, $b_{2\tau} \in G^{p^\tau}$, $\ldots, b_{s\tau} \in G^{p^\tau}$; $u_1, u_2, \ldots, u_s \in N$, and that $1 + r_{1\alpha}(b_{p\tau} - 1) + \ldots + r_{s\alpha}(b_{s\tau} - 1) \in S^{p^\tau}(RG)$, we conclude that $r_{1\alpha}g_{1\alpha} + \ldots + r_{t\alpha}g_{t\alpha} \in S^{p^\tau}(RG) + I(RG; N) = S^{p^\tau}(RG)(1 + I(RG; N))$. So, $x \in S^{p^\tau}(RG)(1 + I_p(RG; N))$ and this ends the inclusion. The proof is over in all generality. $\square$

*Example.* The following example demonstrably shows that in the evidence of Proposition on Niceness the condition for nonidentity of $1 \neq e_1a_1 + \ldots + e_ta_t \in 1 + I_p(RG; N)$ is essential and cannot be ignored. If yes, the proof fails. In fact, it starts with an element $r_{1\alpha}g_{1\alpha} + \ldots + r_{t\alpha}g_{t\alpha} \in S^{p^\tau}(RG) = \cap_{\alpha<\tau}S^{p^\alpha}(RG); \tau$ is a limit ordinal, and by the end of the proof has decomposed this element so that, assuming re-indexing of the subscripts, there is some $s$ with $1 \leq s \leq t$ such that $g_{1\alpha}, \ldots, g_{s\alpha} \in G_p^{p^\tau}N$, and $r_{s+1\alpha}g_{s+1\alpha} + \ldots + r_{t\alpha}g_{t\alpha} \in I(RG; N)$. In order to substantiate our claim, suppose that $G = K \times N$, where $K$ contains a non-trivial torsion-free element $g$ and distinct $p$-torsion elements $c_1$ and $c_2$ such that $gc_i \in G^{p^\tau}; i = 1, 2$. It is really not hard to find such a group $K$. We further see that the concrete element $1 + gc_1 - gc_2 \in S^{p^\tau}(RG) = S(R^{p^\tau}G^{p^\tau})$, thus the proof may have started with this element. But $gc_i \notin G_p^{p^\tau}N$ is clear since otherwise $gc_i \in (G_p^{p^\tau}N) \cap K \subseteq (K_p \times N) \cap K = K_p$ whence $g \in K_p$ and $g = 1$, contrary to the choice. So, only 1 lies in $G_p^{p^\tau}N$. This means that $gc_1 - gc_2$ must lie in $I(RG; N)$. But we observe that this is false because $gc_1 - gc_2 = gc_1(1 - c_1^{-1}c_2) \in I(RG; N)$ implies $c_1^{-1}c_2 \in K \cap N = 1$ hence $c_1 = c_2$, against our hypothesis.

*Remark* 2. W. May showed in [5] that if $N$ is a nice $p$-subgroup of the abelian group $G$, then $1 + I_p(RG; N)$ is nice in $S(RG)$ provided $R$ is a perfect field. The above affirmation extends May's result when $N$ is not $p$-primary, however, and $R$ is not necessarily a field. It may be successfully applied for arguing the simply presented structure of $S(RG)/G_p$, but this is a problem of some other investigation.

**Problem.** *Does it follow that* $1 + I_p(RG; N)$ *is nice in* $S(RG)$ *provided* $N$ *is* $p$*-nice in* $G$ *only?*

## Corrigendum

The misspelled word "droped" from of [1, p. 51] should read as "dropped".

Also, in [2, p. 258, line 20 (+)], the expression "$C_\beta$" must be replaced by "$G_\beta$"; on p. 260, the words "ensure" and "choosen" should be replaced by "ensures" and "chosen", repsectively, and "expanson" on p. 261 should replaced by "expansion."

In [3, p. 223, line 13 (+)], the expression "$B_\alpha$" must be replaced by "$G_\alpha$" and on p. 224, the term "$\coprod_{\alpha<\lambda} \coprod_{\mu<\alpha} G_\mu$" should be replaced by "$\prod_{\alpha<\lambda} \coprod_{\mu<\alpha} G_\mu$."

Finally, in [4, p. 9, line 5 (-)], the sign "=" should be read as "$\subset$"; the choice realized there is possible because any subgroup of a $\sigma$-summable $p$-group with equal length is also $\sigma$-summable. Finally, the second expression "$g_{in}^{\varepsilon_n}$" on p. 12, line 5 (+), should be removed.

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