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A CHARACTERIZATION OF CONE PRESERVING MAPPINGS OF QUASIORDERED SETS

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ABSTRACT. By a cone of a quasiordered set (A, Q), any $U_Q(a) = \{x \in A : \langle a, x\rangle \in Q\}$ is meant. A mapping $f : A \to B$ is a cone preserving mapping of (A, Q) into (B, Q') if $f(U_Q(a)) = U_{Q'}(f(a))$ for each $a \in A$. We characterize these mappings by using certain relational inclusions. The result can be applied for construction of a quotient quasiorder hypergroup.

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Let $H \neq 0$ be a set and "$*$" be a mapping of $H \times H$ into $\mathcal{P}^*(H)$ (the set of all non-void subsets of H). The pair $(H, *)$ is called *a hypergroupoid* (see, e. g., [2, 5]). For $A, B \in \mathcal{P}^*(H)$, we define $A * B = \bigcup\{a * b; a \in A, b \in B\}$.

A hypergroupoid $(H, *)$ is called *a hypergroup*, see [1,2,5], if $(a*b)*c = a*(b*c)$ for all $a, b, c \in H$, and the so-called *reproduction axiom*, i. e., $a * H = H = H * a$ for any $a \in H$, is satisfied.

For a binary relation R on A and $a \in A$ denote $U_R(a) = \{b \in A; \langle a, b\rangle \in R\}$. A binary relation Q on a set A is called *quasiorder* if it is reflexive and transitive. The set $U_Q(a)$ is called a *cone* of a. In the case when a quasiorder Q is an equivalence one can prove that $U_Q(A) = \{x \in H : \exists y \in A, \langle x, y\rangle \in \theta\}$ for any $A \subseteq H$. Analogously, for $M \subseteq A$ we set $U_Q(M) = \bigcap\{U_Q(a) : a \in M\}$. The concept of a quasiorder hypergroup can be found, e. g., in [1]:

Definition 1. Let (H, Q) be a quasiordered set. For $a, b \in H$ define

$$a \star b = U_Q(a) \cup U_Q(b). \tag{1}$$

Then $(H, \star)$ is a hypergroup which is called a *quasiorder hypergroup induced by* (H, Q).

To produce quotient hypergroups from a given general hypergroup, we need a concept of congruence. This was introduced independently by several authors but it was shown by J. Chvalina [4] that all of these definitions are equivalent to the following one:

Definition 2. Let $\mathcal{H} = (H, *)$ be a hypergroupoid. An equivalence θ on H is called a *congruence on* $\mathcal{H}$ if $\langle a, b\rangle \in \theta$ and $\langle c, d\rangle \in \theta$ imply

$$U_\theta(a * c) = U_\theta(b * d).$$

Giving an equivalence θ on a set A, denote by h_θ the so-called *natural mapping* of A onto A/θ defined by $h_\theta(a) = [a]_\theta$.

It can be shown that if θ is a congruence on a hypergroupoid $(H, *)$, then $(H/\theta, *_\theta)$, where

$$[a]_\theta *_\theta [b]_\theta = h_\theta(a * b),$$

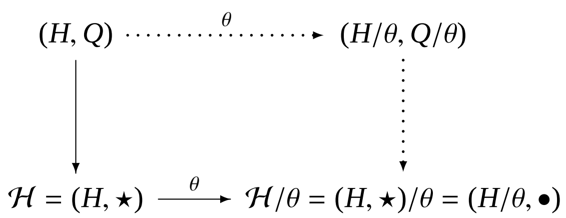
is also a hypegroupoid. If, moreover, $\mathcal{H} = (H, *)$ is a hypergroup, then $\mathcal{H}/\theta = (H/\theta, *_\theta)$ is a quotient hypergroup—see Chvalina [4, p. 155].

The following question was solved in [3]: Let (H, Q) be a quasiordered set and $\mathcal{H} = (H, \star)$ be the induced hypergroup. Let θ be an equivalence on $(H, \star)$.

(i) How to define a quotient relation Q/θ on the quotient set H/θ to be a quasiorder again?

(ii) Under what condition the quasiorder hypergroup $(H/\theta, \bullet)$ induced by the quasiordered set $(H/\theta, Q/\theta)$, i. e., $[a]_\theta \bullet [b]_\theta = U_{Q/\theta}([a]_\theta) \cup U_{Q/\theta}([b]_\theta)$, is the quotient hypergroup $\mathcal{H}/\theta$?

It can be visualized by the following diagram:



In Corollary 1 from [3] it was proved that if θ is a congruence, then the preceding diagram commutes.

We are going to use another approach to solve these problems in terms of relational products. Recall that for two binary relations R and S on a set A we have

$$R \circ S = \{\langle x, y\rangle \in A^2 : \exists z \in A \text{ with } \langle x, z\rangle \in R, \langle z, y\rangle \in S\}.$$

Further, let $f : A \to B$ be a mapping. Denote by θ_f the so-called *induced equivalence* (on A), i. e., $\langle x, y\rangle \in \theta_f$ if and only if $f(x) = f(y)$.

If $f : A \to B$ is a mapping and $M \subseteq A$, denote by $f(M) = \{f(m) : m \in M\}$. It is well-known that for any $X, Y \subseteq A$, $f(X \cup Y) = f(X) \cup f(Y)$.

We have to introduce a quotient quasiordered set. For this purpose, we introduce the following

Definition 3. Let R be a binary relation on a set A and θ be an equivalence on A. Define R/θ on A/θ as follows

$$\langle [a]_\theta, [b]_\theta \rangle \in R/\theta \text{ if and only if there exist } x \in [a]_\theta, y \in [a]_\theta \text{ with } \langle x, y\rangle \in R. \qquad (2)$$

The couple $(A/\theta, R/\theta)$ is called a *quotient of* (A, R) *by* θ.

Lemma 1. *Let* (H, Q) *be a quasiordered set and* $(H, \star)$, *where* $\star$ *is given by* (1), *be a corresponding quasiorder hypergroup. Let* θ *be an equivalence on* H. *Then* θ *is a congruence on* $(H, \star)$ *if and only if*

$$h_\theta(U_Q(a)) = U_{Q/\theta}(h_\theta(a)) \quad (= U_{Q/\theta}([a]_\theta)) \qquad (3)$$

holds for any $a \in H$.

Proof. Assume first that θ is a congruence on $(H, \star)$. If $x \in U_Q(a)$, i. e., $\langle a, x\rangle \in Q$, then $\langle h_\theta(a), h_\theta(x)\rangle \in Q/\theta$, i. e., $h_\theta(x) \in U_{Q/\theta}(h_\theta(a))$, which means that

$$h_\theta(U_Q(a)) \subseteq U_{Q/\theta}(h_\theta(a)).$$

To prove the opposite inclusion consider $z \in U_{Q/\theta}(h_\theta(a))$.The mapping h_θ is a surjection, so there exists $x \in H$ such that $z = h_\theta(x)$. Therefore $h_\theta(x) \in U_{Q/\theta}(h_\theta(a))$, which implies $\langle h_\theta(a), h_\theta(x)\rangle \in Q/\theta$. From the definition of Q/θ there exist $b, y \in H$ such that $\langle b, y\rangle \in Q$, $h_\theta(b) = h_\theta(a)$, $h_\theta(x) = h_\theta(y) = z$. As θ is a congruence we get $U_\theta(a \star a) = U_\theta(b \star b)$, $U_\theta(x \star x) = U_\theta(y \star y)$ that is $h_\theta(U_Q(a)) = h_\theta(U_Q(b))$, $h_\theta(U_Q(x)) = h_\theta(U_Q(y))$. Further from $\langle b, y\rangle \in Q$ we have $U_Q(b) \supseteq U_Q(y)$. From this we obtain $z \in h_\theta(U_Q(y)) \subseteq h_\theta(U_Q(b)) = h_\theta(U_Q(a))$, which gives

$$U_{Q/\theta}(h_\theta(a)) \subseteq h_\theta(U(a)).$$

Assume now that (3) is true. We have to prove that θ is a congruence, i. e., $h_\theta(a \star c) = h_\theta(b \star d)$ for $\langle a, b\rangle \in \theta$, $\langle c, d\rangle \in \theta$. We have

$$\begin{aligned} h_\theta(a \star c) &= h_\theta(U_Q(a) \cup U_Q(c)) = h_\theta(U_Q(a)) \cup h_\theta(U_Q(c)) \\ &= U_{Q/\theta}(h_\theta(a)) \cup U_{Q/\theta}(h_\theta(c)) = U_{Q/\theta}(h_\theta(b)) \cup U_{Q/\theta}(h_\theta(d)) \\ &= h_\theta(U_Q(b)) \cup h_\theta(U_Q(d)) = h_\theta(U_Q(b) \cup U_Q(d)) = h_\theta(b \star d). \end{aligned} \qquad \square$$

Hence, we must solve the above mentioned problem (i) and classify the mappings having the property (ii). For this sake, let us introduce the following

Definition 4. Let $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ be quasiordered sets. A mapping $h\colon A \to B$ is called *cone preserving* if $h(U_R(a)) = U_S(h(a))$ for each $a \in A$.

Let us recall that for $\mathcal{A} = (A, R)$, $\mathcal{B} = (B, S)$ (where R or S is a binary relation on A or B, respectively) a mapping $h\colon A \to B$ is called a *homomorphism* if $\langle a, b\rangle \in R$ implies $\langle h(a), h(b)\rangle \in S$. It is called a *strong homomorphism* if, moreover, for every $\langle h(a), h(b)\rangle \in S$ there exist $c, d \in A$ such that $\langle c, d\rangle \in R$ and $h(c) = h(a)$, $h(d) = h(b)$.

The proof of the following lemma is almost evident (see, e. g., Lemma 2 in [3]):

Lemma 2. *Let $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ be quasiordered sets. If $h\colon A \to B$ is a cone preserving mapping, then h is a strong homomorphism of $\mathcal{A}$ into $\mathcal{B}$.*

It is an easy exercise to verify that the converse assertion is not valid in general.

We are ready to solve our first question.

Lemma 3. *Let $\mathcal{A} = (A, R)$ be a quasiordered set and θ an equivalence on A. If*

$$\theta \circ R \subseteq R \circ \theta,$$

then $\mathcal{A}/\theta = (A/\theta, R/\theta)$ is a quasiordered set.

Proof. It is clear from Definition 3 that if R is reflexive, then R/θ has this property. We have to show transitivity of R/θ. Suppose that $\langle [a]_\theta, [b]_\theta \rangle \in R/\theta$ and $\langle [b]_\theta, [c]_\theta \rangle \in R/\theta$. Then there exist $x \in [a]_\theta$, $y, y' \in [b]_\theta$ and $z \in [c]_\theta$ such that $\langle x, y \rangle \in R$, $\langle y', z \rangle \in R$. Moreover, $\langle y, y' \rangle \in \theta$ thus $\langle x, z \rangle \in R \circ \theta \circ R$. Due to the assumption, we have

$$\langle x, z \rangle \in R \circ \theta \circ R \subseteq R \circ R \circ \theta = R \circ \theta,$$

i. e., there exists $w \in A$ with $\langle x, w \rangle \in R$, $\langle w, z \rangle \in \theta$, thus $w \in [z]_\theta = [c]_\theta$. Altogether, $\langle [a]_\theta, [c]_\theta \rangle \in R/\theta$ proving transitivity of R/θ . □

In the remaining part of our paper, we will solve the problem whether a mapping f of a quasiordered set $\mathcal{A} = (A, R)$ into a quasiordered set $\mathcal{B} = (B, S)$ is a cone preserving mapping.

Theorem 1. *Let $\mathcal{A} = (A, R)$, $\mathcal{B} = (B, S)$ be quasiordered sets and $f\colon A \to B$ a surjective mapping. The following conditions are equivalent:*

(a) *f is a cone preserving mapping,*

(b) *f is a homomorphism of $\mathcal{A}$ onto $\mathcal{B}$ and the relation $\langle f(x), f(y) \rangle \in S$ implies that $\langle x, y \rangle \in R \circ \theta_f$.*

Proof. (a) $\Rightarrow$ (b): By Lemma 2, f is a homomorphism of $\mathcal{A}$ onto $\mathcal{B}$. Suppose $\langle f(x), f(y) \rangle \in S$. Then $f(y) \in U_S(f(x)) = f(U_R(x))$. Hence, there exists $z \in U_R(x)$ with $f(z) = f(y)$, i. e., $\langle x, z \rangle \in R$ and $\langle z, y \rangle \in \theta_f$ giving $\langle x, y \rangle \in R \circ \theta_f$.

(b) $\Rightarrow$ (a): Since f is a homomorphism of $\mathcal{A}$ onto $\mathcal{B}$, we have $f(U_R(x)) \subseteq U_S(f(x))$ for each $x \in A$. Suppose $z \in U_S(f(x))$. Since f is surjective, it follows that $z = f(w)$ and $\langle f(x), f(w) \rangle \in S$. By (b) we have $\langle x, w \rangle \in R \circ \theta_f$, i. e., there exists $c \in A$ with $\langle x, c \rangle \in R$ and $\langle c, w \rangle \in \theta_f$. Hence $f(c) = f(w) = z$ proving that $U_S(f(x)) \subseteq f(U_R(x))$. Thus f is cone preserving. □

As mentioned, an equivalence θ on a quasiordered set is a congruence of the induced quasiorder hypergroup if the quotient relation is a quasiorder again and the natural mapping h_θ is a cone preserving mapping. Hence, we can solve our second problem:

Theorem 2. *Let θ be an equivalence on a quasiordered set $\mathcal{A} = (A, R)$. If*

$$\theta \circ R \subseteq R \circ \theta \tag{4}$$

then $\mathcal{A}/\theta = (A/\theta, R/\theta)$ is a quasiordered set and the natural mapping h_θ is cone preserving.

Proof. By Lemma 3, (4) implies that $\mathcal{A}/\theta = (A/\theta, R/\theta)$ is a quasiordered set. Suppose now that

$$\langle [x]_\theta, [y]_\theta \rangle \in R/\theta.$$

Then there exists $a, b \in A$, such that $a \in [x]_\theta$, $b \in [y]_\theta$ and $\langle a, b\rangle \in R$, i. e., $\langle x, a\rangle \in \theta$ and $\langle x, b\rangle \in \theta \circ R$. By (4), $\langle x, b\rangle \in R \circ \theta$, i. e., there exists $w \in A$ such that $\langle x, w\rangle \in R$ and $\langle w, b\rangle \in \theta$. Hence and $[w]_\theta = [b]_\theta = [y]_\theta$, i. e., $\langle w, y\rangle \in \theta$. Altogether, we obtain $\langle x, y\rangle \in R \circ \theta$. Clearly $\theta_{h_\theta} = \theta$. Applying Theorem 1, we conclude that the natural mapping $h_\theta \colon a \mapsto [a]_\theta$ is cone preserving. □

By Lemma 2, every cone preserving mapping is a strong homomorphism. Hence, we will characterize which strong homomorphisms are cone preserving mappings.

Lemma 4. *Let $\mathcal{A} = (A, R)$, $\mathcal{B} = (B, S)$ be quasiordered sets and $f\colon A \to B$ be a surjective mapping. The following conditions are equivalent:*

(a) *f is a strong homomorphism of $\mathcal{A}$ onto $\mathcal{B}$;*

(b) *$\langle f(x), f(y)\rangle \in S$ if and only if $\langle x, y\rangle \in \theta_f \circ R \circ \theta_f$.*

Proof. (a) ⇒ (b): Suppose $\langle f(x), f(y)\rangle \in S$. Since f is a strong homomorphism, there exist $a, b \in A$ with $\langle a, b\rangle \in R$, $f(a) = f(x)$, $f(b) = f(y)$, thus $\langle x, y\rangle \in \theta_f \circ R \circ \theta_f$. Conversely, if $\langle x, y\rangle \in \theta_f \circ R \circ \theta_f$ then there exist $c, d \in A$ with $f(x) = f(c)$, $\langle c, d\rangle \in R$, $f(d) = f(y)$. Since f is a homomorphism, it follows that $\langle f(c), f(d)\rangle \in S$, i. e., $\langle f(x), f(y)\rangle \in S$ as well.

(b) ⇒ (a): Suppose $\langle a, b\rangle \in R$. Then also $\langle a, b\rangle \in \theta_f \circ R \circ \theta_f$ and, by virtue of (b), $\langle f(a), f(b)\rangle \in S$. Thus f is a homomorphism of $\mathcal{A}$ onto $\mathcal{B}$.

Suppose now $\langle c, d\rangle \in S$. Since f is surjective, there are $x, y \in A$ with $f(x) = c$, $f(y) = d$, i. e., $\langle f(x), f(y)\rangle \in S$. By (b) we have $\langle x, y\rangle \in \theta_f \circ R \circ \theta_f$, i. e., there exist $a, b \in A$ such that $f(x) = f(a)$, $f(y) = f(b)$ and $\langle a, b\rangle \in R$, i. e., f is a strong homomorphism. □

Theorem 3. *Let $\mathcal{A} = (A, R)$, $\mathcal{B} = (B, S)$ be quasiordered sets and f be a surjective strong homomorphism of $\mathcal{A}$ onto $\mathcal{B}$. The following assertions are equivalent:*

(a) *f is cone preserving,*

(b) $\theta_f \circ R \subseteq R \circ \theta_f$.

Proof. (a) ⇒ (b): Suppose $\langle x, y\rangle \in \theta_f \circ R$. Then there exists $z \in A$ with $\langle x, z\rangle \in \theta_f$, $\langle z, y\rangle \in R$. Hence $f(x) = f(z)$ and $\langle f(z), f(y)\rangle \in S$; i. e., also $\langle f(x), f(y)\rangle \in S$. Thus

$f(y) \in U_S(f(x)) = f(U_R(x))$. Hence, there is $a \in A$ with $\langle x, a\rangle \in R$ and $f(a) = f(y)$ thus $\langle x, y\rangle \in R \circ \theta_f$ proving (b).

(b) $\Rightarrow$ (a): Suppose $\langle f(x), f(y)\rangle \in S$. By Lemma 4 we have $\langle x, y\rangle \in \theta_f \circ R \circ \theta_f$. Applying (b) we conclude $\langle x, y\rangle \in \theta_f \circ R \circ \theta_f \subseteq R \circ \theta_f \circ \theta_f = R \circ \theta_f$. By Theorem 1, f is a cone preserving mapping. □

Corollary 1. *Let $\mathcal{A} = (A, R)$, $\mathcal{B} = (B, S)$ be quasiordered sets and f be a surjective strong homomorphism of $\mathcal{A}$ onto $\mathcal{B}$. Suppose R is symmetric (i. e., an equivalence on A). Then f is cone preserving if and only if $\theta_f \circ R = R \circ \theta_f$.*

Proof. Due to Theorem 3, we need only to show the converse inclusion of that in (b). However, by symmetry of R and θ_f we have

$$R \circ \theta_f = R^{-1} \circ \theta_f^{-1} = (\theta_f \circ R)^{-1} \subseteq (R \circ \theta_f)^{-1} = \theta_f^{-1} \circ R^{-1} = \theta_f \circ R. \qquad \square$$

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