Miskolc Mathematical Notes
Vol. 8 (2007), No 2, pp. 123-134

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2007.119

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HU e-ISSN 1787-2413

FINE LIMITS OF GENERALIZED POTENTIAL-TYPE INTEGRAL OPERATORS WITH NON-ISOTROPIC KERNEL

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Received 23 February, 2005

Abstract. This paper deals with the fine limits of generalized potential-type operators with non-isotropic kernels defined for functions on $\mathbb{R}^n$ satisfying appropriate conditions.

1991 *Mathematics Subject Classification:* Primary 31B15; Secondary 31C15, 42B99

Keywords: non-isotropic distance, Riesz potentials

1. INTRODUCTION

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be positive numbers with $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ and $\|x\|_\lambda = (|x_1|^{\frac{1}{\lambda_1}} + \ldots + |x_n|^{\frac{1}{\lambda_n}})^{\frac{|\lambda|}{n}}$, $x \in \mathbb{R}^n$. The expression $\|x - y\|_\lambda$, where $x,\ y \in \mathbb{R}^n$, is called the λ-distance or non-isotropic distance between x and y. This distance is an important concept in the theory of partial differential equations and imbedding theorems. Some problems with the λ-distance were examined in [6, 7].

It can be seen that λ-distance becomes the ordinary Euclidean distance $|x - y|$ for $\lambda_j = \frac{1}{2}$, $j = 1, 2, \ldots, n$. The λ-distance has the following properties.

Using the inequality $(a + b)^m \leq 2^m (a^m + b^m)$, $m > 1$, we obtain

$$\|x - y\|_\lambda \leq M_\lambda (\|x\|_\lambda + \|y\|_\lambda), \tag{1.1}$$

where $M_\lambda = 2^{\left(1+\frac{1}{\lambda_{\min}}\right)\frac{|\lambda|}{n}}$ and $\lambda_{\min} = \min(\lambda_1, \lambda_2, \ldots \lambda_n)$.

Several authors have investigated the properties of classical Riesz potentials and their generalizations. For example, taking some appropriate conditions on the kernel depending on Euclidean distance type of $K(|x - y|)$, Gadjiev [3] proved a variant of the Hardy–Littlewood–Sobolev theorem. He also gave the properties of convergence almost everywhere. In [1], a theorem similar to results of [3] was proved for potential-type integrals with kernel depending on the λ-distance.

Some results on potential-type integral operators and Riesz potentials given by generalized shift operators can be found in [2, 4, 5]. Various generalizations of the Riesz potentials are given in [10].

Supported by NATO PC-B program of TUBİTAK.

A potential-type integral operator depending on the λ-distance and defined for non-negative measurable functions f on $\mathbb{R}^n$ is given by the equality

$$(Lf)(x) = \int_{\mathbb{R}^n} K(\|x-y\|_\lambda) f(y)dy,$$

where K is the kernel function satisfying the following conditions (see [1]):

(K_1) *K is a non-negative continuous and decreasing function on semiaxis* $[0,\infty)$ *and* $\lim_{t\to 0} K(t) = \infty$;

(K_2) $L(r) = \int_r^a K\left(2\beta M_\lambda t^{\frac{2|\lambda|}{n}}\right) t^{2|\lambda|-\delta-1} dt < \infty$ *for* $0<\delta<2|\lambda|$, $\beta\in(0,1)$ *and* $0\le r<a$.

We know that $(Lf)(x) \neq \infty$ if and only if

$$\int_{\mathbb{R}^n} K(\beta(1+\|y\|_\lambda)) f(y)dy, \tag{1.2}$$

where $\beta\in(0,1)$. Hence it is seen that $(Lf)(x)\neq\infty$ when f is integrable on $\mathbb{R}^n$. Note that (1) is equivalent to

$$\int_{\mathbb{R}^n - B_\lambda(x,1)} K(\beta\|x-y\|_\lambda) f(y)dy$$

for every $x\in\mathbb{R}^n$, and $\beta\in(0,1)$, where $B_\lambda(x,1)$ is λ-ball centered at x with radius 1. That is $B_\lambda(x,1) = \{y\in\mathbb{R}^n : \|x-y\|_\lambda < 1\}$.

In what follows, we investigate the fine limits of generalized potential-type integral operators with non-isotropic kernels Lf at $x_0\in\mathbb{R}^n$. Our results are generalizations of the corresponding results for classical Riesz potentials given in [9, 11].

To obtain a general result, we assume the condition

$$\int_{\mathbb{R}^n} \phi_p(f(y))\, w\left(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right) dy < \infty. \tag{1.3}$$

where $x_0\in\mathbb{R}^n$ and $\phi_p(r)$ is positive monotone function on interval $(0,\infty)$ having the following properties:

(φ_1) *$\phi_p(r)$ is of the form $r^p\varphi(r)$, where $1\le p<\infty$ and φ is a positive non-decreasing function on interval* $(0,\infty)$.

(φ_2) *There exists A_1 such that $\varphi(2r)\le A_1\varphi(r)$ whenever $r>0$.*

Throughout this paper, let $w(r)$ be a positive non-increasing function on $(0,\infty)$ satisfying the condition:

(w_1) *There exists $A_2>0$ such that $A_2^{-1}w(r)\le w(2r)\le A_2 w(r)$ whenever $r>0$.*

In this paper we will use some ideas from [9, 11]. By the symbol M, we denote a positive constant whose value may change depending on the context.

2. Preliminary lemmas

First we collect properties which follow from conditions (φ_1) and (φ_2).

Lemma 2.1. *The function φ satisfies the doubling condition, that is, there exists $A_3 > 1$ such that*

$$\varphi(r) \le \varphi(2r) \le A_3\varphi(r) \quad \textit{for } r > 0.$$

Lemma 2.2. *For any $\gamma > 0$, there exists $A_4(\gamma) > 1$ such that*

$$A_4^{-1}(\gamma)\varphi(r) \le \varphi(r^\gamma) \le A_4(\gamma)\varphi(r), \textit{ whenever } r > 0.$$

3. The estimate of Lf

We write $(Lf)(x) = L_1(x) + L_2(x) + L_3(x)$ for $x \in \mathbb{R}^n - \{x_0\}$, where

$$L_1(x) = \int_{\mathbb{R}^n - B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda)} K(\|x-y\|_\lambda) f(y)dy,$$
$$L_2(x) = \int_{B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda) - B_\lambda(x, \|x-x_0\|_\lambda/2M_\lambda)} K(\|x-y\|_\lambda) f(y)dy,$$
$$L_3(x) = \int_{B_\lambda(x, \|x-x_0\|_\lambda/2M_\lambda)} K(\|x-y\|_\lambda) f(y)dy.$$

Using (1.1), then for any $x, y \in \mathbb{R}^n$

$$\|x-y\|_\lambda \ge \frac{1}{M_\lambda}\|y-x_0\|_\lambda - \|x-x_0\|_\lambda.$$

It is obvious that, if $y \in \mathbb{R}^n - B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda)$, then $\|x-y\|_\lambda \ge \frac{1}{2M_\lambda}\|y-x_0\|_\lambda$. Taking into account $L_1(x)$, we have the inequality

$$L_1(x) \le M \int_{\mathbb{R}^n - B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda)} K(\beta\|y-x_0\|_\lambda) f(y)dy \tag{3.1}$$

for any $\beta = \frac{1}{2M_\lambda} \in (0,1)$. For $y \in B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda) - B_\lambda(x, \|x-x_0\|_\lambda/2M_\lambda)$, since $\|y-x\|_\lambda \ge \frac{1}{2M_\lambda}\|x-x_0\|_\lambda$, we have similarly

$$L_2(x) \le K(\beta\|x-x_0\|_\lambda) \int_{B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda) - B_\lambda(x, \|x-x_0\|_\lambda/2M_\lambda)} f(y)dy \tag{3.2}$$

for any $\beta = \frac{1}{2M_\lambda} \in (0,1)$.

Let us begin with the Hölder type inequality.

Lemma 3.1. *Let $p > 1$, $\delta > 0$, and f be a non-negative measurable function on $\mathbb{R}^n$. If $0 \le 2M_\lambda \|x - x_0\|_\lambda < 2M_\lambda a^{\frac{2|\lambda|}{n}} < 1$, then*

$$\begin{aligned} &\int_{\mathbb{R}^n - B_\lambda(x_0, 2M_\lambda \|x-x_0\|_\lambda)} K(\beta \|y - x_0\|_\lambda) f(y) dy \\ &\quad \le \int_{\mathbb{R}^n - B_\lambda(x_0, 2M_\lambda a^{\frac{2|\lambda|}{n}})} K(\beta \|y - x_0\|_\lambda) f(y) dy + ML\left(\|x - x_0\|_\lambda^{\frac{n}{2|\lambda|}} \right) \\ &+ MR_1\left(\|x - x_0\|_\lambda^{\frac{n}{2|\lambda|}} \right) \left(\int_{B_\lambda(x_0, 2M_\lambda a^{\frac{2|\lambda|}{n}})} \phi_p(f(y)) w\left(\|y - x_0\|_\lambda^{\frac{n}{2|\lambda|}} \right) dy \right)^{\frac{1}{p}}, \end{aligned}$$

where $R_1(r) = \left(\int_r^a K^{p'} \left(2\beta M_\lambda t^{\frac{2|\lambda|}{n}} \right) \left[\varphi(t^{-1}) w(t) \right]^{\frac{p'}{p}} t^{2|\lambda|-1} dt \right)^{\frac{1}{p'}}$ if $0 < 2M_\lambda r^{\frac{2|\lambda|}{n}} < 1$ and $R_1(r) = R_1\left((2M_\lambda)^{-\frac{n}{2|\lambda|}} \right)$ in the other cases.

Proof. Without loss of generality we assume that $f = 0$ outside of $B_\lambda(x_0, 2M_\lambda a^{\frac{2|\lambda|}{n}})$. We have

$$\begin{aligned} &\int_{\mathbb{R}^n - B_\lambda(x_0, 2M_\lambda \|x-x_0\|_\lambda)} K(\beta \|y - x_0\|_\lambda) f(y) dy = \int_{A(y)} K(\beta \|y - x_0\|_\lambda) f(y) dy \\ &\qquad \le \int_{\left\{ y \in A(y);\, f(y) > \|y - x_0\|_\lambda^{-\frac{\delta n}{2|\lambda|}} \right\}} K(\beta \|y - x_0\|_\lambda) f(y) dy \\ &\qquad + \int_{\left\{ y \in A(y);\, f(y) \le \|y - x_0\|_\lambda^{-\frac{\delta n}{2|\lambda|}} \right\}} K(\beta \|y - x_0\|_\lambda) f(y) dy =: L_{11} + L_{12}, \end{aligned}$$

where $A(y) = B_\lambda(x_0, 2M_\lambda a^{\frac{2|\lambda|}{n}}) - B_\lambda(x_0, 2M_\lambda \|x - x_0\|_\lambda)$. Consider the integral L_{11}. From Hölder's inequality, we obtain

$$\begin{aligned} L_{11}(x) \le &\left(\int_{U(y)} f^p(y) \varphi(f(y)) w(\|y - x_0\|_\lambda^{\frac{n}{2|\lambda|}}) dy \right)^{\frac{1}{p}} \\ &\times \left(\int_{U(y)} K(\beta \|y - x_0\|_\lambda)^{p'} \left[\varphi(f(y)) w(\|y - x_0\|_\lambda^{\frac{n}{2|\lambda|}}) \right]^{-\frac{P'}{p}} dy \right)^{\frac{1}{p'}}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $U(y) = \left\{ y \in A(y);\; f(y) > \|y - x_0\|_\lambda^{-\frac{\delta n}{2|\lambda|}} \right\}$.

Since φ is a non-decreasing function, we have $\varphi(f(y)) \geq \varphi(\|y-x_0\|_\lambda^{-\frac{\delta n}{2|\lambda|}})$ and therefore, Lemma 2.2 implies $\varphi(\|y-x_0\|_\lambda^{-\frac{\delta n}{2|\lambda|}}) \geq M\varphi(\|y-x_0\|_\lambda^{-\frac{n}{2|\lambda|}})$. Thus,

$$L_{11}(x) \leq M\left(\int_{U(y)} \phi_p(f(y))w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})dy\right)^{\frac{1}{p}} \times\left(\int_{U(y)} K(\beta\|y-x_0\|_\lambda)^{p'}\left[\varphi(\|y-x_0\|_\lambda^{-\frac{n}{2|\lambda|}})w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})\right]^{-\frac{P'}{p}}dy\right)^{\frac{1}{p'}}. \tag{3.3}$$

The right hand side integral with respect to y may be easily calculated. Namely, passing to generalized spherical coordinates by transformation

$$\begin{aligned} y_1 &= x_{01} + (t\cos\theta_1)^{2\lambda_1}, \\ y_2 &= x_{02} + (t\sin\theta_1\cos\theta_2)^{2\lambda_2}, \\ &\vdots \\ y_n &= x_{0n} + (t\sin\theta_1\sin\theta_2\ldots\sin\theta_{n-1})^{2\lambda_n}, \end{aligned}$$

where θ_j, $j=1,2,\ldots,n$, are the coordinates of the point θ on unit sphere. We can see that the Jacobian of this transformation $t^{2|\lambda|-1}\Omega_\lambda(\theta)$, where $\Omega_\lambda(\theta)$ depends on angles $\theta_1,\theta_2,\ldots,\theta_{n-1}$ only $0\leq\theta_1,\ldots,\theta_{n-2}\leq\pi$, $0\leq\theta_{n-1}\leq 2\pi$ and

$$\Omega_\lambda(\theta) = 2^n\prod_{j=1}^{n-1}\left(\cos\theta_j\right)^{2\lambda_j-1}\left(\sin\theta_j\right)^{2|\lambda|-\sum_{k=1}^{j}\lambda_k-1}.$$

Here the integral $\int_{S^{n-1}}\Omega_\lambda(\theta)d\theta$ is finite, where S^{n-1} is the unit ball in $\mathbb{R}^n$. Consequently, from (3.3) we have

$$L_{11}(x) \leq M\left(\int_{(2M_\lambda)^{\frac{n}{2|\lambda|}}\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}}}^{(2M_\lambda)^{\frac{n}{2|\lambda|}}a} K^{p'}\left(\beta t^{\frac{2|\lambda|}{n}}\right)\left[\varphi(t^{-1})w(t)\right]^{-\frac{p'}{p}}t^{2|\lambda|-1}dt\right)^{\frac{1}{p'}} \times\left(\int_{U(y)}\phi_p(f(y))w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})dy\right)^{\frac{1}{p}}. \tag{3.4}$$

Let us now consider the integral L_{12}. By passing to generalized spherical coordinates, we get

$$L_{12}(x) \leq \int_{A(y)} K\left(\|y-x_0\|_\lambda\right)\|y-x_0\|_\lambda^{-\frac{n}{2|\lambda|}\delta} dy$$
$$= ML\left(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right), \tag{3.5}$$

where $L(r)$ is defined in the condition (K_2). Relations (3.4) and (3.5) give the desired conclusion. □

Lemma 3.2. *Let f be a non-negative measurable function on $\mathbb{R}^n$. If $0 < 2M_\lambda\|x-x_0\|_\lambda < 1$ and $0 < \delta < 2|\lambda|$, then there exists a positive M such that*

$$L_2(x) \leq MR_2(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}})\left(\int_{B_\lambda(x_0,2M_\lambda\|x-x_0\|_\lambda)} \phi_p(f(y))w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})dy\right)^{\frac{1}{p}}$$
$$+M\|x-x_0\|_\lambda^{2|\lambda|-\delta},$$

where

$$R_2(r) = K\left(\beta t^{\frac{2|\lambda|}{n}}\right)r^{\frac{2|\lambda|}{n}\frac{2|\lambda|}{p'}}\left[\varphi\left(r^{-\frac{2|\lambda|}{n}}\right)w(r)\right]^{-\frac{1}{p}}.$$

Proof. It follows from (3.2) that

$$L_2(x) \leq K\left(\beta\|x-x_0\|_\lambda\right)\int_{B(x)} f(y)dy$$
$$\leq K\left(\beta\|x-x_0\|_\lambda\right)\left\{\int_{\{y\in B(x);\, f(y)>\|x-x_0\|_\lambda^{-\delta}\}} f(y)dy\right.$$
$$\left.+\int_{\{y\in B(x);\, f(y)\leq\|x-x_0\|_\lambda^{-\delta}\}} f(y)dy\right\} =: L_{21}(x)+L_{22}(x),$$

where $B(x) = B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda) - B_\lambda(x, \|x-x_0\|_\lambda/2M_\lambda)$.

Let us first consider L_{21}. Since φ is a non-decreasing function, by Lemma 3.1, we get

$$L_{21}(x) \leq M\left[\varphi\left(\|x-x_0\|_\lambda^{-1}\right)\right]^{-\frac{1}{p}} K\left(\beta\|x-x_0\|_\lambda\right)\int_{B(x)} f(y)\left[\varphi(f(y))\right]^{\frac{1}{p}} dy.$$

From Hölder's inequality, we obtain

$$L_{21}(x) \leq M\left[\varphi\left(\|x-x_0\|_\lambda^{-1}\right)\right]^{-\frac{1}{p}} K\left(\beta\|x-x_0\|_\lambda\right)$$
$$\times\left(\int_{B_\lambda(x_0,2M_\lambda\|x-x_0\|_\lambda)} dy\right)^{\frac{1}{p'}}\left(\int_{B(x)} f(y)^p\varphi(f(y))dy\right)^{\frac{1}{p}},$$

where $\frac{1}{p}+\frac{1}{p'} = 1$.

Therefore, because w is a non-increasing function, it follows that

$$L_{21}(x) \le M\left[\varphi\left(\|x-x_0\|_\lambda^{-1}\right)\right]^{-\frac{1}{p}} K\left(\beta\|x-x_0\|_\lambda\right)\|x-x_0\|_\lambda^{\frac{2|\lambda|}{P'}} \times\left(\int_{B(x)}\phi_p(f(y))dy\right)^{\frac{1}{p}} \tag{3.6}$$

$$\le M\left(\varphi\left(\|x-x_0\|_\lambda^{-1}\right)w(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}})\right)^{-\frac{1}{p}} K\left(\beta\|x-x_0\|_\lambda\right)\|x-x_0\|_\lambda^{\frac{2|\lambda|}{P'}} \times\left(\int_{B_\lambda(x_0,2M_\lambda\|x-x_0\|_\lambda)}\phi_p(f(y))w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})dy\right)^{\frac{1}{p}}. \tag{3.7}$$

On the other hand, we have

$$L_{22} \le K\left(\beta\|x-x_0\|_\lambda\right)\int_{B_\lambda(x_0,2M_\lambda\|x-x_0\|_\lambda)}\|x-x_0\|_\lambda^{-\delta}dy \le MK\left(\beta\|x-x_0\|_\lambda\right)\|x-x_0\|_\lambda^{2|\lambda|-\delta}. \tag{3.8}$$

We have the desired conclusion from (3.7) and (3.8). □

Lemma 3.3. *Let f be a non-negative measurable function on $\mathbb{R}^n$. If $\delta > 0$, then there exists a positive M such that*

$$L_3(x) \le MR_3\left(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right)\left(\int_{B_\lambda(x,\|x-x_0\|_\lambda/2M_\lambda)}\phi_p(f(y))w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})dy\right)^{\frac{1}{p}} + M\int_0^{(\|x-x_0\|_\lambda/2M_\lambda)^{\frac{n}{2|\lambda|}}} K^{p'}\left(2\beta M_\lambda t^{\frac{2|\lambda|}{n}}\right)t^{2|\lambda|-\delta-1}dt$$

where $R_3(r) = \varphi^*(r)\omega(r)^{-\frac{1}{p}}$ *and* $\varphi^*(r) = \left(\int_0^r K^{p'}\left(t^{\frac{2|\lambda|}{n}}\right)\left[\varphi(t^{-1})\right]^{-\frac{p'}{p}} t^{2|\lambda|-1}dt\right)^{\frac{1}{p'}}$.

Proof. By change of variable, we have

$$L_3(x) = \int_{B_\lambda(0,\|x-x_0\|_\lambda/2M_\lambda)} K\left(\|y\|_\lambda\right)f(x+y)dy.$$

In a way similar to the proof of Lemmas 3.1 and 3.2, we obtain

$$
\begin{aligned}
L_3(x) \le M & \left(\int_0^{(\|x-x_0\|_\lambda/2M_\lambda)^{\frac{n}{2|\lambda|}}} K^{p'}\left(t^{\frac{2|\lambda|}{n}}\right) \left[\varphi(t^{-1})\right]^{-\frac{p'}{p}} t^{2|\lambda|-1} dt \right)^{\frac{1}{p'}} \\
& \times \left(\int_{B_\lambda(0,\|x-x_0\|_\lambda/2M_\lambda)} \phi_p(f(x+y)) dy \right)^{\frac{1}{p}} \\
& + M \int_0^{(\|x-x_0\|_\lambda/2M_\lambda)^{\frac{n}{2|\lambda|}}} K\left(\beta 2M_\lambda t^{\frac{2|\lambda|}{n}}\right) t^{2|\lambda|-\delta-1} dt \\
\le M\varphi^* & \left(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}} \right) w\left(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}} \right)^{-\frac{1}{p}} \\
& \times \left(\int_{B_\lambda(x,\|x-x_0\|_\lambda/2M_\lambda)} \phi_p(f(y)) w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}}) dy \right)^{\frac{1}{p}} \\
& + \int_0^{(\|x-x_0\|_\lambda/2M_\lambda)^{\frac{n}{2|\lambda|}}} K\left(\beta 2M_\lambda t^{\frac{2|\lambda|}{n}}\right) t^{2|\lambda|-\delta-1} dt,
\end{aligned}
$$

as required. □

4. Fine limit of $R_\alpha f$

We consider the function

$$
\begin{aligned}
R(r) &= R_1(r) + R_2(r) + R_3(r) \\
&= R_1(r) + K\left(\beta t^{\frac{2|\lambda|}{n}}\right) r^{\frac{2|\lambda|}{n}\frac{2|\lambda|}{p'}} \left(w(r)\varphi(r^{-\frac{2|\lambda|}{n}}) \right)^{-\frac{1}{p}} + \varphi^*(r) w(r)^{-\frac{1}{p}}.
\end{aligned}
$$

Theorem 4.1. *Let $p > 1$ and f be a non-negative measurable function on $\mathbb{R}^n$ satisfying conditions (1.2) and (1.3). If $\varphi^*(1) < \infty$ and $\lim_{r\to 0} R(r) = \infty$, then*

$$\lim_{x\to x_0} \left[R\left(\|x-x_0\|_\lambda\right)\right]^{-1} (Lf)(x) = 0.$$

If $R(r)$ is bounded, then $(Lf)(x_0)$ is finite and $(Lf)(x)$ tends to $(Lf)(x_0)$ as $x \to x_0$.

Proof. By condition (1.2), the integral

$$\int_{\mathbb{R}^n - B_\lambda(x_0, 2M_\lambda a^{\frac{2|\lambda|}{n}})} K\left(\beta\|y-x_0\|_\lambda\right) f(y) dy$$

is finite. It follows from (3.1), the condition K_2 and Lemma 3.1 that

$$\limsup_{x\to x_0}\left(R\left(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right)\right)^{-1}L_1(x)$$
$$\le M\left(\int_{B_\lambda(x_0,2M_\lambda a^{\frac{2|\lambda|}{n}})}\phi_p(f(y))w\left(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right)dy\right)^{\frac{1}{p}}.$$

Since a is arbitrary, we see that the integral in the left-hand side of the last estimate is equal to zero.

In view of Lemmas 3.2 and 3.3 and condition (1.3), we have

$$\lim_{x\to x_0}\left[R\left(\|x-x_0\|_\lambda\right)\right]^{-1}\left(L_2(x)+L_3(x)\right)=0.$$

If we combine these results, we have

$$\lim_{x\to x_0}\left[R\left(\|x-x_0\|_\lambda\right)\right]^{-1}(Lf)(x)=0.$$

If $R(r)$ is bounded, then Lemmas 3.2 and 3.3 imply that $L_2(x)+L_3(x)$ tends to zero at $x\to x_0$. Furthermore, in view of Lemma 3.1, we have $\limsup_{x\to x_0}L_1(x)<\infty$. Thus it follows that $(Lf)(x_0)$ is finite. Hence

$$L_1(x)+L_2(x)=\int_{\mathbb{R}^n-B_\lambda(x,\|x-x_0\|_\lambda/2M_\lambda)}K\left(\|x-y\|_\lambda\right)f(y)dy.$$

Since $\|y-x_0\|_\lambda\le 2M_\lambda^2\|y-x\|_\lambda$ for $y\in\mathbb{R}^n-B_\lambda(x,\|x-x_0\|_\lambda/2M_\lambda)$, we have by Lebesgue's dominated convergence theorem

$$\lim_{x\to x_0}\left(L_1(x)+L_2(x)\right)=(Lf)(x_0).$$

However, we also know that $\lim_{x\to x_0}L_3(x)=0$. The proof of Theorem 4.1 is thus complete. □

Corollary 4.1 ([8,9])**.** *Let $p=\frac{n}{\alpha}$ and $\varphi^*(1)<\infty$. If f is a non-negative measurable function on $\mathbb{R}^n$ satisfying* (1.2) *and the condition*

$$\int_{\mathbb{R}^n}\phi_p(f(y))dy<\infty,$$

then $L_\alpha f$ is continuous on $\mathbb{R}^n$ with $K(t)=t^{\alpha-n}$, $0<\alpha<n$, and $\lambda_k=\frac{1}{2}$, $k=1,2,\dots,n$.

Corollary 4.2 ([1])**.** *Let f be a non-negative measurable function satisfying conditions* (1.2) *and the condition*

$$\int_{\mathbb{R}^n}\phi_p(f(y))dy<\infty,$$

then $L_\alpha f$ is continuous on $\mathbb{R}^n$.

Proposition 4.1. *Let* $ap = n$, $\varphi^*(1) < \infty$, $x_0 = 0$, $K(t) = t^{\alpha-n}$, *and*

$$\lim_{r\to 0} r^{\frac{2|\lambda|}{p'}} (w(r))^{-\frac{1}{p}} \left(\varphi(r^{-1})\right)^{-\frac{1}{p}} = 0.$$

Then for any positive non-decreasing function $a(r)$ *on* $(0,\infty)$ *such that*

$$\lim_{r\to 0} a(r) = \infty,$$

there exists a non-negative measurable function f *satisfying* (1.2) *and* (1.3) *such that*

$$\limsup_{x\to x_0} a\left(\|x\|_\lambda^{\frac{n}{2|\lambda|}}\right)\left(w\left(\|x\|_\lambda^{\frac{n}{2|\lambda|}}\right)\varphi\left(\|x\|_\lambda^{-\frac{n}{2|\lambda|}}\right)\right)^{-\frac{1}{p}} R_\alpha f(x) = \infty,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Let (j_i) be a sequence of positive integers such that $j_i + 2 < J_{i+1}$ and $\sum_i a_i^{-\frac{1}{p}} < \infty$, where $a_i(r_j) = a_i$ and $r_j = 2^{-j_i}$. We set

$$f(y) = a_i^{-\frac{1}{p}} \left(\varphi(r_j^{-1})\right)^{\frac{1}{p'}} (w(r_j))^{-\frac{1}{p}} \|x_i - y\|_\lambda^{-\alpha} \left[\varphi(\|x_i - y\|_\lambda^{-1})\right]^{-1}$$

if $y \in \bigcup_{i=1}^{\infty} B_\lambda(x_i, (2r_j)^{\frac{2|\lambda|}{n}}) - B_\lambda(x_i, (r_j)^{\frac{2|\lambda|}{n}})$, otherwise $f(y) = 0$, where $x_i = (r_j, 0, \ldots, 0) \in \mathbb{R}^n$.

Let us now show that f meets all the conditions in the proposition. If we use Lemmas 2.1 and 2.2, then we have

$$\begin{aligned}
\int f(y)dy &= \sum_i a_i^{-\frac{1}{p}} (\varphi(r_j^{-1}))^{\frac{1}{p'}} (w(r_j))^{-\frac{1}{p}} \\
&\quad \times \int_{B_\lambda(x_i,(2r_j)^{\frac{2|\lambda|}{n}}) - B_\lambda(x_i,(r_j)^{\frac{2|\lambda|}{n}})} \|x_i - y\|_\lambda^{-\alpha} \left[\varphi(\|x_i - y\|_\lambda^{-1})\right]^{-1} dy \\
&\le M \sum_i a_i^{-\frac{1}{p}} \left(\varphi(r_j^{-1})\right)^{\frac{1}{p'}} (w(r_j))^{-\frac{1}{p}} \int_{r_j}^{2r_j} t^{-\frac{2|\lambda|}{n}\alpha} \left(\varphi(t^{-\frac{2|\lambda|}{n}})\right)^{-1} t^{2|\lambda|-1} dt \\
&\le M \sum_i a_i^{-\frac{1}{p}} \left\{ r_j^{\frac{2|\lambda|}{p'}} \left(\varphi(r_j^{-1})\right)^{\frac{1}{p}} (w(r_j))^{-\frac{1}{p}} \right\} \\
&\le M \sum_i a_i^{-\frac{1}{p}} < \infty.
\end{aligned}$$

Consequently f satisfies (1.2). On the other hand, since the values $\left(a_i^{-\frac{1}{p}}\right)$ and $\left(r_j^{\frac{2|\lambda|}{p'}}\left(\varphi(r_j^{-1})\right)^{-\frac{1}{p}}\left(w(r_j)\right)^{-\frac{1}{p}}\right)$ are bounded, we have

$$\begin{aligned} f(y) &\le M\left(\varphi(r_j^{-1})\right)^{\frac{1}{p'}}\left(w(r_j)\right)^{-\frac{1}{p}}\|x_i-y\|_\lambda^{-\alpha}\left[\varphi(\|x_i-y\|_\lambda)\right]^{-1} \\ &\le M\left(\varphi(r_j^{-1})\right)^{-\frac{1}{p}}\left\{r_j^{\frac{2|\lambda|}{p'}}\varphi(r_j^{-1})^{-\frac{1}{p}}\|x_i-y\|_\lambda^{-\alpha}\right\}^{-1} \\ &\le M\|x_i-y\|_\lambda^{-\alpha-\frac{n}{p'}}. \end{aligned}$$

Thus, the inequality $\varphi(f(y)) \le \varphi(\|x_i-y\|_\lambda^{-1})$ holds.

Now we show that f satisfies (1.3). Using condition (w_1), we get

$$\begin{aligned} &\int_{\mathbb{R}^n}\phi_p(f(y))w\left(\|y\|_\lambda^{\frac{n}{2|\lambda|}}\right)dy \\ &\le \sum_i a_i^{-1}\left(\varphi(r_j^{-1})\right)^{\frac{p}{p'}}\int_{B_\lambda(x_i,(2r_j)^{\frac{2|\lambda|}{n}})-B_\lambda(x_i,(r_j)^{\frac{2|\lambda|}{n}})}\|x_i-y\|_\lambda^{-\alpha p} \\ &\qquad\times\left[\varphi(\|x_i-y\|_\lambda^{-1})\right]^{-\frac{p}{p'}}dy \\ &\le M\sum_i a_i^{-1}\left(\varphi(r_j^{-1})\right)^{\frac{p}{p'}}\int_{r_j}^{2r_j}t^{-\frac{2|\lambda|}{n}\alpha p}\left(\varphi(t^{-\frac{2|\lambda|}{n}})\right)^{-\frac{p}{p'}}t^{2|\lambda|-1}dt \\ &\le M\sum_i a_i^{-1}\left(\varphi(r_j^{-1})\right)^{\frac{p}{p'}}\int_{r_j}^{2r_j}\left(\varphi(t^{-1})\right)^{-\frac{p}{p'}}t^{-1}dt \\ &\le M\sum_i a_i^{-1} < \infty. \end{aligned}$$

Finally,

$$\begin{aligned} R_\alpha f(x_i) &\ge a_i^{-\frac{1}{p}}\left(\varphi(r_j^{-1})\right)^{\frac{1}{p'}}\left(w(r_j)\right)^{-\frac{1}{p}} \\ &\qquad\times\int_{B_\lambda(x_i,(2r_j)^{\frac{2|\lambda|}{n}})-B_\lambda(x_i,(r_j)^{\frac{2|\lambda|}{n}})}\|x_i-y\|_\lambda^{-n}\left[\varphi(\|x_i-y\|_\lambda^{-1})\right]^{-1}dy \\ &\ge Ma_i^{-\frac{1}{p}}\left(\varphi(r_j^{-1})\right)^{\frac{1}{p'}}\left(w(r_j)\right)^{-\frac{1}{p}}\int_{r_j}^{2r_j}\left(\varphi(t^{-1})\right)^{-1}t^{-1}dt \\ &\ge Ma_i^{-\frac{1}{p}}\left(\varphi(r_j^{-1})\right)^{-\frac{1}{p}}\left(w(r_j)\right)^{-\frac{1}{p}}. \end{aligned}$$

Thus we have

$$a_i\left(\varphi(r_j^{-1})\right)^{\frac{1}{p}}\left(w(r_j)\right)^{\frac{1}{p}}R_\alpha f(x^{(i)}) \ge Ma_i^{\frac{1}{p}}.$$

This proves the proposition for $j \to \infty$. □

REFERENCES

[1] A. Aral, "On generalized potentials-type integral operators with non-isotropic kernels," *Int. Math. J.*, vol. 3, no. 12, pp. 1367–1378, 2003.

[2] A. D. Gadjiev and I. A. Aliyev, "Riesz and Bessel potentials generalized by generalized shift operators and its inversion," in *Theory of Functions and Approximation. Proc. IV Math. Conference, Saratov, 1988*, Saratov, 1990, pp. 43–53.

[3] A. D. Gadjiev, "On generalized potential-type integral operators," *Funct. Approx. Comment. Math.*, vol. 25, pp. 37–44, 1997, dedicated to Roman Taberski on the occasion of his 70th birthday.

[4] V. S. Guliev and Z. V. Safarov, "On generalized fractional integrals, associated with the Bessel differential expansions," *Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.*, vol. 22, no. 4, Math. Mech., pp. 75–90, 2002, VIII Republic Conference on Mathematics and Mechanics (Baku, 2002).

[5] V. S. Guliyev and R. C. Mustafayev, "On generalized fractional integrals," *Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.*, vol. 21, no. 4, Math. Mech., pp. 63–71, 237, 2001.

[6] N. S. Landkof, *Foundations of modern potential theory*. New York: Springer-Verlag, 1972, translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180.

[7] P. I. Lizorkin, "Multipliers of Fourier integrals and estimates of convolutions in spaces with mixed norm. Applications," *Izv. Akad. Nauk SSSR Ser. Mat.*, vol. 34, pp. 218–247, 1970.

[8] Y. Mizuta, "Continuity properties of Riesz potentials and boundary limits of Beppo Levi functions," *Math. Scand.*, vol. 63, no. 2, pp. 238–260, 1988.

[9] Y. Mizuta, "Continuity properties of potentials and Beppo-Levi-Deny functions," *Hiroshima Math. J.*, vol. 23, no. 1, pp. 79–153, 1993.

[10] E. Nakai and H. Sumitomo, "On generalized Riesz potentials and space of some smooth functions," preprint.

[11] T. Shimomura and Y. Mizuta, "Taylor expansion of Riesz potentials," *Hiroshima Math. J.*, vol. 25, no. 3, pp. 595–621, 1995.

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