Miskolc Mathematical Notes
Vol. 7 (2006), No 2, pp. 161-168

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2006.122

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Vol. 7 (2006), No. 2, pp. 161–168

HU ISSN 1787-2413
electronic version

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[Received: April 19, 2005]

ABSTRACT. In this paper we study the Kantorovich form of Stancu operators. As particular cases, we shall obtain similar properties of the Kantorovich form for Bernstein, Schurer and Schurer–Stancu operators.

Mathematics Subject Classification: 41A10, 41A25, 41A36

Keywords: linear positive operators, Bernstein operators, Schurer operators, Stancu operators, Kantorovich operators

1. INTRODUCTION

In this section, we recall some notions and results which we will use in this article (see [5]).

We define the natural number m_0 by

$$m_0 = \begin{cases} \max\{1, -[\beta]\}, & \text{iff} \quad \beta \in \mathbb{R} \setminus \mathbb{Z}, \\ \max\{1, 1-\beta\}, & \text{iff} \quad \beta \in \mathbb{Z}. \end{cases} \tag{1.1}$$

For the real number p, we have

$$m + \beta \geq \gamma_\beta = m_0 + \beta \tag{1.2}$$

for any natural number m, $m \geq m_0$, where

$$\gamma_\beta = \begin{cases} \max\left\{1+\beta, \{\beta\}\right\}, & \text{iff} \quad \beta \in \mathbb{R} \setminus \mathbb{Z}, \\ \max\{1+\beta, 1\}, & \text{iff} \quad \beta \in \mathbb{Z}. \end{cases} \tag{1.3}$$

For the real numbers $\alpha, \beta, \alpha \geq 0$, we set

$$\mu^{(\alpha,\beta)} = \begin{cases} 1, & \text{iff } \alpha \leq \beta, \\ 1 + \dfrac{\alpha-\beta}{\gamma_\beta}, & \text{iff } \alpha > \beta. \end{cases} \tag{1.4}$$

Lemma 1. *For the real numbers α and β, $\alpha \geq 0$, we have*

$$0 \leq \frac{k+\alpha}{m+\beta} \leq \mu^{(\alpha,\beta)} \tag{1.5}$$

for any natural number m, $m \geq m_0$ and for any $k \in \{0, 1, \dots, m\}$.

For the real numbers α and β, $\alpha \geq 0$, where m_0 and $\mu^{(\alpha,\beta)}$ are defined by (1.1)–(1.4), let the operators $P_m^{(\alpha,\beta)} : C\big([0, \mu^{(\alpha,\beta)}]\big) \to C\big([0, 1]\big)$ be defined for any function $f \in C\big([0, \mu^{(\alpha,\beta)}]\big)$ by

$$\big(P_m^{(\alpha,\beta)} f\big)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right), \tag{1.6}$$

for any natural number $m, m \geq m_0$ and for any $x \in [0, 1]$, where $p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}$ are the fundamental Bernstein polynomials. These operators are named Bernstein-Stancu operators, introduced and studied in 1969 by D. D. Stancu in the paper [7]. In [7] the domain of definition for the Bernstein–Stancu operators is $C([0, 1])$ and $0 \leq \alpha \leq \beta$.

Remark 1. Because there is no restriction on the real parameter β in our construction, in the following remarks we will explain how to obtain the Bernstein, Schurer and Schurer–Stancu operators from the Stancu operators, through particularization.

Remark 2. If $\alpha = \beta = 0$, then $m_0 = 1$, $\mu^{(0,0)} = 1$, we obtain $P_m^{(0,0)} = B_m$, $m \geq 1$, the Bernstein operators, $B_m : C([0, 1]) \to C([0, 1])$ defined by

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.7}$$

for any function $f \in C([0, 1])$ and any $x \in [0, 1]$.

Remark 3. If p is a natural number, $\alpha = 0$ and $\beta = -p$, then $m_0 = 1 + p$, $\mu^{(0,-\beta)} = 1 + p$. Changing m with $m + p$, we obtain $P_{m+p}^{(0,-\beta)} = \widetilde{B}_{m,p}$, $m \geq 1$, the Schurer operators, $\widetilde{B}_{m,p} : C([0, 1+p]) \to C([0, 1])$ defined by

$$\big(\widetilde{B}_{m,p} f\big)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.8}$$

for any function $f \in C([0, 1+p])$ and any $x \in [0, 1]$, where

$$\widetilde{p}_{m,k}(x) = p_{m+p,k}(x) \binom{m+p}{k} x^k (1-x)^{m+p-k}$$

are the fundamental Schurer polynomials.

Remark 4. If $0 \leq \alpha$, p is a natural number, substituting m with $m + p$ and β with $\beta - p$, we obtain $P_{m+p}^{(\alpha,\beta-p)} = \widetilde{S}_{m,p}^{(\alpha,\beta)}$, $m \geq m_0$, where m_0 is defined in (1.1) for $\beta - p$, the Schurer–Stancu operators, $\widetilde{S}_{m,p}^{(\alpha,\beta)} : C\big([0, \mu^{(\alpha,\beta-p)}]\big) \to C([0, 1])$ defined by

$$\big(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\big)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) \left(\frac{k+\alpha}{m+\beta}\right), \tag{1.9}$$

for any function $f \in C\big([0, \mu^{(\alpha,\beta-p)}]\big)$ and any $x \in [0,1]$ (see [2], where the domain of definition for the Schurer–Stancu operators is $C([0, 1+p])$ and the parameters α and β verify $0 \le \alpha \le \beta$).

Proposition 1. *The operators* $\big(P_m^{(\alpha,\beta)}\big)_{m \ge m_0}$ *satisfy the relations*

$$\big(P_m^{(\alpha,\beta)} e_0\big)(x) = 1, \tag{1.10}$$

$$\big(P_m^{(\alpha,\beta)} e_1\big)(x) = x + \frac{\alpha - \beta x}{m+\beta} \tag{1.11}$$

and

$$\big(P_m^{(\alpha,\beta)} e_2\big)(x) = x^2 + \frac{mx(1-x) + (\alpha - \beta x)(2mx + \beta x + \alpha)}{(m+\beta)^2} \tag{1.12}$$

for any natural number m, $m \ge m_0$, *for any* $x \in [0,1]$.

PROOF. The proof can be found in [7, 8]. □

2. PRELIMINARIES

For a nonzero natural number m, let the operator $K_m : L_1\big([0,1]\big) \to C([0,1])$ be defined for any function $f \in L_1\big([0,1]\big)$ by

$$(K_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)\,dt, \tag{2.1}$$

for any $x \in [0,1]$.

The operators K_m, where m is a nonzero natural number, are named Kantorovich operators, introduced and studied in 1930 by L. V. Kantorovich (see [8]).

In the following, we consider the real numbers α and β, $\alpha \ge 0$, where m_0 and $\mu^{(\alpha,\beta)}$ are defined by (1.1)–(1.4).

Lemma 2. *For a natural number* m, $m \ge m_0$, *we have*

$$0 \le \frac{k+\alpha}{m+\beta+1} \le \frac{k+\alpha+1}{m+\beta+1} \le \mu^{(\alpha,\beta)} \tag{2.2}$$

for any $k \in \{0, 1, \ldots, m\}$.

PROOF. This results from (1.5). □

For a natural number m, $m \ge m_0$, let the operator $K_m^{(\alpha,\beta)} : L_1\big([0, \mu^{(\alpha,\beta)}]\big) \to C([0,1])$ be defined for any function $f \in L_1\big([0, \mu^{(\alpha,\beta)}]\big)$ by

$$\big(K_m^{(\alpha,\beta)} f\big)(x) = (m+\beta+1) \sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k+\alpha}{m+\beta+1}}^{\frac{k+\alpha+1}{m+\beta+1}} f(t)\,dt, \tag{2.3}$$

for any $x \in [0,1]$. These operators are named the Kantorovich–Stancu type operators.

Lemma 3. *The operators* $\left(K_m^{(\alpha,\beta)}\right)_{m\geq m_0}$ *satisfy the relations*

$$\left(K_m^{(\alpha,\beta)}e_0\right)(x) = 1, \tag{2.4}$$

$$\left(K_m^{(\alpha,\beta)}e_1\right)(x) = \frac{m}{m+\beta+1}\,x + \frac{2\alpha+1}{2(m+\beta+1)}, \tag{2.5}$$

and

$$\left(K_m^{(\alpha,\beta)}e_2\right)(x) = \frac{m(m-1)}{(m+\beta+1)^2}\,x^2 + \frac{2m(\alpha+1)}{(m+\beta+1)^2}\,x + \frac{3\alpha^2+3\alpha+1}{3(m+\beta+1)^2} \tag{2.6}$$

for any natural number m, $m \geq m_0$, *and for any* $x \in [0,1]$.

PROOF. Using the definition of the operator $K_m^{(\alpha,\beta)}$ and applying Proposition 1.1, the conclusion follows. □

Lemma 4. *The operators* $\left(K_m^{(\alpha,\beta)}\right)_{m\geq m_0}$ *satisfy the relation*

$$\left(K_m^{(\alpha,\beta)}\varphi_x^2\right)(x) = \frac{-m+(\beta+1)^2}{(m+\beta+1)^2}\,x^2 + \frac{m-(2\alpha+1)(\beta+1)}{(m+\beta+1)^2}\,x + \frac{3\alpha^2+3\alpha+1}{3(m+\beta+1)^2} \tag{2.7}$$

for any natural number m, $m \geq m_0$, *for any* $x \in [0,1]$.

PROOF. We have

$$\left(K_m^{(\alpha,\beta)}\varphi_x^2\right)(x) = \left(K_m^{(\alpha,\beta)}e_2\right)(x) - 2x\left(K_m^{(\alpha,\beta)}e_1\right)(x) + x^2\left(K_m^{(\alpha,\beta)}e_0\right)(x),$$

and applying Lemma 2.2 we get the conclusion. □

Lemma 5. *The operators* $\left(K_m^{(\alpha,\beta)}\right)_{m\geq m_0}$ *are linear and positive.*

PROOF. The conclusion follows immediately. □

3. MAIN RESULTS

Let us recall that if $I \subset \mathbb{R}$ is a given interval, $f \in C_B(I)$, where $B(I) = \{f \mid f : I \to \mathbb{R},\ f \text{ bounded on } I\}$, $C(I) = \{f \mid f : I \to \mathbb{R},\ f \text{ is continuous on } I\}$, and $C_B(I) = B(I) \cap C(I)$. The first order modulus of smoothness is the function $\omega_1 : [0,\infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by the formula

$$\omega_1(f;\delta) = \sup\left\{|f(x') - f(x'')| : x', x'' \in I,\ |x'-x''| \leq \delta\right\}. \tag{3.1}$$

In the sequel, we will use the following result established by O. Shisha and B. Mond (see [1, 6, 8]).

Theorem 1. *Let $L : C(I) \to B(I)$ be a linear and positive operator with the properties $Le_0 = e_0$.*

(i) *If $f \in C_b(I)$, then*

$$\left|(Lf)(x) - f(x)\right| \le \left[1 + \delta^{-1}\sqrt{\left(L\varphi_x^2\right)(x)}\right]\omega_1(f;\delta) \tag{3.2}$$

and

$$\left|(Lf)(x) - f(x)\right| \le \left[1 + \delta^{-2}\left(L\varphi_x^2\right)(x)\right]\omega_1(f;\delta) \tag{3.3}$$

for any $x \in I$, for any $\delta > 0$;

(ii) *If f is a derivable function on I and $f' \in C_B(I)$, then*

$$\left|(Lf)(x) - f(x)\right| \le$$

$$\le \left|f'(x)\right|\left|(Le_1)(x) - x\right| + \sqrt{\left(L\varphi_x^2\right)(x)}\left[1 + \delta^{-1}\sqrt{\left(L\varphi_x^2\right)(x)}\right]\omega_1(f';\delta) \tag{3.4}$$

for any $x \in I$, for any $\delta > 0$.

Theorem 2. *The sequence $\left(K_m^{(\alpha,\beta)} f\right)_{m \ge m_0}$ converges uniformly on $[0,1]$ to f, for any $f \in C\left([0, \mu^{(\alpha,\beta)}]\right)$.*

PROOF. Applying Lemma 2.3 we get $\lim_{m\to\infty}\left(K_m^{(\alpha,\beta)}\varphi_x^2\right)(x) = 0$ uniformly on $[0,1]$. Since $K_m^{(\alpha,\beta)} e_0 = e_0$, using then the well-known Bohman–Korovkin theorem [1, 8], we obtain the result. □

Theorem 3. (i) *If $f \in C\left(\left[0, \mu^{(\alpha,\beta)}\right]\right)$, then*

$$\left|\left(K_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \le \left(1 + \delta^{-1}\sqrt{\left(K_m^{(\alpha,\beta)}\varphi_x^2\right)(x)}\right)\omega_1(f;\delta) \tag{3.5}$$

and

$$\left|\left(K_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \le \left[1 + \delta^{-2}\left(K_m^{(\alpha,\beta)}\varphi_x^2\right)(x)\right]\omega_1(f;\delta) \tag{3.6}$$

for any $x \in [0,1]$, for any $\delta > 0$ and $m \in \mathbb{N}$, $m \ge m_0$.

(ii) *If f is a differentiable function on $\left[0, \mu^{(\alpha,\beta)}\right]$ and $f' \in C\left(\left[0, \mu^{(\alpha,\beta)}\right]\right)$, then*

$$\left|\left(K_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \le |f'(x)|\left|-\frac{\beta+1}{m+\beta+1}x + \frac{2\alpha+1}{2(m+\beta+1)}\right|$$

$$+ \sqrt{\left(K_m^{(\alpha,\beta)}\varphi_x^2\right)(x)}\left(1 + \delta^{-1}\sqrt{\left(K_m^{(\alpha,\beta)}\varphi_x^2\right)(x)}\right)\omega_1(f';\delta) \tag{3.7}$$

for any $x \in [0,1]$, for any $\delta > 0$ and $m \in \mathbb{N}$, $m \ge m_0$.

PROOF. Applying the Theorem 3.1, we obtain the results. □

Theorem 4. *Let* $\delta_m^{(\alpha,\beta)}(x) = \sqrt{\left(K_m^{(\alpha,\beta)}\varphi_x^2\right)(x)}$*, where* $x \in [0,1]$ *and* m *is any natural number,* $m \geq m_0$*. Then*

(1) *If* $f \in C\left(\left[0, \mu^{(\alpha,\beta)}\right]\right)$*, then*

$$\left|\left(K_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \leq 2\omega_1\left(f; \delta_m^{(\alpha,\beta)}(x)\right) \tag{3.8}$$

for any $x \in [0,1]$ *and for any natural number* m*,* $m \geq m_0$*.*

(2) *If* f *is a derivable function on* $\left[0, \mu^{(\alpha,\beta)}\right]$ *and* $f' \in C\left(\left[0, \mu^{(\alpha,\beta)}\right]\right)$*, then*

$$\begin{aligned}\left|\left(K_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \leq |f'(x)| \Bigg| &-\frac{\beta+1}{m+\beta+1}x \\ &+ \frac{2\alpha+1}{2(m+\beta+1)}\Bigg| + 2\delta_m^{(\alpha,\beta)}(x)\omega_1\left(f', \delta_m^{(\alpha,\beta)}(x)\right)\end{aligned} \tag{3.9}$$

for any $x \in [0,1]$ *and for any natural number* m*,* $m \geq m_0$*.*

PROOF. Choosing $\delta = \delta_m^{(\alpha,\beta)}(x)$ in Theorem 3.3, we obtain Theorem 3.4. $\square$

For a natural number $m, m \geq m_1$, let $f_m : [0,1] \to \mathbb{R}$ be a function of second degree defined by

$$f_m(x) = \frac{-m + (\beta+1)^2}{(m+\beta+1)^2}x^2 + \frac{m - (2\alpha+1)(\beta+1)}{(m+\beta+1)^2}x + \frac{3\alpha^2 + 3\alpha + 1}{3(m+\beta+1)^2} \tag{3.10}$$

for any $x \in [0,1]$, where m_1 is the smallest natural number so that

$$m_1 \geq \max\left\{m_0, (2\alpha+1)(\beta+1), (\beta+1)^2 + 1, (\beta+1)(2\beta - 2\alpha + 1)\right\}. \tag{3.11}$$

Lemma 6. *The function* f_m *has a maximum value*

$$M_m^{(\alpha,\beta)} = \frac{3m^2 - 2m(6\alpha\beta + 3\beta + 1 - 6\alpha^2) - (\beta+1)^2}{12\left[m - (\beta+1)^2\right](m+\beta+1)^2} > 0 \tag{3.12}$$

at the point $x_M = \frac{m-(2\alpha+1)(\beta+1)}{2(m-(\beta+1)^2)}$*, where* m *is a natural number,* $m \geq m_1$*.*

PROOF. Let $a = \frac{-m+(\beta+1)^2}{(m+\beta+1)^2}$, $b = \frac{m-(2\alpha+1)(\beta+1)}{(m+\beta+1)^2}$, and $c = \frac{3\alpha^2+3\alpha+1}{3(m+\beta+1)^2}$. Then $f_m = ax^2 + bx + c$. Because $m \geq m_1$, $a < 0$, the function f_m has a maximum value $M_m^{(\alpha,\beta)} = -\frac{\Delta}{4a}$ at the point $x_M = -\frac{b}{2a}$. It follows immediately that $0 \leq x_M \leq 1$, since $m > (\beta+1)^2$, $m \geq (2\alpha+1)(\beta+1)$ and $m \geq (\beta+1)(2\beta - 2\alpha + 1)$. We have $f_m(0) = \frac{3\alpha^2+3\alpha+1}{3(m+\beta+1)^2} > 0$ and from calculations we obtain relation (3.12). $\square$

Lemma 7. *We have*

$$\delta_m^{(\alpha,\beta)}(x) \leq \delta_m^{(\alpha,\beta)} \tag{3.13}$$

for any $x \in [0,1]$ *and for any natural number* m*,* $m \geq m_1$*, where* $\delta_m^{(\alpha,\beta)} = \sqrt{M_m^{(\alpha,\beta)}}$*.*

PROOF. Taking (2.7) into account, the definition of $\delta_m^{(\alpha,\beta)}(x)$ and Lemma 3.1. $\square$

For a natural number m, $m \geq m_0$, let $g_m : [0,1] \to \mathbb{R}$ be a function defined by $g_m(x) = -\frac{\beta+1}{m+\beta+1}x + \frac{2\alpha+1}{2(m+\beta+1)}$, for any $x \in [0,1]$. Because the function g_m is linear, then the extremal value of g_m is $g_m(0)$ and $g_m(1)$. Then

$$\left|g_m(x)\right| \leq \eta_m^{(\alpha,\beta)} \tag{3.14}$$

for any $x \in [0,1]$, for any natural number m, $m \geq m_0$, where

$$\begin{aligned} \eta_m^{(\alpha,\beta)} &= \max\left\{\left|g_m(0)\right|, \left|g_m(1)\right|\right\} = \\ &= \max\left\{\frac{2\alpha+1}{2(m+\beta+1)}, \frac{|-2\beta+2\alpha-1|}{2(m+\beta+1)}\right\}. \end{aligned} \tag{3.15}$$

Corollary 1. *The following assertions are true:*

(1) *If* $f \in C\left(\left[0, \mu^{(\alpha,\beta)}\right]\right)$, *then*

$$\left|\left(K_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \leq 2\omega_1\left(f; \delta_m^{(\alpha,\beta)}\right) \tag{3.16}$$

for any $x \in [0,1]$ *and for any natural number* m, $m \geq m_1$.

(2) *If* f *is a derivable function on* $\left[0, \mu^{(\alpha,\beta)}\right]$ *and* $f' \in C\left(\left[0, \mu^{(\alpha,\beta)}\right]\right)$, *then*

$$\left|\left(K_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \leq M_1 \eta_m^{(\alpha,\beta)} + 2\delta_m^{(\alpha,\beta)} \omega_1\left(f; \delta_m^{(\alpha,\beta)}\right) \tag{3.17}$$

for any $x \in [0,1]$ *and for any natural number* m, $m \geq m_1$, *where* $M_1 = \max_{x \in [0,1]} |f'(x)|$.

PROOF. It results from Theorem 3.4, Lemma 3.2 and relation (3.14). □

Remark 5. Through particularization, in the following applications we obtain known operators which verify the general results proved for the Stancu operators.

Application 1. If $\alpha = \beta = 0$ we obtain the Kantorovich operators.

Application 2. If p is a natural number, $\alpha = \beta = 0$, substituting m with $m + p$, we obtain the Kantorovich form of Schurer type operators (see [4]).

Application 3. If p is a natural number, $0 \leq \alpha \leq \beta$, substituting m with $m + p$, we obtain the Kantorovich form of Schurer–Stancu operators (see [3]).

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