Miskolc Mathematical Notes
Vol. 9 (2008), No 2, pp. 81-89

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2008.132

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HU e-ISSN 1787-2413

NOTES ON THE REPRESENTATION OF DISTRIBUTIVE MODAL ALGEBRAS

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Received 26 September, 2005

Abstract. In this work we give some results on the representation by means of relational structures of distributive lattices endowed with two modal operators $\Box$ and $\Diamond$, and two weak forms of negation Δ and ∇.

2000 *Mathematics Subject Classification:* Primary 03G10, 06D05, 06E25

Keywords: distributive modal algebras, relational structures, representation, canonical varieties

1. INTRODUCTION

In this paper we give some new results on the representation of distributive modal algebras by means of ordered sets with binary relations. Distributive modal algebras are bounded distributive lattices endowed with two modal operators $\Box$ and $\Diamond$, and two weak forms of negation Δ and ∇. These classes of algebras are the algebraic interpretation of certain free-negations modal logics, i. e., logics with modal operators but without classical negation (see, for instance, [2, 4]).

In [1], a relational semantics for Positive Modal Logic is introduced. In this semantics the relational structures, or frames, are triples $\langle X, \leq, R \rangle$ where X is a set, R is a binary relation and $\leq$ is a quasiorder of X, that is, a reflexive and transitive relation on X, such that $\leq \circ R \subseteq R \circ \leq$ and $\leq^{-1} \circ R \subseteq R \circ \leq^{-1}$. This semantics is inherited from a suitable intuitionistic modal logic from which Positive Modal Logic is the positive fragment. Reasons in favour of this semantics are given in [1, 2].

In this paper we study the representation for more extensive classes of distributive modal algebras by means of relational structures of type $\langle X, \leq, R_\Box, R_\Diamond, R_\Delta, R_\nabla \rangle$, where $\leq$ is an ordering of X, and $R_\Box$, $R_\Diamond$, R_Δ, R_∇ are binary relations defined on X.

In Section 2, we introduce the definitions and necessary notions to develop this paper, and we develop the representation by means of relational structures for some subvarieties of the variety of distributive modal algebras $\mathcal{DMA}$. In Section 3we prove that the varieties considered are closed under canonical extensions. Let us recall that the notion of canonical variety is the algebraic interpretation of the notion of canonical modal logic (see [6]).

2. DISTRIBUTIVE MODAL ALGEBRAS

A *distributive modal algebra* is an algebra $\boldsymbol{A} = \langle A, \vee, \wedge, \Box, \Diamond, \Delta, \nabla, 0, 1\rangle$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and $\Box$, $\Diamond$, Δ, and ∇ are unary operations defined on A and satisfying the identities

(M1) $\Box(a \wedge b) = \Box a \wedge \Box b$ and $\Box 1 = 1$;
(M2) $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ and $\Diamond 0 = 0$;
(M3) $\Delta(a \vee b) = \Delta a \wedge \Delta b$ and $\Delta 0 = 1$;
(M4) $\nabla(a \wedge b) = \nabla a \vee \nabla b$ and $\nabla 1 = 0$.

The variety of distributive modal algebras shall be denoted by $\mathcal{DMA}$. A $\Box$-*algebra* is an algebra $\boldsymbol{A} = \langle A, \vee, \wedge, \Box, 0, 1\rangle$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice with an operator $\Box$ satisfying (M1). A $\Diamond$-*algebra* is an algebra $\boldsymbol{A} = \langle A, \vee, \wedge, \Diamond, 0, 1\rangle$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice with an operator $\Diamond$ satisfying (M2). A $\Box\Diamond$-*algebra* is an algebra $\boldsymbol{A} = \langle A, \vee, \wedge, \Box, \Diamond, 0, 1\rangle$ such that it is a $\Box$-algebra and a $\Diamond$-algebra. Similarly, a Δ-*algebra* (resp., ∇-*algebra*) is an algebra $\boldsymbol{A} = \langle A, \vee, \wedge, \Delta, 0, 1\rangle$ (resp., $\boldsymbol{A} = \langle A, \vee, \wedge, \nabla, 0, 1\rangle$) such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice with an operator Δ (resp., ∇) satisfying (M3) (resp., (M4)). A $\Delta\nabla$-*algebra* is an algebra $\boldsymbol{A} = \langle A, \vee, \wedge, \Delta, \nabla, 0, 1\rangle$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice, and it is an Δ-algebra and a ∇-algebra.

A $\Box\Diamond$-*algebra* $\boldsymbol{A}$ is a *Positive Modal algebra*, or PM-algebra, if the following conditions are satisfied:

(P1) $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$;
(P2) $\Box(a \vee b) \leq \Box a \vee \Diamond b$.

A $\Delta\nabla$-*algebra* $\boldsymbol{A}$ is a *Negative Modal algebra*, or NM-algebra, if the following conditions are satisfied:

(N1) $\Delta(a \wedge b) \leq \Delta a \vee \nabla b$;
(N2) $\Delta a \wedge \nabla b \leq \nabla(a \vee b)$.

Let us consider a poset $\langle X, \leq\rangle$, i. e., X is a set and $\leq$ is a reflexive, antisymmetric and transitive binary relation on X. A subset $U \subseteq X$ is said to be *increasing* if for all $x, y \in X$ such that $x \in U$ and $x \leq y$, we have $y \in U$. The set of all increasing subsets of X is denoted by $\mathcal{P}_i(X)$. It is clear that $\langle \mathcal{P}_i(X), \cup, \cap, \varnothing, X\rangle$ is a bounded distributive lattice. For each $Y \subseteq X$, the increasing set (resp., decreasing set) generated by Y is $[Y) = \{x \in X : \exists y \in Y : y \leq x\}$ (resp., $(Y] = \{x \in X : \exists y \in Y : x \leq y\}$). Let Y be a subset of a set X. The theoretical complement of Y is denoted by $Y^c = X - Y$.

Let S and R be binary relations defined on a set X. The composition of R with S is denoted by $R \circ S$. The image of $x \in X$ by means of the relation R is $R(x) = \{y \in X : (x, y) \in R\}$.

Let X be a set and let R be a binary relation on X. We define four operators on $\mathcal{P}(X)$ as follows:

$$\Box_R(U) = \{x \in X : R(x) \subseteq U\}, \qquad \Diamond_R(U) = \{x \in X : R(x) \cap U \neq \varnothing\},$$
$$\Delta_R(U) = \{x \in X : R(x) \cap U = \varnothing\}, \quad \nabla_R(U) = \{x \in X : R(x) \nsubseteq U\}.$$

We note that, in $\mathcal{P}(X)$, all operators can be defined from $\Box_R$, i.e., $\Diamond_R(U) = \Box_R(U^c)^c$, $\Delta_R(U) = \Box_R(U^c)$, and $\nabla_R(U) = \Box_R(U)^c$, but in $\mathcal{P}_i(X)$ this fact is not valid.

Proposition 1. *Let $\langle X, \leq\rangle$ be a poset and let R be a binary relation on X. Then $\Box_R(U) = \Box_{R\circ\leq}(U)$, $\Diamond_R(U) = \Diamond_{R\circ\leq^{-1}}(U)$, $\Delta_R(U) = \Delta_{R\circ\leq^{-1}}(U)$, and $\nabla_R(U) = \nabla_{R\circ\leq}(U)$ for all $U \in \mathcal{P}_i(X)$.*

Proof. Let $U \in \mathcal{P}_i(X)$. We prove that $\Box_R(U) = \Box_{R\circ\leq}(U)$. Let $x, y \in X$ such that $x \in \Box_R(U)$, and $(x, y) \in R\circ\leq$. Then there exists $z \in X$ such that $(x,z) \in R$ and $z \leq y$. Since $R(x) \subseteq U$, $z \in U$, and as U is increasing, $y \in U$. Thus, $x \in \Box_{R\circ\leq}(U)$. Since $\leq$ is reflexive, $R \subseteq R\circ\leq$. So, $\Box_{R\circ\leq}(U) \subseteq \Box_R(U)$.

Let $U \in \mathcal{P}_i(X)$. We prove that $\Delta_R(U) = \Delta_{R\circ\leq^{-1}}(U)$. Let $x \in \Delta_R(U)$. Suppose that $(R\circ\leq^{-1})(x) \cap U \neq \varnothing$. Then there exist $y, z \in X$ such that $(x, y) \in R$, $z \leq y$, and $z \in U$. So, $y \in U \cap R(x)$, which is a contradiction. Thus, $x \in \Delta_{R\circ\leq^{-1}}(U)$. The other inclusion it follows by the fact that $R \subseteq R\circ\leq^{-1}$.

The proofs of the identities $\Diamond_R(U) = \Diamond_{R\circ\leq^{-1}}(U)$ and $\nabla_R(U) = \nabla_{R\circ\leq}(U)$ are similar. $\Box$

In general, the bounded distributive lattice $\mathcal{P}_i(X)$ is not closed under the modal operators or the weak negations. In the next result we shall give conditions on the relation R for that $\mathcal{P}_i(X)$ be closed under these operations.

Proposition 2. *Let $\langle X, \leq\rangle$ be a poset and let R be a binary relation on X. Then*

(1) *$\leq\circ R \subseteq R\circ\leq$ if and only if $\Box_R(U) \in \mathcal{P}_i(X)$, for all $U \in \mathcal{P}_i(X)$.*
(2) *$\leq^{-1}\circ R \subseteq R\circ\leq^{-1}$ if and only if $\Diamond_R(U) \in \mathcal{P}_i(X)$, for all $U \in \mathcal{P}_i(X)$.*
(3) *$\leq\circ R \subseteq R\circ\leq^{-1}$ if and only if $\Delta_R(U) \in \mathcal{P}_i(X)$, for all $U \in \mathcal{P}_i(X)$.*
(4) *$\leq^{-1}\circ R \subseteq R\circ\leq$ if and only if $\nabla_R(U) \in \mathcal{P}_i(X)$, for all $U \in \mathcal{P}_i(X)$.*

Proof. We prove only (1), (3), and (4).

(1) Let us show that if $U \in \mathcal{P}_i(X)$, then $\Box_R(U) \in \mathcal{P}_i(X)$. Let $x \leq y$ and $R(x) \subseteq U$. Let $z \in X$ such that $(y,z) \in R$. Then there exists $k \in X$ such that $(x,k) \in R$ and $k \leq z$. As $R(x) \subseteq U$, $k \in U$, and since U is increasing, $z \in U$.

Assume that $\Box_R(U) \in \mathcal{P}_i(X)$, for all $U \in \mathcal{P}_i(X)$. Let $x, y, z \in X$ such that $x \leq y$ and $(y,z) \in R$. Let us consider the increasing set $[R(x)) = U_x$. As $R(x) \subseteq U_x$, $x \in \Box_R(U_x)$. Then $y \in \Box_R(U_x)$. Thus, $z \in U_x$, i.e. there exists $k \in X$ such that $(x,k) \in R$ and $k \leq z$, i.e., $(x,z) \in R\circ\leq$.

(3) Assume that $\Delta_R(U) \in \mathcal{P}_i(X)$, for all $U \in \mathcal{P}_i(X)$. Let $x, y, z \in X$ such that $x \leq y$ and $(y,z) \in R$. Suppose that for every $k \in R(x)$, $k \notin [z)$, i.e., $R(x) \cap [z) = \varnothing$.

Then $x \in \Delta_R([z))$. So $y \in \Delta_R([z))$. Thus, $R(y) \cap [z) = \varnothing$, which is a contradiction. Therefore, there exists $k \in X$ such that $(x,k) \in R$ and $z \le k$, i. e., $(x,z) \in R \circ \le^{-1}$.

(4) Assume that $\nabla_R(U) \in \mathcal{P}_i(X)$, for all $U \in \mathcal{P}_i(X)$. Let $x,y,z \in X$ such that $z \le x$ and $(z,y) \in R$. Then $R(z) \cap (y] \neq \varnothing$. Since $U_y = (y]^c \in \mathcal{P}_i(X)$, $z \in \nabla_R(U_y)$, and as $\nabla_R(U_y)$ is increasing, $x \in \nabla_R(U_y)$. Then, $R(x) \cap (y] \neq \varnothing$. Thus, there exists $k \in X$ such that $(x,k) \in R$ and $k \le y$, i. e., $(x,y) \in R \circ \le$. The proof of the converse implication is straightforward. □

Theorem 3. *Let $\langle X, \le \rangle$ be a poset and let $R_\Box$, $R_\Diamond$, R_Δ and R_∇ be binary relations on X. Then:*

(1) *The following conditions are equivalent:*
   (a) $R_\Box \subseteq ((R_\Box \circ \le) \cap (R_\Diamond \circ \le^{-1})) \circ \le$;
   (b) $\Box_{R_\Box}(U \cup V) \subseteq \Box_{R_\Box}(U) \cup \Diamond_{R_\Diamond}(V)$, *for all* $U, V \in \mathcal{P}_i(X)$.
(2) *The following conditions are equivalent:*
   (a) $R_\Diamond \subseteq ((R_\Box \circ \le) \cap (R_\Diamond \circ \le^{-1})) \circ \le$;
   (b) $\Box_{R_\Box}(U) \cap \Diamond_{R_\Diamond}(V) \subseteq \Diamond_{R_\Diamond}(U \cap V)$, *for all* $U, V \in \mathcal{P}_i(X)$.
(3) *The following conditions are equivalent:*
   (a) $R_\Delta \subseteq ((R_\Delta \circ \le^{-1}) \cap (R_\nabla \circ \le)) \circ \le^{-1}$;
   (b) $\Delta_{R_\Delta}(U \cap V) \subseteq \Delta_{R_\Delta}(U) \cup \nabla_{R_\nabla}(V)$, *for all* $U, V \in \mathcal{P}_i(X)$.
(4) *The following conditions are equivalent:*
   (a) $R_\nabla \subseteq ((R_\Delta \circ \le^{-1}) \cap (R_\nabla \circ \le)) \circ \le$;
   (b) $\Delta_{R_\Delta}(U) \cap \nabla_{R_\nabla}(V) \subseteq \nabla_{R_\nabla}(U \cup V)$, *for all* $U, V \in \mathcal{P}_i(X)$.

Proof. (1) (a) $\Rightarrow$ (b) Let $x \in X$ and $U, V \in \mathcal{P}_i(X)$ such that $x \in \Box_{R_\Box}(U \cup V)$. Assume that $x \notin \Box_{R_\Box}(U)$. Then there exists $y \in X$ such that $(x,y) \in R_\Box$ and $y \notin U$. By assumption, there exists $z \in X$ such that $(x,z) \in R_\Box \circ \le$, $(x,z) \in R_\Diamond \circ \le^{-1}$, and $z \le y$. As $y \notin U$, $z \notin U$, and since $x \in \Box_{R_\Box}(U \cup V) = \Box_{R_\Box \circ \le}(U \cup V)$, $z \in V$. Then, $z \in (R_\Diamond \circ \le^{-1})(x) \cap V$, and thus $x \in \Diamond_{R_\Diamond \circ \le^{-1}}(V) = \Diamond_{R_\Diamond}(V)$.

(b) $\Rightarrow$ (a) Let $x, y \in X$ such that $(x,y) \in R_\Box$. We shall prove that

$$(R_\Box \circ \le)(x) \cap (R_\Diamond \circ \le^{-1})(x) \cap (y] \neq \varnothing.$$

Let us consider the sets

$$U = X \setminus (R_\Diamond \circ \le^{-1})(x) = (R_\Diamond \circ \le^{-1})(x)^c \quad \text{and} \quad V = X \setminus (y] = (y]^c.$$

It is clear that $V \in \mathcal{P}_i(X)$. We prove that $U \in \mathcal{P}_i(X)$. Let $a \le b$ and $a \in U$. If $b \notin U$, there exists $c \in X$ such that $(x,c) \in R_\Diamond$ and $b \le c$. As $a \le b \le c$, $(x,a) \in (R_\Diamond \circ \le^{-1})$, which is a contradiction. Then, $U \in \mathcal{P}_i(X)$.

Moreover, since $y \in R_\Box(x)$, $R_\Box(x) \cap (y] \neq \varnothing$. So, $x \notin \Box_{R_\Box}(V)$. Also, $x \notin \Diamond_{R_\Diamond}(U)$, because $(R_\Diamond \circ \le^{-1})(x) \cap (R_\Diamond \circ \le^{-1})(x)^c = \varnothing$. Thus, $x \notin \Box_{R_\Box}(V) \cup \Diamond_{R_\Diamond}(U)$, and consequently $x \notin \Box_{R_\Box}(U \cup V)$, i. e.,

$$(R_\Box \circ \le)(x) \nsubseteq U \cup V = ((R_\Diamond \circ \le^{-1})(x) \cap (y])^c.$$

(3) (a) $\Rightarrow$ (b) Let $x \in X$ and $U, V \in \mathcal{P}_i(X)$ such that $x \in \Delta_{R_\Delta}(U \cap V)$. Then $R_\Delta(x) \cap U \cap V = \varnothing$. Assume that $x \notin \Delta_{R_\Delta}(U)$. Then there exists $y \in X$ such that $(x, y) \in R_\Delta$ and $y \in U$. By assumption, there exists $z \in X$ such that $(x, z) \in (R_\Delta \circ \leq^{-1}) \cap (R_\nabla \circ \leq)$ and $y \leq z$. Then $z \in U$, and since $x \in \Delta_{R_\Delta}(U \cap V) = x \in \Delta_{R_\Delta \circ \leq^{-1}}(U \cap V)$, $z \notin V$. As $z \in (R_\nabla \circ \leq)(x)$, $x \in \nabla_{R_\nabla \circ \leq}(V) = \nabla_{R_\nabla}(V)$.

(b) $\Rightarrow$ (a) Let $x, y \in X$ such that $(x, y) \in R_\Delta$. We shall prove that

$$(R_\Delta \circ \leq^{-1})(x) \cap (R_\nabla \circ \leq)(x) \cap [y) \neq \varnothing.$$

Let us consider the sets

$$U = (R_\nabla \circ \leq)(x) \quad \text{and} \quad V = [y).$$

It is easy to see that $(R_\nabla \circ \leq)(x) \in \mathcal{P}_i(X)$. Moreover, $x \notin \nabla_{R_\nabla}(U)$, because $R_\nabla(x) \subseteq (R_\nabla \circ \leq)(x)$. Also, $x \notin \Delta_{R_\Delta}(V)$, because $y \in R_\Delta(x) \cap [y)$. Thus, $x \notin \Delta_{R_\Delta}(U) \cup \nabla_{R_\nabla}(V)$, and consequently $x \notin \Delta_{R_\Delta}(U \cap V) = \Delta_{R_\Delta \circ \leq^{-1}}(U \cap V)$. Therefore,

$$(R_\Delta \circ \leq^{-1})(x) \cap U \cap V = (R_\Delta \circ \leq^{-1})(x) \cap (R_\nabla \circ \leq)(x) \cap [y) \neq \varnothing.$$

The proofs of the assertions (2) and (4) are similar. $\square$

Theorem 4. *Let $\langle X, \leq \rangle$ be a poset. The following conditions are equivalent:*

1. *There exists a binary relation R on X such that*
   (a) $\leq \circ R \subseteq R \circ \leq$,
   (b) $\leq^{-1} \circ R \subseteq R \circ \leq^{-1}$.
2. *There exist two binary relations $R_\Box$ and $R_\Diamond$ on X such that*
   (a) $R_\Box \subseteq (R_\Box \circ \leq) \cap (R_\Diamond \circ \leq^{-1}) \circ \leq$,
   (b) $R_\Diamond \subseteq (R_\Box \circ \leq) \cap (R_\Diamond \circ \leq^{-1}) \circ \leq^{-1}$,
   (c) $\leq \circ R_\Box \subseteq R_\Box \circ \leq$,
   (d) $\leq^{-1} \circ R_\Diamond \subseteq R_\Diamond \circ \leq^{-1}$.
3. *The $\Box\Diamond$-algebra $\langle \mathcal{P}_i(X), \cup, \cap, \Box_R, \Diamond_R, \varnothing, X \rangle$ is a PM-algebra.*
4. *The $\Box\Diamond$-algebra $\langle \mathcal{P}_i(X), \cup, \cap, \Box_{R_\Box}, \Diamond_{R_\Diamond}, \varnothing, X \rangle$ is a PM-algebra.*

Proof. (1) $\Rightarrow$ (2) Define the binary relations $R_\Box = R \circ \leq$ and $R_\Diamond = R \circ \leq^{-1}$. Since $\leq$ is reflexive,

$$R \subseteq R_\Box \cap R_\Diamond \subseteq (R_\Box \circ \leq) \cap (R_\Diamond \circ \leq^{-1}).$$

So,

$$R_\Box = (R \circ \leq) \subseteq (R_\Box \circ \leq) \cap (R_\Diamond \circ \leq^{-1}) \circ \leq,$$

and

$$\leq \circ R_\Box = (\leq \circ R \circ \leq) \subseteq R \circ \leq \circ \leq = R_\Box \circ \leq.$$

The proof for the relation $R_\Diamond$ is similar.

(2) $\Rightarrow$ (1) Given the relations $R_\Box$ and $R_\Diamond$, we define the relation

$$R = (R_\Box \circ \leq) \cap (R_\Diamond \circ \leq^{-1}).$$

Let $x \leq y$ and $(y,z) \in R = (R_{\square} \circ \leq) \cap (R_{\Diamond} \circ \leq^{-1})$. Then there exists $k \in X$ such that $(y,k) \in R_{\square}$ and $k \leq z$. As $(x,k) \in \leq \circ R_{\square} \subseteq R_{\square} \circ \leq$, there exists $w \in X$ such that $(x,w) \in R_{\square}$ and $w \leq k$. Since $R_{\square} \subseteq (R_{\square} \circ \leq) \cap (R_{\Diamond} \circ \leq^{-1}) \circ \leq$, there exists $d \in X$ such that $(x,d) \in R$ and $d \leq w$. Thus $(x,z) \in R \circ \leq$.

The proof of the inclusion $\leq^{-1} \circ R \subseteq R \circ \leq^{-1}$ is similar. The equivalence of (1) and (3) follows from Proposition 2. The equivalence of (2) and (4) follows from Theorem 3 and Proposition 2. □

Theorem 5. *Let $\langle X, \leq \rangle$ be a poset. The following conditions are equivalent:*

(1) *There exists a binary relation R on X such that*
   (a) $\leq \circ R \subseteq R \circ \leq^{-1}$,
   (b) $\leq^{-1} \circ R \subseteq R \circ \leq^{-1}$.

(2) *There exist two binary relations R_{Δ} and R_{∇} on X such that*
   (a) $R_{\Delta} \subseteq ((R_{\Delta} \circ \leq^{-1}) \cap (R_{\nabla} \circ \leq)) \circ \leq^{-1}$,
   (b) $R_{\nabla} \subseteq ((R_{\Delta} \circ \leq^{-1}) \cap (R_{\nabla} \circ \leq)) \circ \leq$,
   (c) $\leq \circ R_{\Delta} \subseteq R_{\Delta} \circ \leq^{-1}$,
   (d) $\leq^{-1} \circ R_{\nabla} \subseteq R_{\nabla} \circ \leq$.

(3) *There exists a binary relation R on X such that the $\square\Diamond$-algebra*

$$\langle \mathcal{P}_i(X), \cup, \cap, \Delta_R, \nabla_R, \varnothing, X \rangle$$

   is a NM-algebra.

(4) *There exist two binary relations R_{Δ} and R_{∇} on X such that the $\square\Diamond$-algebra $\langle \mathcal{P}_i(X), \cup, \cap, \Delta_{R_{\Delta}}, \nabla_{R_{\nabla}}, \varnothing, X \rangle$ is a NM-algebra.*

Proof. The proof of the direction (1) ⇒ (2) is similar to the proof of the previous theorem, but taking $R_{\Delta} = R \circ \leq^{-1}$ and $R_{\nabla} = R \circ \leq$. For the direction (2) ⇒ (1) we need define the relation R as $R = (R_{\Delta} \circ \leq^{-1}) \cap (R_{\nabla} \circ \leq)$. □

3. Canonical varieties

Let $\boldsymbol{A}$ be a bounded distributive lattice. By $X(\boldsymbol{A})$ we shall denote the set of all prime filters of $\boldsymbol{A}$. The filter (resp., ideal) generated by a subset $H \subseteq A$ will be denoted by $[H)$ (resp., $(H]$). The lattice of all filters (resp., ideals) of $\boldsymbol{A}$ is denoted by $\mathrm{Fi}(\boldsymbol{A})$ (resp., $\mathrm{Id}(\boldsymbol{A})$).

Let $\boldsymbol{A}$ be a distributive modal algebra. We define binary relations $R_{\square}^{\boldsymbol{A}}$, $R_{\Diamond}^{\boldsymbol{A}}$, $R_{\Delta}^{\boldsymbol{A}}$, and $R_{\nabla}^{\boldsymbol{A}}$ on $X(\boldsymbol{A})$ as follows:

(1) $(P,Q) \in R_{\square}^{\boldsymbol{A}} \Leftrightarrow \square^{-1}(P) \subseteq Q$,
(2) $(P,Q) \in R_{\Diamond}^{\boldsymbol{A}} \Leftrightarrow Q \subseteq \Diamond^{-1}(P)$,
(3) $(P,Q) \in R_{\Delta}^{\boldsymbol{A}} \Leftrightarrow Q \subseteq (\Delta^{-1}(P))^c$,
(4) $(P,Q) \in R_{\nabla}^{\boldsymbol{A}} \Leftrightarrow (\nabla^{-1}(P))^c \subseteq Q$

with $P, Q \in X(\boldsymbol{A})$.

Lemma 6. *Let $\boldsymbol{A}$ be a distributive modal algebra. Then, for each $P \in X(\boldsymbol{A})$ and for each $a \in A$, the following assertions hold:*

(1) $\Box a \in P$ *if and only if for every $Q \in X(\boldsymbol{A})$ such that $(P,Q) \in R^{\boldsymbol{A}}_{\Box}$ it holds $a \in Q$.*
(2) $\Diamond a \in P$ *if and only if there exists $Q \in X(A)$ such that $(P,Q) \in R^{\boldsymbol{A}}_{\Diamond}$ and $a \in Q$.*
(3) $\Delta a \in P$ *if and only if for every $Q \in X(\boldsymbol{A})$ such that $(P,Q) \in R^{\boldsymbol{A}}_{\Delta}$ it holds $a \notin Q$.*
(4) $\nabla a \in P$ *if and only if there exists $Q \in X(\boldsymbol{A})$ such that $(P,Q) \in R^{\boldsymbol{A}}_{\nabla}$ and $a \notin Q$.*

Proof. For the proof for the operators $\Box$ and $\Diamond$, see [1,5]. For the weak negation Δ, see [3]. Let $\nabla a \in P$, and let us consider the filter $(\nabla^{-1}(P))^c$. As $a \notin (\nabla^{-1}(P))^c$, there exists $Q \in X(\boldsymbol{A})$ such that $(\nabla^{-1}(P))^c \subseteq Q$ and $a \notin Q$. The converse implication is immediate. $\Box$

Let us consider the relational structure $\mathcal{F}(\boldsymbol{A}) = \langle X(\boldsymbol{A}), R^{\boldsymbol{A}}_{\Box}, R^{\boldsymbol{A}}_{\Diamond}, R^{\boldsymbol{A}}_{\Delta}, R^{\boldsymbol{A}}_{\nabla}\rangle$. It is easy to see that

$$\subseteq \circ R^{\boldsymbol{A}}_{\Box} \subseteq R^{\boldsymbol{A}}_{\Box} \circ \subseteq, \qquad \subseteq^{-1} \circ R^{\boldsymbol{A}}_{\Diamond} \subseteq R^{\boldsymbol{A}}_{\Diamond} \circ \subseteq^{-1},$$

$$\subseteq \circ R^{\boldsymbol{A}}_{\Delta} \subseteq R^{\boldsymbol{A}}_{\Delta} \circ \subseteq^{-1}, \qquad \subseteq^{-1} \circ R^{\boldsymbol{A}}_{\nabla} \subseteq R^{\boldsymbol{A}}_{\nabla} \circ \subseteq .$$

Thus, by Proposition 2, we get that the algebra

$$\mathrm{Ec}(\boldsymbol{A}) = \langle \mathcal{P}_i(X(\boldsymbol{A})), \cup, \cap, \Box, \Diamond, \Delta, \nabla, \varnothing, X(\boldsymbol{A})\rangle$$

is a distributive modal algebra, and the map $\beta_{\boldsymbol{A}} : A \to \mathcal{P}_i(X(\boldsymbol{A}))$ defined by the equality $\beta_{\boldsymbol{A}}(a) = \{P \in X(\boldsymbol{A}) : a \in P\}$ is an injective homomorphism of distributive modal algebras. The algebra $\mathrm{Ec}(\boldsymbol{A})$ is called the *canonical extension* of $\boldsymbol{A}$.

Definition 1. A variety $\mathcal{V}$ of distributive modal algebras is canonical if for any $\boldsymbol{A} \in \mathcal{V}$, $\mathrm{Ec}(\boldsymbol{A}) \in \mathcal{V}$.

The notion of canonical varieties is the algebraic interpretation of canonical modal logics (see [6]). Let us recall that a modal logic is canonical if their canonical frame is a frame of the logic.

Now, we shall see that if $\boldsymbol{A}$ is a PM-algebra (resp., NM-algebra), then $\mathrm{Ec}(\boldsymbol{A})$ is a PM-algebra (resp., NM-algebra).

Theorem 7. *Let $\boldsymbol{A}$ be distributive modal algebra. Then:*

(1) $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$ *is valid in $\boldsymbol{A}$ if and only if* $R^{\boldsymbol{A}}_{\Diamond} \subseteq (R^{\boldsymbol{A}}_{\Box} \cap R^{\boldsymbol{A}}_{\Diamond}) \circ \subseteq^{-1}$.
(2) $\Box(a \vee b) \leq \Box a \vee \Diamond b$ *is valid in $\boldsymbol{A}$ if and only if* $R^{\boldsymbol{A}}_{\Box} \subseteq (R^{\boldsymbol{A}}_{\Box} \cap R^{\boldsymbol{A}}_{\Diamond}) \circ \subseteq$.
(3) $\Delta(a \wedge b) \leq \Delta a \vee \nabla b$ *is valid in $\boldsymbol{A}$ if and only if* $R^{\boldsymbol{A}}_{\Delta} \subseteq (R^{\boldsymbol{A}}_{\Delta} \cap R^{\boldsymbol{A}}_{\nabla}) \circ \subseteq^{-1}$.
(4) $\Delta a \wedge \nabla b \leq \nabla(a \vee b)$ *is valid in $\boldsymbol{A}$ if and only if* $R^{\boldsymbol{A}}_{\nabla} \subseteq (R^{\boldsymbol{A}}_{\Delta} \cap R^{\boldsymbol{A}}_{\nabla}) \circ \subseteq$.

Proof. We prove only (3) and (4).

For (3), we assume that $\Delta(a\wedge b)\leq \Delta a\vee \nabla b$ is valid in $\boldsymbol{A}$. Let $P,Q\in X(\boldsymbol{A})$ such that $Q\subseteq \Delta^{-1}(P)^c$. Let us consider the filter $F(\nabla^{-1}(P)^c\cup Q)$. We prove that

$$F(\nabla^{-1}(P)^c\cup Q)\cap \Delta^{-1}(P)=\varnothing.$$

Suppose the opposite. Then there are $a\in\nabla^{-1}(P)^c$, $b\in Q$ and $c\in\Delta^{-1}(P)$ such that $a\wedge b\leq c$. So,

$$\Delta c\leq \Delta(a\wedge b),$$

and by the hypothesis we obtain

$$\Delta c\leq \Delta a\vee\nabla b.$$

Since $\Delta c\in P$, $\Delta a\vee\nabla b\in P$, and as P is a prime filter, $\Delta a\in P$ or $\nabla b\in P$, which is a contradiction. Thus, there exists $D\in X(\boldsymbol{A})$ such that

$$\nabla^{-1}(P)^c\subseteq D,\quad Q\subseteq D,\quad \text{and}\quad D\cap\Delta^{-1}(P)=\varnothing,$$

i. e., $(P,Q)\in(R^{\boldsymbol{A}}_{\Delta}\cap R^{\boldsymbol{A}}_{\nabla})\circ\subseteq^{-1}$.

Assume that $R^{\boldsymbol{A}}_{\Delta}\subseteq(R^{\boldsymbol{A}}_{\Delta}\cap R^{\boldsymbol{A}}_{\nabla})\circ\subseteq^{-1}$. Let $a,b\in A$ and suppose that $\Delta(a\wedge b)\nleq \Delta a\vee\nabla b$. Then there exits $P\in X(\boldsymbol{A})$ such that $\Delta(a\wedge b)\in P$, and $\Delta a,\nabla b\notin P$. So there exists $Q\in X(\boldsymbol{A})$ such that $(P,Q)\in R^{\boldsymbol{A}}_{\Delta}$ and $a\in Q$. By assumption, there exist $D\in X(\boldsymbol{A})$ such that $(P,D)\in R^{\boldsymbol{A}}_{\Delta}\cap R^{\boldsymbol{A}}_{\nabla}$ and $Q\subseteq D$. Since $\Delta(a\wedge b)\in P$ and $(P,D)\in R^{\boldsymbol{A}}_{\Delta}$, $a\wedge b\notin D$, but as $(P,D)\in R^{\boldsymbol{A}}_{\nabla}$, and $\nabla b\notin P$, $b\in D$. Also, $c\in D$, because $c\in Q\subseteq D$. Then, $b\wedge c\in D$, which is a contradiction. Thus, $\Delta(a\wedge b)\leq\Delta a\vee\nabla b$ for all $a,b\in A$.

(4) Suppose that $\Delta a\wedge\nabla b\leq\nabla(a\vee b)$ is valid in $\boldsymbol{A}$. Let $P,Q\in X(\boldsymbol{A})$ such that $\nabla^{-1}(P)^c\subseteq Q$. Let us consider the ideal $I=I(\Delta^{-1}(P)\cup Q^c)$. We prove that

$$\nabla^{-1}(P)^c\cap I=\varnothing.$$

Suppose the contrary. Then, there exists $c\in\nabla^{-1}(P)^c$, $a\notin Q$, and $b\in\Delta^{-1}(P)$ such that $c\leq a\vee b$. So, $\nabla(a\vee b)\leq\nabla c$. By hypothesis we have

$$\Delta b\wedge\nabla a\leq\nabla c.$$

Since $\Delta b\in P$ and $a\in\nabla^{-1}(P)$, $\nabla c\in P$, which is a contradiction. Thus, there exists $D\in X(\boldsymbol{A})$ such that $(P,D)\in R^{\boldsymbol{A}}_{\Delta}\cap R^{\boldsymbol{A}}_{\nabla}$ and $D\subseteq Q$, i. e., $R^{\boldsymbol{A}}_{\nabla}\subseteq(R^{\boldsymbol{A}}_{\Delta}\cap R^{\boldsymbol{A}}_{\nabla})\circ\subseteq$. The proof of the other implication is omitted. □

Let $\boldsymbol{A}$ be a PM-algebra. Let us define the relation $R_A=R^A_{\Box}\cap R^A_{\Diamond}$. Then it is easy to see that $R^A_{\Box}=R_A\circ\subseteq$, $R^A_{\Diamond}=R_A\circ\subseteq^{-1}$, and $\subseteq\circ R_A\subseteq R_A\circ\subseteq$ and $\subseteq^{-1}\circ R_A\subseteq R_A\circ\subseteq^{-1}$. Consequently, by Theorem 7 we can deduce that the axioms that relate the operators $\Box$ and $\Diamond$ allow us to define the single relation $R_A=R^A_{\Box}\cap R^A_{\Diamond}$.

Let $\boldsymbol{A}$ be a NM-algebra. Let us define the relation $R_A=R^A_{\Delta}\cap R^A_{\nabla}$. Then it is easy to see that $R^A_{\Delta}=R_A\circ\subseteq^{-1}$, $R^A_{\nabla}=R_A\circ\subseteq$, and $\subseteq\circ R_A\subseteq R_A\circ\subseteq$ and

$\subseteq^{-1} \circ R_A \subseteq R_A \circ \subseteq^{-1}$. So, by Theorem 7 we can deduce that the axioms that relate the operators Δ and ∇ allow us to define the single relation $R_A = R^A_\Delta \cap R^A_\nabla$.

Let $\mathcal{V}$ be a variety and let Φ be a set of identities in the algebraic language of $\mathcal{V}$. We denote by $\mathcal{V} + \Phi$ the variety generated by $\mathcal{V}$ and by Φ. We note that the previous results imply that, whenever $\boldsymbol{A}$ belongs to some of the varieties $\mathcal{DML} + \Phi$, where $\Phi \subseteq \{(P1), (P2), (N1), (N2)\}$, then the relational structure $\langle X(\boldsymbol{A}), R^{\boldsymbol{A}}_\Box, R^{\boldsymbol{A}}_\Diamond, R^{\boldsymbol{A}}_\Delta, R^{\boldsymbol{A}}_\nabla \rangle$ satisfies the appropriate first-order conditions that correspond to the identities of Φ.

Theorem 8. *Let $\mathcal{DMA}$ be the variety of distributive modal algebras. The variety $\mathcal{DML}$ and any extension of it that is obtained by adding any subset of the set of identities $\{(P1), (P2), (N1), (N2)\}$ is canonical.*

Proof. It is clear that if $\boldsymbol{A} \in \mathcal{DMA}$, then $\mathrm{Ec}(\boldsymbol{A}) \in \mathcal{DMA}$. Thus, $\mathcal{DMA}$ is canonical. If $\boldsymbol{A}$ is an algebra in some of the varieties $\mathcal{DMA} + \Phi$, where Φ is any subset of $\{(P1), (P2), (N1), (N2)\}$, then we should verify that the canonical extension $\mathrm{Ec}(\boldsymbol{A})$ belong to $\mathcal{DMA} + \Phi$. But this it follows from Theorem 7 and the results of the previous section. □

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