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Miklós Mincsovics

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ONE-STEP ITERATIVE METHODS AND THEIR QUALITATIVE ANALYSIS

MIKLÓS MINCSOVICS

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ABSTRACT. We analyze some qualitative properties of the one-step iterative methods which serve as a mathematical model for the discretized heat conduction problem. These properties are a discrete analogues of the qualitative properties of continuous problems, and we give algebraic conditions of the step-matrix under which the above basic properties are also preserved at the discrete level. We also construct the corresponding step-matrices.

Mathematics Subject Classification: 65F10

Keywords: qualitative analysis, one-step method, heat conduction problem

1. INTRODUCTION

In this paper, we investigate the qualitative properties of the sequence $\{y^k\}$, generated by the linear algebraic iterative process

$$My^{k+1} = Ny^k + b, \quad k = 0, 1, \dots \tag{1.1}$$

where $M, N \in \mathbb{R}^{n\times n}$, $b, y^k \in \mathbb{R}^n$ and y^0 is a given vector, y^k denotes the successive iterates. Model (1.1) is called one-step iterative method.

The discretisation of many physical and engineering problems leads to one-step methods of the above form (1.1). In this model, as a necessary condition, we have to guarantee the convergence of the iteration. On the other hand, we must also investigate the model from the point of view of the preservation of the qualitative properties of the continuous solution, like conservation of the non-negativity and the concavity of the initial vector y^0 (the discretization of the initial function), monotonicity in time, etc. There are several papers which deal with the second problem (see, e. g., [3–8]). But all those papers investigate this problem as a preservation property of a given matrix splitting of some fixed matrix A. In this paper, we approach this question from the other side, that is, the iterative model (1.1) is given *a priori* and we investigate its qualitative properties. As a physical model, we consider the heat conduction problem in dimension 1.

The paper is organized as follows. In Section 2 we give those basic mathematical preliminaries which are used in the paper. Then we collect the important properties of the continuous solution of the physical problem in Section 3. In the next section we investigate the algebraic properties of the step-matrix in the iteration, which guarantee the preservation of the basic qualitative properties of the continuous solution. Namely, in Section 4 we consider the invariant subsets, in Section 5 the monotonicity property and in Section 6 we define the relation between the different subspaces. Finally, in Section 7 we construct a corresponding step-matrix by using the symmetric Latin squares.

2. Mathematical preliminaries

Throughout the paper all matrices are real, square matrices in the vector space $\mathbb{R}^{n\times n}$, i. e., $A, B \in \mathbb{R}^{n\times n}$. The ordering relation is defined in the usual way, i. e., element-wise. This means that A is non-negative (in notation: $A \geq 0$) when all elements of A are non-negative. The partial ordering between two matrices is defined as $A \geq B$ when $A - B \geq 0$. The strict ordering is the following: A is positive ($A > 0$) when each of its elements is positive. Hence, $A > B$ if $A - B > 0$. For a vector in the vector space $\mathbb{R}^n$, the definitions are similar. This means that a vector $x \in \mathbb{R}^n$ is called non-negative ($x \geq 0$) if all the components of x are non-negative. We say that $x \geq y$ if $x - y \geq 0$. Analogously, x is positive ($x > 0$) if each of its elements is positive. Therefore, $x > y$ when $x - y > 0$.

A matrix $A \in \mathbb{R}^{n\times n}$ is said to be positively diagonally dominant, or shortly PDD (resp., strictly positively diagonally dominant, or shortly SPDD) when the relation $a_{i,i} \geq \sum_{j\neq i} |a_{i,j}|$ (resp., $a_{i,i} \geq \sum_{j\neq i} |a_{i,j}|$) is fulfilled for every $i = 1, 2, \ldots, n$. This means that they are diagonally dominant or strictly diagonally dominant with non-negative diagonal elements.

If $A \in \mathbb{R}^{n\times n}$, then $\varrho(A)$ denotes the spectral radius of A; $\varrho(A) = \max_i |\lambda_i|$, where λ_i, $i = 1, 2, \ldots, n$ are the eigenvalues of A.

The matrix $A \in \mathbb{R}^{n\times n}$, $A = (a_{i,j})$, $i, j = 1, 2, \ldots, n$ is reducible if there exists a non-empty set $R \subsetneq \{1, 2, .., n\}$ such that $a_{i,j} = 0$ for $i \in R$, $j \in \{1, 2, \ldots, n\} \setminus R$, otherwise the matrix A is irreducible. Consequently, every positive matrix is irreducible.

In the sequel, $I \in \mathbb{R}^{n\times n}$ is the unit matrix. $E \in \mathbb{R}^{n\times n}$ denotes the matrix with all elements equal to 1. Similarly, $e \in \mathbb{R}^n$, $e = (1, 1, \ldots, 1)^T$.

Let $A \in \mathbb{R}^{n\times n}$ be a matrix with the property

$$A = M - N. \tag{2.1}$$

The iterative scheme (1.1) is convergent to the unique solution $y = A^{-1}b$ for each y^0 if and only if M is nonsingular, and the corresponding step-matrix of the iteration $H = M^{-1}N$ has the property $\varrho(H) < 1$. Relation (1.1) can be rewritten in the following form: $y^{k+1} = Hy^k + M^{-1}b$ or

$$x^{k+1} = Hx^k, \tag{2.2}$$

where $x^k = y^k - A^{-1}b$ is the so-called defect vector.

3. Some basic qualitative properties of the continuous solution of the heat conduction problem

In the following we list the most important qualitative properties of the following one-dimensional heat conduction problem:

$$\begin{gathered} \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in (0,1), \ \ t > 0, \\ u(x,0) = u_0(x), \\ u(0,t) = u(1,t) = 0, \quad t > 0, \end{gathered} \tag{3.1}$$

with a given (sufficiently smooth) initial function $u_0(x)$. We denote the stationary solution of problem (3.1) by $u_{st}(x)$, i. e., $u_{st}(x) = \lim_{t\to\infty} u(x,t)$. We denote by $h(x,t)$ the difference of the solution of the problem (3.1) and the stationary solution $u_{st}(x)$, i. e., $h(x,t) = u(x,t) - u_{st}(x)$. Then this function $h(x,t)$ is expected to possess the following qualitative properties.

(1) $h(x,t)$ exists, and $\lim_{t\to\infty} h(x,t) = 0$.
(2) If $E(0) = \int_0^1 h(x,0)dx \geq 0$, then $E(t) = \int_0^1 h(x,t)dx \geq 0$ for every t.
(3) If $h(x,t^*) \geq 0$, then $h(x,t) \geq 0$ for every $t \geq t^*$.
(4) If $h(x,t') \geq h(x,t)$ for every $t \geq t'$, then for every $t'' \geq t'$ the inequality $h(x,t'') \geq h(x,t^*)$ holds for every $t^* \geq t''$.
(5) If $h(x,t') \geq h(x,t)$ for every $t \geq t'$, then $h(x,t') \geq 0$. If $h(x,t'') \geq 0$, then $E(t'') = \int_0^1 h(x,t'')dx \geq 0$.
(6) If $E(0) = \int_0^1 h(x,0)dx \geq 0$, then $E(t)$ is monotonically decreasing.
(7) $M(t) = \max_x |h(x,t)|$ is monotonically decreasing.
(8) $\int_0^\infty |h(x,t)|dt < \infty$ for every given x.
(9) If $E(0) = \int_0^1 h(x,0)dx > 0$, then there exist $t', t'', t' \leq t''$, such that $h(x,t') > 0$ and $h(x,t'') \geq h(x,t)$ for every $t \geq t''$.

In the sequel, under iteration we understand formula (2.2).

4. Invariant subsets

We start the investigation of the iteration with a definition.

Definition 1. A subset $S \subset \mathbb{R}^n$ is said to be invariant with respect to the iteration if the relation $x^k \in S$ implies that $x^{k+1} \in S$ for all $k = 0, 1, \ldots$.

Let $Y \in \mathbb{R}^{n\times n}$ be a matrix, then we define the subset $S(Y) \subset \mathbb{R}^n$ as follows:

$$S(Y) := \{x \in \mathbb{R}^n : Yx \geq 0\}.$$

We note that in the analysis of the qualitative properties of the iteration (2.2) the subsets $S(E)$, $S(I)$, $S(I - H)$, and $S(A)$ are of a special importance (see [3, 4]).

Lemma 1. *If the step-matrix H is non-negative, and H and Y commute, then the subset $S(Y)$ is invariant with respect to the iteration.*

Proof. Assume that $x^k \in S$. Then by use of the definition of the subset $S(Y)$ and the non-negativity of H, the relation $HYx^k \geq 0$ holds. Hence using the commutativity assumption and the definition of the iteration (2.2), we get $Yx^{k+1} \geq 0$, which proves the statement. □

Corollary 1. *Assume that the step-matrix H is non-negative.*

(a) *Then the subset $S(p(H))$, where $p(H)$ is a polynomial of H, is invariant with respect to the iteration. Consequently $S(I)$ and $S(I-H)$ are invariant with respect to the iteration.*
(b) *If H and E/A commute, then the subset $S(E)/S(A)$ is invariant with respect to the iteration.*

In the following we investigate the relation between the subsets $S(I), S(I-H), S(E)$ and $S(A)$.

Lemma 2. *The following relations hold between the subsets $S(I), S(I-H), S(E)$, and $S(A)$:*

(a) *If $T \geq 0$, then $S(T) \supset S(I)$. Consequently, $S(E) \supset S(I)$.*
(b) *If $T^{-1} \geq 0$, then $S(I) \supset S(T)$. Consequently, if $A^{-1} \geq 0$, then $S(I) \supset S(A)$.*
(c) *If $M^{-1} \geq 0$, then $S(I-H) \supset S(A)$.*
(d) *If $(I-H)^{-1} \geq 0$, then $S(I) \supset S(I-H)$.*

Proof. (a) $T \geq 0$, therefore $x \geq 0$ clearly results in the relation $Tx \geq 0$.
The proof of (b) and (d) is straightforward.
The statement (c) follows directly from the fact that (2.1) can be rewritten in the following form: $A = M(I-H)$. □

Remark 1. Assume that the step-matrix H is a non-negative convergent matrix, i. e., $H \geq 0$ and $\varrho(H) < 1$. Then $(I-H)^{-1} = \sum_{k=1}^{\infty} H^k$, and hence $(I-H)^{-1} \geq 0$. Therefore $S(I)$ and $S(I-H)$ are invariant with respect to the iteration, see part (a) in Corollary 1, and $S(I) \supset S(I-H)$, according to assertion (d) of Lemma 2.

Let us investigate the set of the matrices which commute with some given matrix E.

We define the subspace $C_E \subset \mathbb{R}^{n\times n}$ as follows:

$$C_E := \{X \in \mathbb{R}^{n\times n} : EX = XE\}.$$

Remark 2. The subset C_E forms both a vector space and a ring.

The properties of the operations are obviously true. Moreover, the subset C_E is closed for the operations, because for any $A, B \in C_E$ the following relations are valid: $(A \pm B)E = AE \pm BE = EA \pm EB = E(A \pm B)$, $(\lambda A)E = \lambda EA = E(\lambda A)$, $ABE = AEB = EAB$. Clearly, the matrices 0 and $-A$ also belong to C_E.

Lemma 3. *The following statements are equivalent:*

(a) $X \in C_E$;

(b) *For all fixed* $1 \le i, j \le n$*, the relation*

$$\sum_{k=1}^{n} x_{k,i} = \sum_{k=1}^{n} x_{j,k}$$

holds.

Proof. The statement follows by direct calculation of the elements of the matrices XE and EX, respectively. □

5. Monotonicity

First we define monotonicity in the subset S.

Definition 2. Let $Z \in \mathbb{R}^{n \times n}$ be a given matrix. The iteration is said to be Z-monotone in a subset $S \subset \mathbb{R}^n$ if the following two conditions are fulfilled:*

(I) The subset $S \subset \mathbb{R}^n$ is invariant with respect to the iteration;

(II) For any $x^k \in S$, the relation $Zx^k \ge Zx^{k+1}$ holds.

Lemma 4. *For* $H \ge 0$ *the iteration is* I*-monotone in a subset* $S(I - H)$*.*

Proof. The invariance of $S(I - H)$ follows from Corollary 1 (a).

In order to show the second property, assume that $x^k \in S(I-H)$, i. e., $(I-H)x^k \ge 0$. Then $x^k \ge Hx^k$, which, according to (2.2) means the required relation $x^k \ge x^{k+1}$. □

Remark 3. The above proof also shows that for the iteration the relation $x^k \ge x^{k+1}$ implies the inclusion $x^k \in S(I - H)$. This means that the I-monotonicity property of the iteration is valid only in the subset $S(I - H)$.

Corollary 2. *Let us suppose that* $A^{-1} \ge 0$*,* $H \ge 0$*,* $M^{-1} \ge 0$ *and* H *and* A *commute. Then the iteration is* I*-monotone in the subset* $S(A)$*.*

The first condition follows from the commutativity of H and A, see Lemma 1, about the second condition statement (c) in Lemma 2 and Lemma 4.

Lemma 5. *Let us suppose that the step-matrix* H *is non-negative,* H *and* E *commute and* $I - H$ *is PDD. Then the iteration is* E*-monotone in the subset* $S(E)$*.*

Proof. The invariance of the subset $S(E)$ follows from the statement (b) in Corollary 1.

For all $x \in S(E)$ the inequality $(I - H)Ex \ge 0$ is fulfilled, because $Ex \ge 0$, and for any $1 \le i, j \le n$ the relation $(Ex)_i = (Ex)_j = \sum_{l=1}^{n} x_l$ holds. Moreover, $I - H$ is PDD. Hence, for any $x^k \in S(E)$ we have $(I - H)Ex^k \ge 0$. So, by use of the commutativity property, we have $E(I - H)x^k \ge 0$, which implies the required relation $Ex^k \ge Ex^{k+1}$. □

*For more details, we refer to [3, 4].

Lemma 6. *If the diagonal elements of H are non-negative, then the following two statements are equivalent:*

(a) *$I - H$ is PDD (resp., SPDD);*
(b) *$\|H\|_\infty \le 1$ (resp., $\|H\|_\infty < 1$).*

Proof. Using the definition of PDD (resp., SPDD) and the non-negativity of the diagonal elements of H, we see that the properties $1 - a_{i,i} \ge \sum_{j\ne i} |a_{i,j}|$ (resp., $1 - a_{i,i} > \sum_{j\ne i} |a_{i,j}|$) and $1 \ge \sum_{j=1}^{n} |a_{i,j}|$ (resp., $1 > \sum_{j=1}^{n} |a_{i,j}|$) are equivalent, which proves the statement. □

Lemma 7. *Let $x^0 \in \mathbb{R}^n$, $x^0 \ne 0$. Then the following statements are equivalent:*

(a) *$\|H\|_\infty \le 1$ (resp., $\|H\|_\infty < 1$);*
(b) *$\|x^k\|_\infty \ge \|x^{k+1}\|_\infty$ (resp., $\|x^k\|_\infty > \|x^{k+1}\|_\infty$) for every $x^0 \in \mathbb{R}^n$, $x^0 \ne 0$ and all $k = 0, 1, \ldots$.*

Proof. The relation $\|x^{k+1}\|_\infty = \|Hx^k\|_\infty \le \|H\|_\infty \|x^k\|_\infty \le$ (resp., $<$) $\|x^k\|_\infty$ shows that (a) implies (b).

In order to obtain the converse implication, assume that (b) is true. Then, for any $x^0 \in \mathbb{R}^n$ with the property $x^0 \ne 0$, the inequality $\|x^0\|_\infty \ge \|x^1\|_\infty$ is true. Since $x^1 = Hx^0$, one has

$$\frac{\|Hx^0\|_\infty}{\|x^0\|_\infty} \le 1.$$

Consequently,

$$\max_{x\in\mathbb{R}^n,\, x\ne 0} \frac{\|Hx^0\|_\infty}{\|x^0\|_\infty} \le 1,$$

which yields (a). The proof of the strict inequality is similar. □

Lemma 8. *Assume that H is a non-negative convergent matrix, i. e., $H \ge 0$ and $\varrho(H) < 1$. Then $\sum_{k=0}^{\infty} |(x^k)_i| < \infty$ for every $1 \le i \le n$.*

Proof. Under the assumptions made we have

$$\sum_{k=0}^{\infty} \left|(x^k)_i\right| = \sum_{k=0}^{\infty} \left|\left(H^k x^0\right)_i\right| \le \sum_{k=0}^{\infty} \left(H^k \left\|x^0\right\|_\infty e\right)_i = \left\|x^0\right\|_\infty \left(\sum_{k=0}^{\infty} H^k e\right)_i$$
$$= \|x^0\|_\infty \left(\left(\sum_{k=0}^{\infty} H^k\right) e\right)_i = \|x^0\|_\infty \left((I - H)^{-1} e\right)_i = \|x^0\|_\infty \|(I - H)^{-1}\|_\infty.$$

The right-hand side here is finite, and this proves the statement. □

6. Relation between subspaces

In this Section we analyze the relations between the subspaces $S(E)$, $S(I)$, and $S(I - H)$ with respect to the iteration. The analysis is based on the so-called power method.

Lemma 9. *If the step-matrix H is non-negative, irreducible and $H \in C_E$, then*

(1) *the vector e is an eigenvector of H;*
(2) *$\varrho(H)$ is an eigenvalue of H;*
(3) *$\varrho(H)$ corresponds to the eigenvector e;*
(4) *$\varrho(H)$ is in absolute value a simple eigenvalue.*

Proof. Let us note that $He = \|H\|_\infty e$, hence $\|H\|_\infty$ is an eigenvalue of H. Since $\|H\|_\infty \geq \varrho(H)$, it follows that $\|H\|_\infty$ is, in absolute value, the largest eigenvalue with the eigenvector e. For non-negative and irreducible matrices the Perron–Frobenius theorem (see [9]) guarantees that this eigenvalue is in absolute value simple. □

Theorem 1. *Let us suppose that H is a non-negative, irreducible matrix, $H \in C_E$ and H has an orthonormal system of eigenvectors. Assume that $x^0, y^0 \in \mathbb{R}^n$ are two arbitrary vectors with the property $(x^0, e) = (y^0, e) = z \neq 0$. Then*

$$\lim_{k\to\infty} \frac{\max_{1\leq i\leq n}\{(x^k)_i\}}{\min_{1\leq i\leq n}\{(y^k)_i\}} = 1.$$

Proof. We denote the eigenvalues of H by λ_m, $m = 1, 2, \ldots, n$. Using the proof of Lemma 9 and the notation $\|H\|_\infty = s$, we can write $s = \lambda_1 > |\lambda_2| \geq \cdots \geq |\lambda_n|$. Furthermore, we denote the corresponding orthonormal eigenvectors by v^m, $m = 1, 2 \ldots, n$, respectively. Then, using again the Lemma 9 and the relation $\left\|\frac{1}{\sqrt{n}}e\right\|_2 = 1$, we get $v^1 = \frac{1}{\sqrt{n}}e$. Considering the decomposition with respect to the system (v^m), we get $x^0 = \sum_{m=1}^n a_m v^m$, where $a_m = (x^0, v^m)$, and $y^0 = \sum_{m=1}^n b_m v^m$, where $b_m = (y^0, v^m)$. Note that $a_1 = b_1 = \frac{z}{\sqrt{n}}$. Hence,

$$\frac{(x^k)_i}{(y^k)_j} = \frac{(H^k x^0)_i}{(H^k y^0)_j} = \frac{\left(H^k \sum_{m=1}^n a_m v^m\right)_i}{(H^k \sum_{m=1}^n b_m v^m)_j} = \frac{\left(\sum_{m=1}^n \lambda_m^k a_m v^m\right)_i}{\left(\sum_{m=1}^n \lambda_m^k b_m v^m\right)_j}$$

$$= \frac{\left(s^k \frac{z}{\sqrt{n}} e + \sum_{m=2}^n \lambda_m^k a_m v^m\right)_i}{\left(s^k \frac{z}{\sqrt{n}} e + \sum_{m=2}^n \lambda_m^k b_m v^m\right)_j} = \frac{\left(e + \frac{\sqrt{n}}{z} \sum_{m=2}^n \left(\frac{\lambda_m}{s}\right)^k a_m v^m\right)_i}{\left(e + \frac{\sqrt{n}}{z} \sum_{m=2}^n \left(\frac{\lambda_m}{s}\right)^k b_m v^m\right)_j}.$$

Since $|\lambda_m| < s$, for $m = 2, \ldots, n$, therefore $\lim_{k\to\infty} \left(\frac{\lambda_m}{s}\right)^k = 0$, for $m = 2, \ldots, n$. This completes the proof. □

We denote the maximum element of H by $\max(H)$, and the minimum element of H by $\min(H)$, respectively.

Corollary 3. Let us apply the Theorem 1 to the columns of the step-matrix H. Since H is non-negative, irreducible, $H \in C_E$ and H has an orthonormal system of eigenvectors, therefore

$$\lim_{k\to\infty} \frac{\max(H^k)}{\min(H^k)} = 1.$$

Theorem 2. *Suppose that the step-matrix H is non-negative, irreducible, $H \in C_E$ and H has an orthonormal system of eigenvectors. Let $x^0 \in \mathbb{R}^n, x^0 \in S(E)$ and $Ex^0 \neq 0$. Then there exists an index $k_0 \in \mathbb{N}_0^+$ such that for every $k \in \mathbb{N}_0^+$, $k \geq k_0$, the inequality $x^k \geq 0$ holds (in other words $x^k \in S(I)$).*

Proof. Corollary 3 means that for all $\varepsilon \in \mathbb{R}^+$, there exists an index $k_0 \in \mathbb{N}_0^+$ such that for each $k \in \mathbb{N}_0^+, k \geq k_0$. the inequality

$$1 \leq \frac{\max(H^k)}{\min(H^k)} \leq 1 + \varepsilon$$

holds. We introduce the notation p, p^+, and p^- as follows. We put $Ex^0 = pe$, where $p \in \mathbb{R}^+$; in other words, p denotes the sum of the elements of x^0. We denote the sum of the non-negative elements of x^0 by p^+, and the sum of the negative elements of x^0 by p^-. Hence, $p^+ + p^- = p$. We consider the following estimate:

$$\begin{aligned}(x^k)_i = (H^k x^0)_i \geq \min(H^k)\, p^+ + \max(H^k)\, p^- \geq \min(H^k)\, p^+ + (1+\varepsilon)\min(H^k)\, p^- \\ = \min(H^k)\,(p^+ + (1+\varepsilon)p^-) = \min(H^k)\,(p + \varepsilon p^-).\end{aligned}$$

The number $\min(H^k)$ is non-negative. The choice of an index k_0 large enough (which depends on $\varepsilon = -p/p^-$) guarantees the non-negativity of x^k for each $k \geq k_0$. □

Theorem 3. *Suppose that the step-matrix H is non-negative, irreducible, $H \in C_E$, $I - H$ is PDD, and H has an orthonormal system of eigenvectors. Let $x^0 \in \mathbb{R}^n, x^0 \in S(I)$. Then there exists a $k_0 \in \mathbb{N}_0^+$ such that the inequality $(I - H)x^k \geq 0$ holds for every $k \in \mathbb{N}_0^+$, $k \geq k_0$ (in other words, $x^k \in S(I - H)$ for such indices).*

Proof. In the same way as proof of the Theorem 2, and finally using the proof of Lemma 5. □

We summarize the results in what follows.

Corollary 4. *Suppose that the step-matrix H is non-negative, irreducible, $H \in C_E$, $\|H\|_\infty < 1$, and H has an orthonormal system of eigenvectors. Let $x^0 \in S(E)$ and $Ex^0 \neq 0$. Then*

(a) *the iteration is convergent and* $\lim_{k\to\infty} x^k = 0$;
(b) $S(E)$, $S(I)$, *and* $S(I - H)$ *are invariant with respect to the iteration;*
(c) $S(E) \supset S(I) \supset S(I - H)$;
(d) *the iteration is E-monotone (resp., I-monotone) in $S(E)$ (resp., $S(I - H)$);*
(e) *the iteration is strictly monotonically decreasing in maximum norm;*
(f) $\sum_{k=0}^{\infty} |(x^k)_i| < \infty$, *for every* $1 \leq i \leq n$;
(g) $\exists k_0, l_0 \in \mathbb{N}_0^+$, $k \leq l$, $\forall k \geq k_0$, $x^k \in S(I)$, $\forall l \geq l_0$, $x^l \in S(I - H)$.

The above-listed properties (a)–(g) are the discrete analogues of (1)–(9) from Section 3.

7. Constructing corresponding *H*

A Latin square is an $\mathbb{R}^{n\times n}$ matrix which consists of n sets of n numbers arranged in such a way that no orthogonal (row or column) contains the same number twice.

Lemma 10. *There exists a symmetric Latin square.*

Proof. Assume that $a_1, \ldots, a_n$ are arbitrary numbers. The choice

$$h_{i,j} := a_k \quad \text{if } \; i + j = k + 1 \pmod n$$

for the elements of the matrix H creates a symmetric Latin square. □

Corollary 5. *Let a_i, $i = 1, 2, \ldots, n$, be arbitrary positive numbers with the property $\sum_{i=1}^{n} a_i < 1$. With the help of a_i we construct the step-matrix H, which is a symmetric Latin square. Then the properties (a)–(g) are satisfied because such a matrix H is non-negative, irreducible, $H \in C_E$, $\|H\|_\infty < 1$, and H has an orthonormal system of eigenvectors.*

8. Concluding remarks

In this paper we listed the basic qualitative properties of the continuous solution of the heat conduction problem (3.1), arising from the physical process. After that we investigated the algebraic properties of the step-matrix in the corresponding one-step iteration, and we have determined conditions under which the above qualitative properties of the continuous solution are preserved. Finally we constructed a suitable corresponding step-matrix.

However, there are different open problems and possible extensions which can be considered in our future works:

(1) Our aim is to analyze the independence of the important qualitative properties of the continuous solution and the important properties of the discrete solution as well.
(2) We constructed a corresponding step-matrix. But naturally it would be useful to define the necessary (and sufficient) conditions for the step-matrix in order to fulfil the qualitative properties (a)–(g).
(3) We analyzed the heat conduction problem in 1D. We can extend this analysis both for higher dimensions and for other time-dependent (parabolic type) physical problems as well.
(4) Our goal is to investigate this method from the point of view of the matrix splitting theory, namely how we can get a corresponding step-matrix from some given matrix A with some splitting procedure (regular matrix splitting, weak regular matrix splitting). These splitting procedures can be found in several papers, e. g., [1, 2, 10–12]. However, their qualitative properties are less investigated.

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Author's address

Miklós Mincsovics:

Eötvös Loránd University, Department of Applied Analysis, 1/c P. Pázmány St., H-1117, Budapest, Hungary

E-mail address: mikka76@freemail.hu