Miskolc Mathematical Notes
Vol. 7 (2006), No 2, pp. 189-203

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2006.144

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Miskolc Mathematical Notes
Vol. 7 (2006), No. 2, pp. 189–203

HU ISSN 1787-2413
electronic version

# COMPARISONS AND COMPOSITIONS OF GALOIS-TYPE CONNECTIONS

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[Received: December 7, 2005]

ABSTRACT. In a former paper, motivated by a recent theory of relators (families of relations), the first author has investigated increasingly regular and normal functions of one preordered set into another instead of Galois connections and residuated mappings of partially ordered sets.

A function $f$ of one preordered set $X$ into another $Y$ has been called

(1) increasingly $g$-normal, for some function $g$ of $Y$ into $X$ if for any $x \in X$ and $y \in Y$ we have $f(x) \leq y$ if and only if $x \leq g(y)$;

(2) increasingly $\varphi$-regular, for some function $\varphi$ of $X$ into itself if for any $x_1, x_2 \in X$ we have $x_1 \leq \varphi(x_2)$ if and only if $f(x_1) \leq f(x_2)$.

In the present paper, for instance, we shall show that if $\varphi$ is an increasingly $\psi$-regular function of $X$ into itself, then $\varphi \leq \psi$ if and only if $\varphi \circ \psi \leq \psi$, and if $f_i$ is an increasingly $g_i$-normal function of $X$ into $Y$ for each $i = 1, 2$, then $f_1 \leq f_2$ if and only if $g_2 \leq g_1$.

Moreover, for instance, we shall show that if $f$ is an increasingly $\varphi_i$-regular function of $X$ into $Y$ for each $i = 1, 2$, then $f$ is increasingly $\varphi_1 \circ \varphi_2$-regular, and if $f$ is an increasingly $g$-normal function of $X$ into $Y$ and $h$ is an increasingly $k$-normal function of $Y$ into $Z$, then $h \circ f$ is $g \circ k$-normal.

*Mathematics Subject Classification:* Primary 06A06, 06A15; Secondary 04A05, 54E15

*Keywords:* preordered sets, Galois connections, residuated mappings, closure and interior operations

## INTRODUCTION

In a former paper [13], motivated by a recent theory of relators (see [9] and [7]), the first author has investigated increasingly regular and normal functions of one preordered set into another instead of Galois connections [3, p. 55] and residuated mappings [2, p. 11] of partially ordered sets.

A function $f$ of one preordered set $X$ into another $Y$ has been called

(1) increasingly $g$-normal, for some function $g$ of $Y$ into $X$ if for any $x \in X$ and $y \in Y$ we have $f(x) \leq y$ if and only if $x \leq g(y)$;

(2) increasingly $\varphi$-regular, for some function $\varphi$ of $X$ into itself if for any $x_1, x_2 \in X$ we have $x_1 \leq \varphi(x_2)$ if and only if $f(x_1) \leq f(x_2)$.

In the present paper, for instance, we shall show that on the one hand if $\varphi$ is an increasingly $\psi$-regular function of $X$ into itself, then $\varphi \leq \psi \iff \varphi \circ \psi \leq \psi \iff \psi \circ \varphi \leq \psi$, and the other hand if $f_i$ is an increasingly $g_i$-normal function of $X$ into $Y$ for each $i = 1, 2$, then $f_1 \leq f_2 \iff f_1 \circ g_2 \leq \Delta_Y \iff g_2 \leq g_1$.

Moreover, for instance, we shall show that if $f$ is an increasingly $\varphi_i$-regular function of $X$ into $Y$ for each $i = 1, 2$, then $f$ is increasingly $\varphi_1 \circ \varphi_2$-regular. While, if $f$ is an increasingly $g$-normal function of $X$ into $Y$ and $h$ is an increasingly $k$-normal function of $Y$ into $Z$, then $h \circ f$ is $g \circ k$-normal.

The results obtained naturally extend and supplement some former results not only on Galois-connections and residuated mappings, but also on closure and interior operations. Namely, for instance, we shall show that an increasingly $\psi$-normal function $\varphi$ of $X$ into itself is a closure operation if and only if it is an interior operation.

## 1. A FEW BASIC FACTS ON RELATIONS

A subset $F$ of a product set $X \times Y$ is called a relation on $X$ to $Y$. If in particular $F \subset X^2$, then we may simply say that $F$ is a relation on $X$. Thus, $\Delta_X = \{(x, x) : x \in X\}$ is a relation on $X$.

If $F$ is a relation on $X$ to $Y$, then for any $x \in X$ the set $F(x) = \{y \in Y : (x, y) \in F\}$ is called the image of $x$ under $F$. And the set $D_F = \{x \in X : F(x) \neq \emptyset\}$ is called the domain of $F$.

In particular, a relation $F$ on $X$ to $Y$ is called a function if for each $x \in D_F$ there exists $y \in Y$ such that $F(x) = \{y\}$. In this case, by identifying singletons with their elements, we may usually write $F(x) = y$ in place of $F(x) = \{y\}$.

More generally, if $F$ is a relation on $X$ to $Y$, then for any $A \subset X$ the set $F[A] = \bigcup_{x \in A} F(x)$ is called the image of $A$ under $F$. And the set $R_F = F[D_F]$ is called the range of $F$.

If $F$ is a relation on $X$ to $Y$ such that $D_F = X$, then we say that $F$ is a relation of $X$ to $Y$. While, if $F$ is a relation on $X$ to $Y$ such that $R_F = Y$, then we say that $F$ is a relation on $X$ onto $Y$.

If $F$ is a relation on $X$ to $Y$, then the values $F(x)$, where $x \in X$ uniquely determine $F$ since we have $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the inverse $F^{-1}$ of $F$ can be defined such that $F^{-1}(y) = \{x \in X : \ y \in F(x)\}$ for all $y \in Y$.

Moreover, if $F$ is a relation on $X$ to $Y$ and $G$ is a relation on $Y$ to $Z$, then the composition $G \circ F$ of $G$ and $F$ can be defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G[F[A]]$ for all $A \subset X$.

A relation $R$ on $X$ is called reflexive, antisymmetric, and transitive if $\Delta_X \subset R$, $R \cap R^{-1} \subset \Delta_X$, and $R \circ R \subset R$, respectively. Moreover, a reflexive and transitive relation is called a preorder. And an antisymmetric preorder is called a partial order.

## 2. A FEW BASIC FACTS ON ORDERED SETS

If $\leq$ is a relation on a nonvoid set $X$, then having in mind the terminology of Birkhoff [1, p. 2] the ordered pair $X(\leq) = (X, \leq)$ is called a goset (generalized ordered set). And we usually write $X$ in place of $X(\leq)$.

If $X(\leq)$ is a goset, then by taking $X^* = X$ and $\leq^* = \leq^{-1}$ we can form another goset $X^*(\leq^*)$. This is called the dual of $X(\leq)$. And we usually write $\geq$ in place of $\leq^*$.

The goset $X$ is called reflexive, transitive, and antisymmetric if the inequality relation in it has the corresponding property. Moreover, for instance, $X$ is called preordered if it is reflexive and transitive.

In particular, a preordered set will be called a proset, and a partially ordered set will be called a poset. The usual definitions on posets can be naturally extended to gosets [10, 11].

For instance, for any subset $A$ of a goset $X$, the members of the families

$$\mathrm{lb}\,(A) = \big\{x \in X : \forall\, a \in A : \ x \leq a\big\}$$

and

$$\mathrm{ub}\,(A) = \big\{x \in X : \forall\, a \in A : \ a \leq x\big\}$$

are called the lower and upper bounds of $A$ in $X$, respectively.

Moreover, the members of the families

$$\min\,(A) = A \cap \mathrm{lb}\,(A) \qquad \max\,(A) = A \cap \mathrm{ub}\,(A)$$
$$\inf\,(A) = \max\,(\mathrm{lb}\,(A)) \qquad \sup\,(A) = \min\,(\mathrm{ub}\,(A))$$

are called the minima, maxima, infima and suprema of $A$ in $X$, respectively.

Thus, for any $A, B \subset X$, we have $A \subset \mathrm{lb}\,(B)$ if and only if $B \subset \mathrm{ub}\,(A)$. Moreover, a reflexive goset $X$ is antisymmetric if and only if $\mathrm{card}\,(\max\,(A)) \leq 1$ (resp., $\mathrm{card}\,(\sup\,(A)) \leq 1$) for all $A \subset X$; see [12].

## 3. CLOSURE OPERATIONS AND REGULAR STRUCTURES

**Definition 3.1.** A function $\varphi$ of a proset $X$ into itself is called an operation on $X$. More generally, a function $f$ of $X$ into another proset $Y$ is called a structure on $X$.

*Remark* 3.2. The latter terminology has been mainly motivated by the various structures derived from relators (see [8, 10]).

**Definition 3.3.** An operation $\varphi$ on $X$ is called

(1) expansive if $\Delta_X \leq \varphi$;
(2) quasi-idempotent if $\varphi^2 \leq \varphi$.

Moreover, a structure $f$ on $X$ is called increasing if for any $x_1, x_2 \in X$, with $x_1 \leq x_2$, we have $f(x_1) \leq f(x_2)$.

*Remark* 3.4. Note that if (1) holds, then $\varphi(x) = \Delta_X(\varphi(x)) \leq \varphi(\varphi(x)) = \varphi^2(x)$ for all $x \in X$, and thus $\varphi \leq \varphi^2$. Therefore, if both (1) and (2) hold and $X$ is a poset, then $\varphi$ is actually idempotent in the sense that $\varphi^2 = \varphi$.

Now, as a natural extension of the corresponding definition of [1, p. 111], we may also have the following.

**Definition 3.5.** An increasing and expansive operation is called a preclosure operation. And a quasi-idempotent preclosure operation is called a closure operation.

Moreover, an expansive and quasi-idempotent operation is called a semiclosure operation. And an increasing and idempotent operation is called a modification operation.

*Remark* 3.6. Now, an operation $\varphi$ on $X$ may be naturally called an interior operation if it is a closure operation on $X^*$.

In [13], having in mind the ideas of [7], the first author has also introduced the following

**Definition 3.7.** A structure $f$ on $X$ is called increasingly $\varphi$-regular, for some operation $\varphi$ on $X$, if for any $x_1, x_2 \in X$ we have

$$x_1 \leq \varphi(x_2) \iff f(x_1) \leq f(x_2).$$

*Remark* 3.8. Now, a structure $f$ on $X$ to $Y$ may be naturally called decreasingly $\varphi$-regular if it is an increasingly $\varphi$-regular structure on $X$ to $Y^*$.

The above definition closely resembles a recent definition of the Galois connections [3, p. 155]. However, instead of the Galois connections, it has been more convenient to use residuated mappings [2, p. 11] in the following relevant form.

**Definition 3.9.** A structure $f$ on $X$ to $Y$ is called increasingly $g$-normal, for some structure $g$ on $Y$ to $X$, if for any $x \in X$ and $y \in Y$ we have

$$f(x) \leq y \iff x \leq g(y).$$

*Remark* 3.10. Now, a structure $f$ on $X$ to $Y$ may be naturally decreasingly $g$-normal if it is an increasingly $g$-normal structure on $X$ to $Y^*$.

For an easy illustration of the latter definition, we can note here

*Example* 3.11. Consider the family $\mathcal{P} = \mathcal{P}(X)$ of all subsets of a generalized ordered set $X$ to be partially ordered by inclusion. Moreover, define $F(A) = \mathrm{ub}\,(A)$ and $G(A) = \mathrm{lb}\,(A)$ for all $A \subset X$.

Then, by the corresponding definitions, it is clear that $F$ is a decreasingly $G$-normal structure on $\mathcal{P}$. Hence, by defining $\Phi = G \circ F$ and using the duals of the forthcoming Theorems 4.5 and 4.3, we can easily see that $F$ is a decreasingly $\Phi$-regular structure and $\Phi$ is a closure operation on $\mathcal{P}$.

To appreciate the importance of this example, note that if in particular $X$ is a poset, then by [3, p. 166] the poset $\Phi[\mathcal{P}]$ is just the Dedekind–MacNeille completion of $X$ by the cuts $\Phi(A)$.

## 4. Relationships between closure operations and regular structures

By using the above definitions, in [13], the first author has proved the following theorems.

**Theorem 4.1.** *If $f$ is an increasingly $\varphi$-regular structure on $X$, then*

(1) *$\varphi$ is expansive;*
(2) *$f$ is increasing;*
(3) *$f \leq f \circ \varphi \leq f$.*

**Corollary 4.2.** *If $f$ is an increasingly $\varphi$-regular structure on $X$ to a poset $Y$, then $f = f \circ \varphi$.*

**Theorem 4.3.** *If $\varphi$ is an operation on $X$, then the following assertions are equivalent:*

(1) *$\varphi$ is a closure operation;*
(2) *$\varphi$ is increasingly $\varphi$-regular;*
(3) *there exists an increasingly $\varphi$-regular structure $f$ on $X$.*

**Corollary 4.4.** *If $f$ is a structure and $\varphi$ is an operation on $X$, then $f$ is increasingly $\varphi$-regular if and only if $\varphi$ is a closure operation and for any $x_1, x_2 \in X$ we have $\varphi(x_1) \leq \varphi(x_2)$ if and only if $f(x_1) \leq f(x_2)$.*

**Theorem 4.5.** *If $f$ is an increasingly $g$-normal structure on $X$ to $Y$ and $\varphi$ is an operation on $X$ such that $\varphi \leq g \circ f \leq \varphi$, then $f$ is increasingly $\varphi$-regular.*

**Theorem 4.6.** *If $f$ is an increasingly $\varphi$-regular structure on $X$ onto $Y$ and $g$ is a structure on $Y$ to $X$ such $\varphi \leq g \circ f \leq \varphi$, then $f$ is increasingly $g$-normal.*

**Theorem 4.7.** *$f$ is an increasingly $g$-normal structure on $X$ to $Y$ if and only if $g$ is an increasingly $f$-normal structure on $Y^*$ to $X^*$.*

**Theorem 4.8.** *If $f$ is an increasingly $g$-normal structure on $X$ to $Y$, then $f$ and $g$ are increasing. Moreover, $\varphi = g \circ f$ is a closure operation on $X$ and $\psi = f \circ g$ is an interior operation on $Y$.*

## 5. Characterization of increasingly normal structures

**Definition 5.1.** For a structure $f$ on $X$ to $Y$, we define two relations $\Gamma_f$ and $g_f$ on $Y$ to $X$ such that

$$\Gamma_f(y) = \{x \in X : f(x) \leq y\}$$

and $g_f(y) = \max(\Gamma_f(y))$ for all $y \in Y$.

*Remark* 5.2. Note that $\Gamma_f(y) = f^{-1}[\mathrm{lb}(y)] = (f^{-1} \circ \mathrm{lb})(y)$ for all $y \in Y$.

Moreover, note that if in particular $X$ is a poset, then $g_f$ is already a function of a subset of $Y$ into $X$.

Concerning the relation $g_f$, in [13], the first author has, for instance, proved the following.

**Theorem 5.3.** *For any structures $f$ on $X$ to $Y$ and $g$ on $Y$ to $X$, the following assertions are equivalent:*

(1) *$f$ is increasingly $g$-normal;*
(2) *$f$ is increasing and $g \subset g_f$.*

*Remark* 5.4. Hence, by using that for any increasing structure $f$ on $X$ to $Y$ and $y \in Y$ we have $g_f(y) = \{x \in X : \operatorname{lb}(x) = \Gamma_f(y)\}$, we can easily get a further useful equivalent of (1).

**Definition 5.5.** For a structure $f$ on $X$ to $Y$, we define

$$\mathcal{Q}_f = \big\{g \in X^Y : f \text{ is increasingly } g\text{-normal}\big\}.$$

Moreover, if in particular $\mathcal{Q}_f \neq \emptyset$, then we say that $f$ is increasingly normal.

Concerning increasingly normal structures, in [13], the first author has, for instance, proved the following theorems.

**Theorem 5.6.** *If $f$ is a structure on $X$ to $Y$, then the following assertions are equivalent:*

(1) *$f$ is increasingly normal;*
(2) *$f$ is increasing and $X = D_{g_f}$.*

**Theorem 5.7.** *If $f$ is an increasingly normal structure on $X$ to $Y$, then $g_f = \bigcup \mathcal{Q}_f$.*

**Theorem 5.8.** *If $f$ is an increasingly normal structure on a poset $X$ to $Y$, then $g_f$ is an increasing structure on $Y$ to $X$ and $\mathcal{Q}_f = \{g_f\}$.*

## 6. Characterization of increasingly regular structures

**Definition 6.1.** For a structure $f$ on $X$, we define two relations $\Lambda_f$ and $\varphi_f$ on $X$ such that

$$\Lambda_f(x) = \{u \in X : f(u) \leq f(x)\}$$

and $\varphi_f(x) = \max(\Lambda_f(x))$ for all $x \in X$.

*Remark* 6.2. Note that, in contrast to Remark 5.2, now we simply have $\Lambda_f = f^{-1} \circ \operatorname{lb} \circ f$.

Moreover, note also that if in particular $X$ is a poset, then $\varphi_f$ is already a function of a subset of $X$ into $X$.

Concerning the above relations, in [13], the first author has, for instance, proved the following theorems.

**Theorem 6.3.** *If $f$ is a structure on $X$, then $\Lambda_f$ is a preorder relation on $X$. Moreover, $\Lambda_f = \Gamma_f \circ f$ and $\varphi_f = g_f \circ f$.*

**Theorem 6.4.** *If $\varphi$ is an operation and $f$ is a structure on $X$, then the following assertions are equivalent:*

(1) *$f$ is increasingly $\varphi$-regular;*
(2) *$f$ is increasing and $\varphi \subset \varphi_f$.*

*Remark* 6.5. Hence, by using that for any increasing structure on $X$ and $x \in X$ we have $\varphi_f(x) = \{u \in X : \mathrm{lb}(u) = \Lambda_f(x)\}$, we can easily get a further useful equivalent of (1).

**Definition 6.6.** For a structure $f$ on $X$, we define

$$\mathcal{O}_f = \big\{\varphi \in X^X : f \text{ is increasingly } \varphi\text{-regular}\big\}.$$

Moreover, if in particular $\mathcal{O}_f \neq \varnothing$, then we say that $f$ is increasingly regular.

Concerning increasingly regular structures, in [13], the first author has, for instance, proved the following theorems.

**Theorem 6.7.** *If $f$ is an increasingly normal structure on $X$ to $Y$, then $f$ is, in particular, increasingly regular.*

**Theorem 6.8.** *If $f$ is a structure on $X$, then the following assertions are equivalent:*

(1) *$f$ is increasingly regular;*
(2) *$f$ is increasing and $X = D_{\varphi_f}$.*

**Theorem 6.9.** *If $f$ is an increasingly regular structure on $X$, then $\varphi_f = \bigcup \mathcal{O}_f$.*

**Theorem 6.10.** *If $f$ is an increasingly regular structure on $X$ onto $Y$, then $f$ is increasingly normal.*

**Theorem 6.11.** *If $f$ is an increasingly regular structure on a poset $X$, then $\varphi_f$ is a closure operation on $X$ and $\mathcal{O}_f = \{\varphi_f\}$.*

## 7. Comparison of closure operations

**Definition 7.1.** For any operation $\varphi$ on $X$, we set

$$A_\varphi = \{x \in X : \varphi(x) \leq x\}.$$

*Remark* 7.2. Note that if in particular $\varphi$ is expansive, then $x \leq \varphi(x)$ for all $x \in X$. Therefore, if $\varphi$ is an expansive operation on a poset $X$, then $A_\varphi$ is just the family of all fixed points of $\varphi$.

Moreover, note that the operation $\varphi$ is quasi-idempotent if and only if $\varphi(x) \in A_\varphi$ for all $x \in X$. Therefore, if in particular $\varphi$ is a semiclosure operation on a poset $X$, then $A_\varphi$ is just the range of $\varphi$.

Now, as a natural extension of [6, Theorem 2] and [2, Theorem 4.4, p. 30], we can also prove the following

**Theorem 7.3.** *If $\varphi$ is increasing and $\psi$ is a closure operation on $X$, then the following assertions are equivalent:*

(1) $\varphi \leq \psi$;
(2) $A_\psi \subset A_\varphi$;
(3) $\varphi \circ \psi \leq \psi$;
(4) $\psi \circ \varphi \leq \psi$.

PROOF. If $x \in A_\psi$, then $\psi(x) \leq x$. Moreover, if (1) holds, then we also have $\varphi(x) \leq \psi(x)$. Hence, by the transitivity of $X$, it follows that $\varphi(x) \leq x$. Therefore, $x \in A_\varphi$, and thus (2) also holds.

Moreover, if $x \in X$, then by the quasi-idempotency of $\psi$ we have $\psi(\psi(x)) \leq \psi(x)$, and thus $\psi(x) \in A_\psi$. Hence, if (2) holds, we can infer that $\psi(x) \in A_\varphi$. Therefore, $(\varphi \circ \psi)(x) = \varphi(\psi(x)) \leq \psi(x)$, and thus (3) also holds.

Furthermore, if $x \in X$, then by the expansivity of $\psi$ we have $x \leq \psi(x)$. Hence, by the increasingness of $\varphi$, it follows that $\varphi(x) \leq \varphi(\psi(x))$. Moreover, if (3) holds, then we also have $\varphi(\psi(x)) \leq \psi(x)$. Hence, by the transitivity of $X$, it follows that $\varphi(x) \leq \psi(x)$. Therefore, (1) also holds.

Now, it remains only to prove that (1) and (4) are also equivalent. For this, note that if $x \in X$ and (1) holds, then $\varphi(x) \leq \psi(x)$. Hence, by using the increasingness of $\psi$, we can infer that $(\psi \circ \varphi)(x) = \psi(\varphi(x)) \leq \psi(\psi(x)) = \psi^2(x)$. Moreover, by the quasi-idempotency of $\psi$, we also have $\psi^2(x) \leq \psi(x)$. Hence, by the transitivity of $X$, it follows that $(\psi \circ \varphi)(x) \leq \psi(x)$. Therefore, (4) also holds.

On the other hand, if $x \in X$, then by the expansivity of $\psi$, we also have $\varphi(x) \leq \psi(\varphi(x))$. Moreover, if (4) holds, then we also have $\psi(\varphi(x)) \leq \psi(x)$. Hence, by the transitivity of $X$, it follows that $\varphi(x) \leq \psi(x)$. Therefore, (1) also holds. □

Now, as an immediate consequence of the above theorem, we can also state.

**Corollary 7.4.** *If $\varphi$ and $\psi$ are closure operations on a poset $X$ such that $A_\varphi = A_\psi$, then $\varphi = \psi$.*

Moreover, from Theorem 7.3, we can also easily get the following.

**Theorem 7.5.** *If $\varphi$ is a preclosure and $\psi$ is a closure operation on a poset $X$, then the following assertions are equivalent:*

(1) $\varphi \leq \psi$;
(2) $\psi = \varphi \circ \psi$;
(3) $\psi = \psi \circ \varphi$.

PROOF. If $x \in X$, then by the expansivity of $\varphi$, we have $\psi(x) \leq \varphi(\psi(x)) = (\varphi \circ \psi)(x)$. Moreover, if (1) holds, then by Theorem 7.3, we also have $(\varphi \circ \psi)(x) \leq \psi(x)$. Hence, by the antisymmetry of $X$, it follows that $\psi(x) = (\varphi \circ \psi)(x)$. Therefore, (2) also holds.

On the other hand, if $x \in X$, then by the expansivity of $\varphi$ we also have $x \leq \varphi(x)$. Hence, by the increasingness of $\psi$, it follows that $\psi(x) \leq \psi(\varphi(x)) = (\psi \circ \varphi)(x)$.

Moreover, if (1) holds, then by Theorem 7.3 we also have $(\psi \circ \varphi)(x) \leq \psi(x)$. Hence, by the antisymmetry of $X$, it follows that $\psi(x) = \varphi(\psi(x))$. Therefore, (3) also holds.

Now, since the converse implications (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (1) are immediate from Theorem 7.3, the proof is complete. □

Now, to include a further part of [5, Proposition 4], we can also prove the following.

**Theorem 7.6.** *If $\varphi$ and $\psi$ are closure operations on a poset $X$, then the following assertions are equivalent:*

(1) $\varphi \leq \psi$;
(2) $\varphi^{-1} \circ \varphi \subset \psi^{-1} \circ \psi$.

PROOF. If $(x, y) \in \varphi^{-1} \circ \varphi$, then $y \in (\varphi^{-1} \circ \varphi)(x) = \varphi^{-1}(\varphi(x))$, and thus $\varphi(y) = \varphi(x)$. Moreover, if (1) holds, then by Theorem 7.5, we have $\psi = \psi \circ \varphi$. Now, we can already see that $\psi(y) = (\psi \circ \varphi)(y) = \psi(\varphi(y)) = \psi(\varphi(x)) = (\psi \circ \varphi)(x) = \psi(x)$. Hence, it follows that $y \in \psi^{-1}(\psi(x)) = (\psi^{-1} \circ \psi)(x)$, and thus $(x, y) \in \psi^{-1} \circ \psi$. Therefore, (2) also holds.

On the other hand, if $x \in X$, then under the notation $y = \varphi(x)$ we have $(\psi \circ \varphi)(x) = \psi(\varphi(x)) = \psi(y)$. Moreover, by the idempotency of $\varphi$, we also have $\varphi(y) = \varphi(\varphi(x)) = \varphi^2(x) = \varphi(x)$. Hence, quite similarly the above, we can infer that $(x, y) \in \varphi^{-1} \circ \varphi$. Now, if (2) holds, then we can also state that $(x, y) \in \psi^{-1} \circ \psi$. Hence, quite similarly the above, we can infer that $\psi(y) = \psi(x)$. Now, by the equality $(\psi \circ \varphi)(x) = \psi(y)$, we can also state that $(\psi \circ \varphi)(x) = \psi(x)$, and thus $\psi \circ \varphi = \psi$. Therefore, by Theorem 7.5, (1) also holds. □

From the above results, by using Theorems 4.1 and 4.3, we can easily derive the following theorems.

**Theorem 7.7.** *If $\varphi$ is an increasingly $\psi$-regular operation on $X$, then the following assertions are equivalent:*

(1) $\varphi \leq \psi$;
(2) $A_\psi \subset A_\varphi$;
(3) $\varphi \circ \psi \leq \psi$;
(4) $\psi \circ \varphi \leq \psi$.

PROOF. Now, by Theorems 4.1 and 4.3, $\varphi$ is increasing and $\psi$ is a closure operation on $X$. Therefore, Theorem 7.3 can be applied. □

**Theorem 7.8.** *If $\varphi$ is a semiclosure and an increasingly $\psi$-regular operation on a poset $X$ such that $A_\varphi = A_\psi$, then $\varphi = \psi$.*

PROOF. Now, by Theorems 4.1 and 4.3, $\varphi$ and $\psi$ are closure operations on $X$. Therefore, Corollary 7.4 can be applied. □

**Theorem 7.9.** *If $\varphi$ is an expansive increasingly $\psi$-regular operation on a poset $X$ such that $\varphi \leq \psi$, then $\varphi = \psi$.*

PROOF. Now, by Theorems 4.1 and 4.3, $\varphi$ is a preclosure and $\psi$ is a closure operation on $X$. Therefore, by Theorem 7.5 and Corollary 4.2, $\psi = \varphi \circ \psi = \varphi$. □

## 8. COMPARISON OF INCREASINGLY NORMAL STRUCTURES

By using the corresponding definitions, we can easily prove the following.

**Theorem 8.1.** *If $f_i$ is a structure on $X$ to $Y$ for each $i = 1, 2$, then the following assertions are equivalent:*

(1) $f_1 \leq f_2$;
(2) $\Gamma_{f_2} \subset \Gamma_{f_1}$.

PROOF. Suppose that (1) holds and $(y, x) \in \Gamma_{f_2}$. Then, $x \in X$, and thus $f_1(x) \leq f_2(x)$. Moreover, $x \in \Gamma_{f_2}(y)$, and thus $f_2(x) \leq y$. Hence, by the transitivity of $X$, it follows that $f_1(x) \leq y$, and thus $x \in \Gamma_{f_1}(y)$. Therefore, $(y, x) \in \Gamma_{f_1}$, and thus (2) also holds.

Suppose now that (2) holds and $x \in X$. Then, by the reflexivity of $Y$, we have $f_2(x) \leq f_2(x)$. Therefore, $x \in \Gamma_{f_2}(f_2(x))$, and thus $(f_2(x), x) \in \Gamma_{f_2}$. Hence, by using (2), we can infer that $(f_2(x), x) \in \Gamma_{f_1}$, and thus $x \in \Gamma_{f_1}(f_2(x))$. Therefore, $f_1(x) \leq f_2(x)$, and thus (1) also holds. □

Moreover, as a straightforward extension of [3, Exercise 7.16, p. 172], we can prove the following.

**Theorem 8.2.** *If $f_i$ is an increasingly $g_i$-normal structure on $X$ to $Y$ for each $i = 1, 2$, then the following assertions are equivalent:*

(1) $f_1 \leq f_2$;
(2) $f_1 \circ g_2 \leq \Delta_Y$;
(3) $g_2 \leq g_1$.

PROOF. If (1) holds, then we also have $f_1 \circ g_2 \leq f_2 \circ g_2$. Moreover, since $f_2$ is increasingly $g_2$-normal, by Theorem 4.8 we also have $f_2 \circ g_2 \leq \Delta_Y$. Hence, by the transitivity of $Y$, it is clear that (2) also holds.

If (2) holds, then for any $y \in Y$ we have $f_1(g_2(y)) = (f_1 \circ g_2)(y) \leq \Delta_Y(y) = y$. Hence, by using that $f_1$ is increasingly $g_1$-normal, we can already infer that $g_2(y) \leq g_1(y)$. Therefore, (3) also holds.

Now, to prove the remaining implication (3) $\Longrightarrow$ (1), it is enough to note only that, by Theorem 4.7, $g_i$ is an increasingly $f_i$-normal structure on $Y^*$ to $X^*$ for each $i = 1, 2$. Therefore, by the corresponding definitions and the implication (1) $\Longrightarrow$ (3), we can state that $g_2 \leq g_1 \implies g_1 \leq^* g_2 \implies f_2 \leq^* f_1 \implies f_1 \leq f_2$. □

Now, as an immediate consequence of the above theorem, we can also state the following reformulation of [5, Theorem 1].

**Corollary 8.3.** *If $f_i$ is an increasingly $g_i$-normal structure on poset $X$ to another $Y$ for each $i = 1, 2$, then $f_1 = f_2$ if and only if $g_1 = g_2$.*

Moreover, by using Theorem 8.2, we can also prove the following.

**Theorem 8.4.** *If $f_i$ is an increasingly normal structure on $X$ to $Y$ for each $i = 1, 2$, then the following assertions are equivalent:*

(1) $f_1 \leq f_2$;
(2) $g_{f_2}(y) \subset \operatorname{lb}(g_{f_1}(y))$ *for all* $y \in Y$.

PROOF. Suppose that (1) holds, and moreover $y \in Y$ and $x_i \in g_{f_i}(y)$ for each $i = 1, 2$. Then, by Theorem 5.7, for each $i = 1, 2$ there exists $g_i \in \mathcal{Q}_{f_i}$ such that $x_i = g_i(y)$. Moreover, by Theorem 8.2, we have $x_2 = g_2(y) \leq g_1(y) = x_1$. Hence, it is clear that $x_2 \in \operatorname{lb}(g_{f_1}(y))$, and thus (2) also holds.

To prove the converse implication, note that now, for each $i = 1, 2$, there exists $g_i \in \mathcal{Q}_{f_i}$. Moreover, by Theorem 5.3, we have $g_i(y) \in g_{f_i}(y)$ for all $y \in Y$. Hence, if (2) holds, we can infer that $g_2(y) \leq g_1(y)$ for all $y \in Y$. Now, by Theorem 8.2, it is clear that (1) also holds. □

## 9. Compositions of increasingly regular structures

**Theorem 9.1.** *If $f$ is an increasingly $\varphi$-regular structure on $X$, then*

$$\Lambda_f = \Lambda_f \circ \varphi \qquad \text{and} \qquad \varphi_f = \varphi_f \circ \varphi.$$

PROOF. If $x \in X$, then by the corresponding definitions, Theorem 4.1 and the transitivity of $R_f$, it is clear that

$$u \in (\Lambda_f \circ \varphi)(x) \iff u \in \Lambda_f(\varphi(x)) \iff$$
$$\iff f(u) \leq f(\varphi(x)) \iff f(u) \leq f(x) \iff u \in \Lambda_f(x).$$

Therefore, $(\Lambda_f \circ \varphi)(x) = \Lambda_f(x)$. Moreover, now we can also easily see that

$$(\varphi_f \circ \varphi)(x) = \varphi_f(\varphi(x)) =$$
$$= \max(\Lambda_f(\varphi(x))) = \max((\Lambda_f \circ \varphi)(x)) = \max(\Lambda_f(x)) = \varphi_f(x).$$

Therefore, the required equalities are also true. □

From the above theorem, by using Theorem 6.9, we can easily derive

**Corollary 9.2.** *If $f$ is an increasingly regular structure on $X$, then $\Lambda_f = \Lambda_f \circ \varphi_f$ and $\varphi_f = \varphi_f \circ \varphi_f$.*

PROOF. If $x \in X$, then by Theorem 6.9, we have

$$\varphi_f(x) = \{\varphi(x) : \varphi \in \mathcal{O}_f\}.$$

Moreover, if $F = \Lambda_f$ or $\varphi_f$, then by Theorem 9.1 we have $F \circ \varphi = F$ for all $\varphi \in \mathcal{O}_f$. Hence, it is clear that

$$(F \circ \varphi_f)(x) = F[\varphi_f(x)] = F[\{\varphi(x)\}_{\varphi \in \mathcal{O}_f}] = \\ = \bigcup_{\varphi \in \mathcal{O}_f} F(\varphi(x)) = \bigcup_{\varphi \in \mathcal{O}_f} (F \circ \varphi)(x) = \bigcup_{\varphi \in \mathcal{O}_f} F(x) = F(x).$$

Therefore, $F \circ \varphi_f = F$, and thus the required equalities are also true. □

**Theorem 9.3.** *If $f$ is an increasingly regular structure on $X$, then $\mathcal{O}_f$ is closed under composition.*

PROOF. Suppose that $\varphi, \psi \in \mathcal{O}_f$, and define $\chi = \psi \circ \varphi$. Then, by the corresponding definitions, Theorem 4.1 and the transitivity of $R_f$, it is clear that for any $x_1, x_2 \in X$ we have

$$x_1 \leq \chi(x_2) \iff x_1 \leq (\psi \circ \varphi)(x_2) \iff x_1 \leq \psi\big(\varphi(x_2)\big) \iff \\ \iff f(x_1) \leq f\big(\varphi(x_2)\big) \iff f(x_1) \leq f(x_2).$$

Therefore, $\chi \in \mathcal{O}_f$ also holds. □

*Remark* 9.4. Note that if in particular $f$ is an increasingly regular structure on a poset $X$, then by Theorem 6.11 we only have $\mathcal{O}_f = \{\varphi_f\}$.

As an extension of [2, Theorem 2.8, p. 14], we also have the following.

**Theorem 9.5.** *If $f$ is an increasingly $g$-normal structure on $X$ to $Y$ and $h$ is an increasingly $k$-normal structure on $Y$ to $Z$, then $h \circ f$ is a $g \circ k$-normal structure on $X$ to $Z$.*

PROOF. For any $x \in X$ and $z \in Z$, we have

$$(h \circ f)(x) \leq z \iff h(f(x)) \leq z \iff \\ \iff f(x) \leq k(z) \iff x \leq g(k(z)) \iff x \leq (g \circ k)(z).$$

Therefore, the required assertion is true. □

Now, as an immediate consequence of the above theorem, we can also state

**Corollary 9.6.** *If $f$ is a increasingly normal structure on $X$ to $Y$ and $h$ is an increasingly normal structure on $Y$ to $Z$, then $h \circ f$ is an increasingly normal structure on $X$ to $Z$.*

Hence, by Theorem 6.10, it is clear that in particular we also have

**Corollary 9.7.** *If $f$ is an increasingly regular structure on $X$ onto $Y$ and $h$ is an increasingly regular structure on $Y$ onto $Z$, then $h \circ f$ is an increasingly regular structure on $X$ onto $Z$.*

## 10. Some further results on closure operations

**Theorem 10.1.** *If $\varphi$ is an increasingly $\psi$-normal structure on $X$ to itself, then*

(1) $\Delta_X \leq \varphi \iff \psi \leq \Delta_X$;
(2) $\varphi^2 \leq \varphi \iff \psi \leq \psi^2$;
(3) $\varphi \leq \Delta_X \iff \Delta_X \leq \psi$;
(4) $\varphi \leq \varphi^2 \iff \psi^2 \leq \psi$.

PROOF. Clearly, $\Delta_X$ is an increasingly $\Delta_X$-normal structure on $X$ to itself. Hence, by Theorem 8.2, it is clear that (1) and (3) are true.

Moreover, from Theorem 9.5, we can see that $\varphi^2$ is an increasingly $\psi^2$-normal structure on $X$ to itself. Hence, by Theorem 8.2, it is clear that (2) and (4) are also true. □

Now, as an immediate consequence of the above theorem, we can also state

**Corollary 10.2.** *If $\varphi$ is an increasingly $\psi$-normal structure on a poset $X$ to itself, then*

(1) $\varphi = \Delta_X \iff \psi = \Delta_X$;
(2) $\varphi = \varphi^2 \iff \psi = \psi^2$.

Moreover, as a natural extension of the first part of [2, Theorem 2.10, p. 15], we can also prove

**Theorem 10.3.** *If $\varphi$ is an increasingly $\psi$-normal structure on $X$ to itself, then the following assertions are equivalent:*

(1) *$\varphi$ is a closure operation;*
(2) $\varphi \leq \psi \circ \varphi \leq \varphi$;
(3) *$\psi$ is an interior operation;*
(4) $\psi \leq \varphi \circ \psi \leq \psi$.

PROOF. From Theorem 4.8, we can see that both $\varphi$ and $\psi$ are increasing. Hence, by Theorem 10.1 and the corresponding definitions, it is clear that (1) and (3) are equivalent.

Moreover, from Theorem 4.8, we can see that $\psi \circ \varphi$ is expansive. Furthermore, if (1) holds, then $\varphi$ is quasi-idempotent. Hence, it is clear that

$$\varphi = \Delta_X \circ \varphi \leq (\psi \circ \varphi) \circ \varphi = \psi \circ \varphi^2 \leq \psi \circ \varphi.$$

On the other hand, from Theorem 4.5 we can see that $\varphi$ is increasingly $\psi \circ \varphi$-regular. Moreover, if (1) holds, then $\varphi$ is expansive. Hence, by Theorem 4.1, it is clear that

$$\psi \circ \varphi = \Delta_X \circ (\psi \circ \varphi) \leq \varphi \circ (\psi \circ \varphi) \leq \varphi.$$

Now, by the transitivity of $X$, it is clear that (2) also holds.

Conversely, if (2) holds, then from Theorem 4.5 we can see that $\varphi$ is increasingly $\varphi$-regular. Therefore, by Theorem 4.3, (1) also holds.

Now, to prove the equivalence of (1) and (4), it is enough to note only that, by Theorem 4.7, $\psi$ is an increasingly $\varphi$-normal structure on $X^*$ to itself. Therefore, by the equivalences (1) $\Longleftrightarrow$ (3) and (1) $\Longleftrightarrow$ (2), we can state that

$$\varphi \text{ is a closure on } X \iff \psi \text{ is an interior on } X \iff$$
$$\iff \psi \text{ is a closure on } X^* \iff \psi \leq^* \varphi \circ \psi \leq^* \psi \iff \psi \leq \varphi \circ \psi \leq \psi,$$

as required. □

Now, as an immediate consequence of Theorems 4.8 and 10.3, we can also state

**Corollary 10.4.** *If $\varphi$ is a semiclosure operation and an increasingly $\psi$-normal structure on a poset $X$ to itself, then $\varphi = \psi \circ \varphi$ and $\psi = \varphi \circ \psi$.*

Hence, it is clear in particular we also have

**Corollary 10.5.** *If $\varphi$ is a semiclosure operation and an increasingly $\psi$-normal structure on a poset $X$ to itself, then $R_\varphi = R_\psi$.*

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