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ON THE ACTION OF THE STEENROD SQUARES ON POLYNOMIAL ALGEBRA

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Abstract. In this work we gather and formulize some useful tools for handling the action of the Steenrod squares on monomials. In particular, we introduce some matrices and call them Sq-matrices, which are sufficient tools in algorithmic calculations with the Steenrod squares on polynomials.

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1. INTRODUCTION

In this section we introduce briefly the hit problem for the polynomial algebra $\mathbf{P}(n) = \mathbb{F}_2[x_1, x_2, \ldots, x_n] = \bigoplus_{d \geq 0} \mathbf{P}^d(n)$, viewed as a graded module over the Steenrod algebra $\mathcal{A}$ at prime 2. The grading is by the homogeneous polynomials $\mathbf{P}^d(n)$ of degree d in the variables $x_1, x_2, \ldots, x_n$ of grading 1. We refer to [1, 6] in cohomology operations, to [6, 7] in the Steenrod algebra, and to [3] and the comprehensive reference [8] for the hit problem.

The Steenrod algebra $\mathcal{A}$ is defined to be the graded algebra over the field $\mathbb{F}_2$, generated by the Steenrod squares Sq^k, in grading $k \geq 0$, subject to the Adem relations [3, 8].

From a topological point of view, the Steenrod algebra is the algebra of stable cohomology operations for ordinary cohomology $\mathbf{H}^*$ over $\mathbb{F}_2$. The polynomial algebra $\mathbf{P}(n)$ realizes the cohomology of products of n copies of infinite real projective spaces.

For the present purpose we only need to know that the Steenrod algebra acts by composition of linear operators on $\mathbf{P}(n)$ and the action of the Steenrod squares $Sq^k : \mathbf{P}^d(n) \to \mathbf{P}^{d+k}(n)$ is determined by the following rules [8].

Proposition 1.1. *For homogeneous elements f, g in $\mathbf{P}(n)$ we have*

(i) Sq^0 *is the identity homomorphism;*
(ii) $Sq^k(f) = f^2$ *if* $\deg(f) = k$ *and* $Sq^k(f) = 0$ *if* $\deg(f) < k$;

(iii) *The Cartan formula* $Sq^k(fg) = \sum_{0\le r\le k} Sq^r(f)Sq^{k-r}(g)$.

The Cartan formula can be expressed in a more concise form by defining the *total* Steenrod square by $Sq = Sq^0 + Sq^1 + \cdots$. This acts on $\mathbf{P}(n)$ since by property (ii) in the above proposition, only a finite number of Sq^k's can be nonzero on a given polynomial. The Cartan formula then says that $Sq(fg) = Sq(f)Sq(g)$, so Sq is a ring homomorphism $Sq : \mathbf{P}(n) \to \mathbf{P}(n)$. Now, we can use Sq to compute the operator Sq^k via the following lemmas [7].

Lemma 1.2. *If* $\deg(x) = 1$, *then* $Sq^k(x^\alpha) = \binom{\alpha}{k}x^{k+\alpha}$, *for any non-negative integer* α.

Proof. Properties (i), (ii) in Proposition 1.1 give $Sq(x) = x + x^2 = x(1+x)$, so

$$Sq(x^\alpha) = Sq(x)^\alpha = x^\alpha(1+x)^\alpha = \sum_k \binom{\alpha}{k} x^{k+\alpha}$$

and hence $Sq^k(x^\alpha) = \binom{\alpha}{k}x^{k+\alpha}$. □

The following lemma is now immediate.

Lemma 1.3. *If* $\deg(x) = 1$, *then*

$$Sq^k(x^{2^\tau}) = \begin{cases} x^{2^\tau} & \text{if} \quad k = 0, \\ 0 & \text{if} \quad 0 < k < 2^\tau, \\ x^{2^{\tau+1}} & \text{if} \quad k = 2^\tau. \end{cases}$$

Remark. It is clear by Proposition 1.1 that $Sq^k(x^{2^\tau}) = 0$ if $k > 2^\tau$.

2. Some results in the n variables

Our main goal in this section is to extend Lemmas 1.2 and 1.3 and get some useful tools for handling the Steenrod operations. In particular, we show that given $\tau \ge 1$, the $Sq^k(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n})$, where $1 \le k < 2^\tau$ and $1 \le \alpha_i \le 2^\tau$, determine all $Sq^\ell(x_1^{\beta_1}x_2^{\beta_2}\cdots x_n^{\beta_n})$ for any $\beta_i \ge 1$ and any ℓ. On the other hand if we change the places of α_i and α_j in $Sq^k(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n})$, the results will be a permutation of x_i and x_j. So, to handle the Sq's it is sufficient to know only

$$Sq^k(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}) \quad \text{with } \alpha_1 \le \alpha_2 \cdots \le \alpha_n \le 2^\tau \text{ and } 1 \le k < 2^\tau,$$

for some $\tau > 0$.

Throughout the paper we shall adopt the following notations for any positive integer τ.

$$\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}, \tag{1}$$

$$\boldsymbol{x}^{\boldsymbol{m}(2^\tau)} = x_1^{m_1(2^\tau)}x_2^{m_2(2^\tau)}\cdots x_n^{m_n(2^\tau)}, \tag{2}$$

where, α_i and m_i are non negative integers for $1 \leq i \leq n$.

The following lemma is an extension of Lemma 1.3.

Lemma 2.1.

$$Sq^k(\boldsymbol{x}^{2^\tau}) = \begin{cases} \boldsymbol{x}^{2^\tau} & \text{if} \quad k = 0, & \text{(i)} \\ 0 & \text{if} \quad 0 < k < 2^\tau, & \text{(ii)} \\ \boldsymbol{x}^{2^\tau} \sum_{j=1}^{n} x_j^{2^\tau} & \text{if} \quad k = 2^\tau. & \text{(iii)} \end{cases}$$

Proof. Item (i) is trivial. For (ii) and (iii) use induction on n, noting the fact that the one variable case is consistent with Lemma 1.3 in each case. □

We prove even more general results.

Lemma 2.2.

$$Sq^k(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}) = \begin{cases} \boldsymbol{x}^{\boldsymbol{m}(2^\tau)} & \text{if } k = 0, & \text{(i)} \\ 0 & \text{if } 0 < k < 2^\tau, & \text{(ii)} \\ 0 & \text{if } k = 2^\tau \text{ and all the } m_i \text{ even}, & \text{(iii)} \\ \boldsymbol{x}^{\boldsymbol{m}(2^\tau)} \sum_{j=1}^{h} x_j^{2^\tau} & \text{if } k = 2^\tau,\ m_1, m_2, \dots m_h \text{ are odd, and the other } m_i\text{'s are even.} & \text{(iv)} \end{cases}$$

Proof. Item (i) is trivial. To prove (ii), expand notation (2) as

$$\boldsymbol{x}^{\boldsymbol{m}(2^\tau)} = \big(\overbrace{x_1^{2^\tau} \dots x_1^{2^\tau}}^{m_1 \text{ times}} \overbrace{x_2^{2^\tau} \dots x_2^{2^\tau}}^{m_2 \text{ times}} \dots \overbrace{x_n^{2^\tau} \dots x_n^{2^\tau}}^{m_n \text{ times}}\big).$$

Now, the result follows from Lemma 2.1(ii) taking $n = m_1 + m_2 + \cdots + m_n$.

We prove (iii) by induction on n, the number of variables. In the one variable case, put $m_1 = 2n_1$. Then, using (ii) we get $Sq^k(x_1^{m_1(2^\tau)}) = 0$. Assume now the result for smaller variables than n. Then

$$\begin{aligned} Sq^k(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}) =& Sq^0\big(x_1^{m_1(2^\tau)} x_2^{m_2(2^\tau)} \dots x_{n-1}^{m_{n-1}(2^\tau)}\big) Sq^{2^\tau}\big(x_n^{m_n(2^\tau)}\big) \\ &+ Sq^{2^\tau}\big(x_1^{m_1(2^\tau)} x_2^{m_2(2^\tau)} \dots x_{n-1}^{m_{n-1}(2^\tau)}\big) Sq^0\big(x_n^{m_n(2^\tau)}\big) \quad \text{(by (ii))} \\ =& 0, \end{aligned}$$

by assumption.

Finally, we prove (iv) by induction, this time, on h. Let m_1 be odd and the other m_i's even. Then, from (ii) and (iii) it follows that

$$Sq^k(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}) = \boldsymbol{x}^{\boldsymbol{m}(2^\tau)} x_1^{2^\tau}.$$

Now assume the result is true for smaller values than h. Then

$$\begin{aligned}
Sq^k(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}) =& Sq^0\left(x_h^{m_h(2^\tau)}\right)Sq^{2^\tau}\left(\prod_{h\neq i=1}^{n} x_i^{m_i(2^\tau)}\right)\\
&+ Sq^{2^\tau}(x_h^{m_h(2^\tau)})Sq^0\left(\prod_{h\neq i=1}^{n} x_i^{m_i(2^\tau)}\right) \quad \text{(by (ii))}\\
=& x_h^{m_h(2^\tau)}\left(\prod_{h\neq i=1}^{n} x_i^{m_i(2^\tau)}\right)\left(\sum_{j=1}^{h-1} x_j^{2^\tau}\right)\\
&+ x_h^{m_h(2^\tau)+2^\tau}\prod_{h\neq i=1}^{n} x_i^{m_i(2^\tau)} \quad \text{(by assumption)}\\
=& \prod_{i=1}^{n} x_i^{m_i(2^\tau)}\left(\sum_{j=1}^{h-1} x_j^{2^\tau} + x_h^{2^\tau}\right)\\
=& \boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\sum_{j=1}^{h} x_j^{2^\tau}.
\end{aligned}$$

The proof is complete. □

Corollary 2.3. *In our earlier notations* (1) *and* (2)*, where $\alpha_i, m_i \geq 0$ and $\tau > 0$, assume in addition that $1 \leq \alpha_i < 2^\tau$ for $1 \leq i \leq n$. Let also $0 \leq k < 2^\tau$. Then*

$$Sq^k(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\boldsymbol{x}^{\boldsymbol{\alpha}}) = \boldsymbol{x}^{\boldsymbol{m}(2^\tau)}Sq^k(\boldsymbol{x}^{\boldsymbol{\alpha}}). \tag{3}$$

If, in addition, for $1 \leq h \leq n$, we assume $m_1, m_2, \ldots, m_h$ are odd and other m_i's even, then

$$Sq^{2^\tau}(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\boldsymbol{x}^{\boldsymbol{\alpha}}) = \boldsymbol{x}^{\boldsymbol{m}(2^\tau)}Sq^{2^\tau}(\boldsymbol{x}^{\boldsymbol{\alpha}}) + \boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\boldsymbol{x}^{\boldsymbol{\alpha}}\sum_{j=1}^{h} x_j^{2^\tau} \tag{4}$$

and

$$Sq^{k+2^\tau}(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\boldsymbol{x}^{\boldsymbol{\alpha}}) = \boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\sum_{j=1}^{h} x_j^{2^\tau}Sq^k(\boldsymbol{x}^{\boldsymbol{\alpha}}). \tag{5}$$

In particular, if all m_i's are even, then

$$Sq^{2^\tau}(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\boldsymbol{x}^{\boldsymbol{\alpha}}) = \boldsymbol{x}^{\boldsymbol{m}(2^\tau)}Sq^{2^\tau}(\boldsymbol{x}^{\boldsymbol{\alpha}}) \tag{6}$$

and

$$Sq^{k+2^\tau}(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\boldsymbol{x}^{\boldsymbol{\alpha}}) = 0. \tag{7}$$

Proof. For $k = 0$ the relation (3) is trivial. Let $1 \leq k < 2^\tau$. Then by the Cartan formula

$$Sq^k(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\boldsymbol{x}^{\boldsymbol{\alpha}}) = \boldsymbol{x}^{\boldsymbol{m}(2^\tau)}Sq^k(\boldsymbol{x}^{\boldsymbol{\alpha}}) + \sum_{r=1}^{k} Sq^r(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)})Sq^{k-r}(\boldsymbol{x}^{\boldsymbol{\alpha}}).$$

But if $1 \leq r \leq k$, then $0 < r < 2^\tau$, and hence by Lemma 2.2 (ii), we have $Sq^r(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}) = 0$. This proves the relation (3).

To prove the relation (4), by Lemma 2.2 (ii) and the Cartan formula we have

$$Sq^{2^\tau}(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\boldsymbol{x}^{\boldsymbol{\alpha}}) = Sq^0(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)})Sq^{2^\tau}(\boldsymbol{x}^{\boldsymbol{\alpha}}) + Sq^{2^\tau}(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)})Sq^0(\boldsymbol{x}^{\boldsymbol{\alpha}}).$$

Now the result follows from Lemma 2.2 (iv).

Finally, to prove the relation (5) expand the left hand side of (5) using the Cartan formula. Now, by Lemma 2.2 (ii) and Lemma 1.1 (ii), we see that all the terms in the expansion are zero except $Sq^{2^\tau}(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)})Sq^k(\boldsymbol{x}^{\boldsymbol{\alpha}})$, which is the right hand side of (5) by Lemma 2.2.

The previous corollary shows clearly the main object stated at the beginning of this section. In the following example we illustrate all the cases in Corollary 2.3, i. e., relations (3)–(7). □

Example 2.4.

$$\begin{aligned}
Sq^3(x^{14}y^{11}) &= Sq^3(x^{3\cdot2^2}y^{2\cdot2^2}x^2y^3) = x^{3\cdot2^2}y^{2\cdot2^2}Sq^3(x^2y^3)\\
&= x^{12}y^8(x^2y^6 + x^4y^4) = x^{14}y^{14} + x^{16}y^{12},\\
Sq^4(x^{14}y^{11}) &= Sq^{2^2}(x^{3\cdot2^2}y^{2\cdot2^2}x^2y^3) = x^{3\cdot2^2}y^{2\cdot2^2}(Sq^{2^2}(x^2y^3) + x^2y^3\cdot x^{2^2})\\
&= x^{12}y^8(x^2y^7 + x^4y^5 + x^6y^3) = x^{14}y^{15} + x^{16}y^{13} + x^{18}y^{11},\\
Sq^7(x^{14}y^{11}) &= Sq^{3+2^2}(x^{3\cdot2^2}y^{2\cdot2^2}x^2y^3) = x^{3\cdot2^2}y^{2\cdot2^2}\cdot x^{2^2}Sq^3(x^2y^3)\\
&= x^{16}y^8(x^2y^6 + x^4y^4) = x^{18}y^{14} + x^{20}y^{12},\\
Sq^4(x^{18}y^{11}) &= Sq^{2^2}(x^{4\cdot2^2}y^{2\cdot2^2}x^2y^3) = x^{4\cdot2^2}y^{2\cdot2^2}Sq^{2^2}(x^2y^3)\\
&= x^{16}y^8(x^2y^7 + x^4y^5) = x^{18}y^{15} + x^{20}y^{13},\\
Sq^7(x^{18}y^{11}) &= Sq^{3+2^2}(x^{4\cdot2^2}y^{2\cdot2^2}x^2y^3) = 0.
\end{aligned}$$

Since every n-variable polynomial over $\mathbb{F}_2$ is the sum of n-variable monomials, the following result is concluded directly from equation (3) in Corollary 2.3.

Corollary 2.5. *Let $\tau \geq 1$ and $0 \leq k < 2^\tau$. Then given integers $m_i \geq 0$ for $1 \leq i \leq n$, and any n-variable polynomial f,*

$$Sq^k(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}f) = \boldsymbol{x}^{\boldsymbol{m}(2^\tau)}Sq^k(f).$$

In particular, if $\deg(f) < k$, *then*

$$Sq^k(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)} f) = 0.$$

3. Application 1

A homogeneous element f of grading d in a graded module $\mathbf{M}$ over $\mathcal{A}$ is said to be *hit* if it can be written as

$$f = \sum_{k>0} Sq^k(f_k),$$

where the pre-image elements f_k have a degree less than d. The hit problem is to discover criteria for elements of $\mathbf{M}$ to be hit and find minimal generating sets for $\mathbf{M}$ as an $\mathcal{A}$-module. However, we shall not go deeply into the hit problem. The following result is a direct consequence of Corollary 2.5.

Proposition 3.1. *Let* $\tau \geq 1$, *and let* f, g *be* n*-variable polynomials. Then* f *is hit via*

$$f = \sum_{0<k<2^\tau} Sq^k(f_k),$$

if and only if $g = \boldsymbol{x}^{\boldsymbol{m}(2^\tau)} f$ *is hit via*

$$g = \sum_{0<k<2^\tau} Sq^k(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)} f_k).$$

Example 3.2. Consider the hit polynomial

$$f = xy^5 = Sq^1(xy^4) + Sq^2(xy^3).$$

Then, by Proposition 3.1, $g_1 = x^{2^2} y^{2^3} f = x^5 y^{13}$ is hit and

$$x^5 y^{13} = Sq^1(x^5 y^{12}) + Sq^2(x^5 y^{11}).$$

But $g_2 = x^{2^1} f = x^3 y^5$ is not hit. Here $k = 2 = 2^1 = 2^\tau$ and we cannot use the lemma to conclude that $x^3 y^5 = Sq^1(x^3 y^4) + Sq^2(x^3 y^3)$.

Let the monomial $\boldsymbol{x}^{\boldsymbol{\alpha}'}$ be a permutation of the monomial $\boldsymbol{x}^{\boldsymbol{\alpha}}$. Then by equation (3) in Corollary 2.3 we have

$$Sq^k[\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}(\boldsymbol{x}^{\boldsymbol{\alpha}} + \boldsymbol{x}^{\boldsymbol{\alpha}'})] = \boldsymbol{x}^{\boldsymbol{m}(2^\tau)} Sq^k(\boldsymbol{x}^{\boldsymbol{\alpha}} + \boldsymbol{x}^{\boldsymbol{\alpha}'}).$$

Using this fact we can state the same results as Corollary 2.5 and Proposition 3.1 for symmetric polynomials. In particular, in Proposition 3.1 both f and g may be chosen symmetric if we take $\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}$ symmetric n-variable, i. e.,

$$\boldsymbol{x}^{\boldsymbol{m}(2^\tau)} = x_1^{m_1(2^\tau)} x_2^{m_2(2^\tau)} \cdots x_n^{m_n(2^\tau)},$$

where $m_1(2^\tau) = \cdots = m_n(2^\tau) > 0$. If this is the case, f_k will be symmetric as well. For the symmetric hit problem we refer to [4, 5].

4. Application 2

In this section we get some tools for handling the Steenrod squares in the 2-variable case and, since higher variables are determined recursively from two variables, these tools apply for general n.

Proposition 4.1. *Given $\tau \geq 1$, let*

(i) $0 \leq \alpha < 2^\tau$,
(ii) $2^\tau \leq \beta < 2^{\tau+1} - 1$,
(iii) $2^\tau < \alpha + \beta < 2^{\tau+1}$.

Then

$$Sq^{2^\tau}(x^\alpha y^\beta) = x^\alpha y^{\beta+2^\tau}.$$

Proof. Put $\beta = \beta' + 2^\tau$, where $0 \leq \beta' \leq 2^\tau - 1$. By the Cartan formula and Lemma 1.2 we have

$$\begin{aligned} Sq^{2^\tau}(x^\alpha y^\beta) &= Sq^0(x^\alpha) Sq^{2^\tau}(y^\beta) + \sum_{r=1}^{2^\tau} Sq^r(x^\alpha) Sq^{2^\tau - r}(y^\beta) \\ &= x^\alpha y^{\beta+2^\tau} + y^{2^\tau} \sum_{r=1}^{2^\tau} Sq^r(x^\alpha) Sq^{2^\tau - r}(y^{\beta'}). \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{r=1}^{2^\tau} Sq^r(x^\alpha) Sq^{2^\tau - r}(y^{\beta'}) = &\sum_{1 \leq r \leq \alpha} Sq^r(x^\alpha) Sq^{2^\tau - r}(y^{\beta'}) \\ &+ \sum_{\alpha < r \leq 2^\tau} Sq^r(x^\alpha) Sq^{2^\tau - r}(y^{\beta'}). \end{aligned}$$

If $r \leq \alpha$, then $2^\tau - r > \beta'$ and hence $Sq^{2^\tau - r}(y^{\beta'}) = 0$, and if $r > \alpha$, then $Sq^r(x^\alpha) = 0$. Therefore,

$$\sum_{r=1}^{2^\tau} Sq^r(x^\alpha) Sq^{2^\tau - r}(y^{\beta'}) = 0,$$

and the proof is completed. □

Proposition 4.2. *Let $m \geq n+2$, $n \geq 1$. Let*

(i) $2^{m-2} \leq \alpha \leq 2^{m-2} + 2^{n-1} - 1$,
(ii) $2^{m-2} + 2^{n-1} - 1 \leq \beta \leq 2^{m-2} + 2^n - 2$,
(iii) $\alpha + \beta = 2^{m-1} + 2^n - 2$.

Then

$$Sq^{2^{m-1}}(x^\alpha y^\beta) = \left(x^{\alpha+2^{m-2}} y^{\beta+2^{m-2}}\right).$$

Proof. By the Cartan formula we have

$$Sq^{2^{m-1}}(x^{\alpha}y^{\beta}) = \sum_{0\le r<2^{m-1}-\beta} Sq^{r}(x^{\alpha})Sq^{2^{m-1}-r}(y^{\beta}).$$

If $0 \le r < 2^{m-1}-\beta$, then $2^{m-1}-r > \beta$ and $Sq^{2^{m-1}-r}(y^{\beta}) = 0$.
If $2^{m-1}-\beta \le r < 2^{m-2}$, then $r > \alpha - 2^{m-2}$. Hence, by Corollary 2.5,

$$Sq^{r}(x^{\alpha}) = Sq^{r}\big(x^{2^{m-2}}x^{\alpha-2^{m-2}}\big) = 0.$$

If $2^{m-2} < r \le \alpha$, then $2^{m-1}-r > \beta - 2^{m-2}$. Once again,

$$Sq^{2^{m-1}-r}(y^{\beta}) = Sq^{2^{m-1}-r}\big(y^{2^{m-2}}y^{\beta-2^{m-2}}\big) = 0.$$

Finally, if $\alpha < r \le 2^{m-1}$, then $Sq^{r}(x^{\alpha}) = 0$. So, by splitting the summation, one sees that only the middle term is non-zero. Thus,

$$Sq^{2^{m-1}}(x^{\alpha}y^{\beta}) = Sq^{2^{m-2}}(x^{\alpha})Sq^{2^{m-2}}(y^{\beta}) = \big(x^{\alpha+2^{m-2}}y^{\beta+2^{m-2}}\big).$$

The proof is complete. □

5. Application 3

The subject of this section is to introduce some particular matrices, which we call *Sq-matrices*, and apply them to simplify the action of the Steenrod squares. To do this, we need some preliminaries.

Definition 5.1. Let M be an $m \times n$ matrix. By a *reverse transpose of* M, denoted M^{rt}, we mean an $n \times m$ matrix obtained by reversing the order of rows of M^{t}, the transpose of M. Therefore,

$$M^{rt}_{ji} = M^{t}_{(n+1-j)i} = M_{i(n+1-j)},$$

for $1 \le i \le m, 1 \le j \le n$.

The following result follows directly from the definition.

Proposition 5.2. *Given any $m \times n$ matrix M, the product MM^{rt} is a symmetric $m \times m$ matrix.*

The next lemma in [2] describes how binomial coefficients can be computed modulo a prime.

Lemma 5.3. *If p is a prime, then $\binom{m}{n} = \prod_i \binom{m_i}{n_i}$ mod p, where $m = \sum_i m_i p^i$ and $n = \sum_i n_i p^i$, with $0 \le m_i < p$ and $0 \le n_i < p$, are the p-adic expansions of m and n.*

When $n = 2$, for example, the extreme cases of a dyadic expansion consisting of a single 1 or all 1's give

$$Sq(x^{2^k}) = x^{2^k} + x^{2^{k+1}}$$

and

$$Sq(x^{2^k-1}) = x^{2^k-1} + x^{2^k} + x^{2^k+1} + \cdots + x^{2^{k+1}-2}$$

for all x with degree 1. More generally, the coefficient of $Sq(x^n)$ can be read from the $(n+1)^{th}$ row of the mod 2 Pascal triangle, a portion of which is shown in Figure 1, where dots denote zeros [2].

Definition 5.4. Let m be a positive integer. For $1 \le k \le 2^m - 1$ the *Sq-matrix* $\mathcal{S}_k$ is a $2^m \times (k+1)$ matrix defined by

$$(\mathcal{S}_k)_{ij} = Sq^{j-1}(x^i).$$

In other words, the terms of $Sq(x^n)$ can be read from the nth row of $\mathcal{S}_k$.

Without any confusion, if convenient, in expression of Sq-matrices each non zero entry may be denoted only by the power of x in it. Figure 2 shows the Sq-matrix $\mathcal{S}_{31}$ where, as in Figure 1, dots denote zeros. Note that it contains the Sq-matrices $\mathcal{S}_{15}, \mathcal{S}_7, \mathcal{S}_3$, and $\mathcal{S}_1$ as sub-blocks.

As seen comparing Figures 1 and 2, if we remove the top row of Figure 1 then, up to arrangement, the position of zeros in both figures are the same.

In the following algorithm, given a positive integer m, we construct the Sq-matrix $\mathcal{S}_{2^m-1}$ using Corollary 2.3. For $1 \le k \le 2^m - 1$ the Sq-matrix $\mathcal{S}_k$ can be obtained from $\mathcal{S}_{2^m-1}$ by choosing the first k columns.

Algorithm 5.5.

1) Define

$$\mathcal{S}_1 = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix};$$

2) For $t = 1$ to $m-1$ do
   - 2.1) For $i, j = 1$ to 2^t do
     - 2.1.1) Define $\mathcal{T}_{ij} = (\mathcal{S}_{2^t-1})_{ij} + 2^t$;
     - 2.1.2) Define $\mathcal{U}_{ij} = (\mathcal{S}_{2^t-1})_{ij} + 2^{t+1}$;
     - 2.1.3) Define $\mathbf{0}$ to be the $2^t \times 2^t$ zero matrix;
   - 2.2) Define the $2^{t+1} \times 2^{t+1}$ matrix

$$\mathcal{S} = \left(\begin{array}{c|c} \mathcal{S}_{2^t-1} & \mathbf{0} \\ \hline \mathcal{T} & \mathcal{U} \end{array}\right);$$

   - 2.3) Define $\mathcal{S}_{2^{t+1}-1}$ to be the matrix obtained from $\mathcal{S}$ by substituting $\mathbf{0}_{2^t1}$ by 2^t, and $\mathcal{U}_{2^t1}$ by 0.

```
                1
               1 1
              1 . 1
             1 1 1 1
            1 . . . 1
           1 1 . . 1 1
          1 . 1 . 1 . 1
         1 1 1 1 1 1 1 1
        1 . . . . . . . 1
       1 1 . . . . . . 1 1
      1 . 1 . . . . . 1 . 1
     1 1 1 1 . . . . 1 1 1 1
    1 . . . 1 . . . 1 . . . 1
   1 1 . . 1 1 . . 1 1 . . 1 1
  1 . 1 . 1 . 1 . 1 . 1 . 1 . 1
 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 . . . . . . . . . . . . . . . 1
```

FIGURE 1. Mod 2 Pascal triangle

```
 1 2 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
 2 . 4 . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
 3 4 5 6 . . . . . . . . . . . . . . . . . . . . . . . . . . . .
 4 . . . 8 . . . . . . . . . . . . . . . . . . . . . . . . . . .
 5 6 . . 910 . . . . . . . . . . . . . . . . . . . . . . . . . .
 6 . 8 .10 .12 . . . . . . . . . . . . . . . . . . . . . . . . .
 7 8 91011121314 . . . . . . . . . . . . . . . . . . . . . . . .
 8 . . . . . . .16 . . . . . . . . . . . . . . . . . . . . . . .
 910 . . . . . .1718 . . . . . . . . . . . . . . . . . . . . . .
10 .12 . . . . .18 .20 . . . . . . . . . . . . . . . . . . . . .
11121314 . . . .19202122 . . . . . . . . . . . . . . . . . . . .
12 . . .16 . . .20 . . .24 . . . . . . . . . . . . . . . . . . .
1314 . .1718 . .2122 . .2526 . . . . . . . . . . . . . . . . . .
14 .16 .18 .20 .22 .24 .26 .28 . . . . . . . . . . . . . . . . .
15161718192021222324252627282930 . . . . . . . . . . . . . . . .
16 . . . . . . . . . . . . . . .32 . . . . . . . . . . . . . . .
1718 . . . . . . . . . . . . . .3334 . . . . . . . . . . . . . .
18 .20 . . . . . . . . . . . . .34 .36 . . . . . . . . . . . . .
19202122 . . . . . . . . . . . .35363738 . . . . . . . . . . . .
20 . . .24 . . . . . . . . . . .36 . . .40 . . . . . . . . . . .
2122 . .2526 . . . . . . . . . .3738 . .4142 . . . . . . . . . .
22 .24 .26 .28 . . . . . . . . .38 .4. .42 .44 . . . . . . . . .
2324252627282930 . . . . . . . .3940414243444546 . . . . . . . .
24 . . . . . . .32 . . . . . . .40 . . . . . . .48 . . . . . . .
2526 . . . . . .3334 . . . . . .4142 . . . . . .495. . . . . . .
26 .28 . . . . .34 .36 . . . . .42 .44 . . . . .50 .52 . . . . .
27282930 . . . .35363738 . . . .43444546 . . . .51525354 . . . .
28 . . .32 . . .36 . . .40 . . .44 . . .48 . . .52 . . .56 . . .
2930 . .3334 . .3738 . .4142 . .4546 . .4950 . .5354 . .5758 . .
30 .32 .34 .36 .38 .40 .42 .44 .46 .48 .50 .52 .54 .56 .58 .60 .
3132333435363738394041424344454647484950515253545556575859606162
32 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
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FIGURE 2. The Sq-matrix $\mathcal{S}_{31}$

The following observation follows directly from Definition 5.4 and the Cartan formula, where in $\mathcal{S}_k^{rt}$ the symbol y is used instead of x.

Proposition 5.6. *Let m be a positive integer and $1 \le k \le 2^m - 1$. Then*

$$(\mathcal{S}_k \mathcal{S}_k^{rt})_{ij} = Sq^k(x^i y^j), \quad 1 \le i, j \le 2^m.$$

For example

$$\mathcal{S}_1 = \begin{pmatrix} x^1 & x^2 \\ x^2 & 0 \end{pmatrix}, \qquad \mathcal{S}_1^{rt} = \begin{pmatrix} y^2 & 0 \\ y^1 & y^2 \end{pmatrix}$$

$$\mathcal{S}_1 \mathcal{S}_1^{rt} = \begin{pmatrix} Sq^1(x^1y^1) & Sq^1(x^1y^2) \\ Sq^1(x^2y^1) & Sq^1(x^2y^2) \end{pmatrix} = \begin{pmatrix} x^1y^2 + x^2y^1 & x^2y^2 \\ x^2y^2 & 0 \end{pmatrix}.$$

We extend the argument above. To do this, suppose that X, Y are monomials in positive grading with distinct variables. Let x, y be distinct variables different from those in X, Y. Given $m > 0$ and $1 \le k \le 2^m - 1$, define the $2^m \times (k+1)$ Sq-matrices

$$(\mathcal{X}_k)_{ij} = Sq^{j-1}(Xx^i), \qquad (\mathcal{Y}_k)_{ij} = Sq^{j-1}(Yy^i).$$

The following observation is analogous to Proposition 5.6.

Proposition 5.7. *Let m be a positive integer and $1 \le k \le 2^m - 1$. Then*

$$(\mathcal{X}_k \mathcal{Y}_k^{rt})_{ij} = Sq^k(Xx^iYy^j), \quad 1 \le i, j \le 2^m.$$

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