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MINIMAL PERIODS OF PERIODIC SOLUTIONS

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ABSTRACT. We derive minimal periods of non-constant periodic solutions for semilinear damped wave equations on Hilbert spaces. Similar estimates are obtained for symmetric nonconstant periodic solutions of $\mathbb{Z}_p$-symmetric autonomous ordinary differential equations.

Mathematics Subject Classification: 34C14, 34C25, 35B10

Keywords: minimal periods, weak solutions, symmetries

1. INTRODUCTION

In this note, we present some estimates, lower bounds, concerning periods of nontrivial periodic orbits for certain differential equations. First, in Section 2, we study damped semilinear wave equations on Hilbert spaces. Here we are inspired by [14], where semilinear parabolic equations are studied. We apply our method to the equation of a damped buckled beam [9] and also to the equation of a buckled elastic panel excited by a fluid flow over its upper surface [3, 10].

Then, in Sections 3 and 4, we investigate $\mathbb{Z}_p$-symmetric autonomous ordinary differential equations which are generalizations of odd systems and related antiperiodic solutions (see [1]).

We note that recent results on minimal periods are also derived in [5, 12, 13, 15], where discrete, continuous and delay dynamical systems are considered. Minimal periods for ordinary differential equations on one-dimensional lattices are studied in [6, 7]. Finally, we refer the reader to [14, 15] for the history of these topics.

2. SEMILINEAR DAMPED WAVE EQUATIONS

Let H be a Hilbert space with a norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. Let A be an unbounded linear self-adjoint operator on H with an orthonormal basis of eigenvectors $\{w_j\}_{j\geq 1}$ on H and with corresponding eigenvalues λ_j, $Aw_j = \lambda_j w_j$ such that $\lambda_j \to +\infty$ as $j \to \infty$ (see [16]).

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Let $0 \le \alpha \le 1$. We put

$$H^\alpha := \left\{ x = \sum_{j\ge 1} x_j w_j \mid \sum_{j\ge 1} (\lambda_j^{2\alpha} + 1)\, x_j^2 < \infty \right\}$$

with an inner product $(\cdot,\cdot)_\alpha$ and corresponding norm $\|\cdot\|_\alpha$ defined by the formulae

$$(x, y)_\alpha := \sum_{j\ge 1} (\lambda_j^{2\alpha} + 1)\, x_j y_j, \quad \|x\|_\alpha := \sqrt{(x,x)_\alpha}$$

for any $x = \sum_{j\ge 1} x_j w_j$ and $y = \sum_{j\ge 1} y_j w_j$. Here, we consider $\lambda_j^{2\alpha} = (\lambda_j^2)^\alpha = |\lambda_j|^{2\alpha}$.

Let $f : H^\alpha \to H$ be globally Lipschitz continuous, i. e., $\exists L > 0$ such that

$$\|f(u_1) - f(u_2)\| \le L\|u_1 - u_2\|_\alpha \quad \forall u_{1,2} \in H^\alpha.$$

We consider a damped abstract wave equation

$$\ddot{u} + \delta\dot{u} + Au + f(u) = 0. \tag{2.1}$$

We put

$$L_T^2(\mathbb{R}, H) = \left\{ h \in L_{\mathrm{loc}}^2(\mathbb{R}, H) : h \text{ is } T\text{-periodic} \right\},$$
$$L_{T,0}^2(\mathbb{R}, H) = \left\{ h \in L_T^2(\mathbb{R}, H) : \int_0^\infty h(t)\, dt = 0 \right\}.$$

Similarly, we define Hilbert spaces $L_T^2(\mathbb{R}, H^\alpha)$ and $L_{T,0}^2(\mathbb{R}, H^\alpha)$. For $\alpha > 0$, the usual integral norm on $L_T^2(\mathbb{R}, H^\alpha)$ is denoted by $\|\cdot\|_{\alpha,2}$, while for $\alpha = 0$, we take the standard integral norm $\|\cdot\|_2$ on $L_T^2(\mathbb{R}, H)$.

Definition 2.1. By a weak T-periodic solution of (2.1), we mean any function $u \in L_T^2(\mathbb{R}, H^\alpha)$ satisfying the relation

$$\int_0^T \{(u(t), \ddot{v}(t)) - \delta(u(t), \dot{v}(t)) + (u(t), Av(t)) + (f(u(t)), v(t))\}\, dt = 0$$

for all $v \in C_T^2(\mathbb{R}, H^1)$, where $C_T^2(\mathbb{R}, H^1)$ is defined as above and $H^1 = D(A)$.

First we study the linear equation

$$\ddot{u} + \delta\dot{u} + Au = h \tag{2.2}$$

for $h \in L_T^2(\mathbb{R}, H)$.

Lemma 2.2. *Let $0 \le \alpha \le \frac{1}{2}$. For any $h \in L_{T,0}^2(\mathbb{R}, H)$, equation* (2.2) *has a unique weak solution $u \in L_{T,0}^2(\mathbb{R}, H^\alpha)$ satisfying the estimate*

$$\|u\|_{\alpha,2} \le \Psi_{\delta,\alpha}(T)\|h\|_2,$$

for a function $\Psi_{\delta,\alpha} \in C([0,\infty),\ (0,\infty))$ *with*

$$\begin{aligned} \lim_{T\to 0_+} \Psi_{\delta,\alpha}(T)/T^{1-2\alpha} &= c_{\alpha,0}, \\ \lim_{T\to\infty} \Psi_{\delta,\alpha}(T)/T &= c_{\alpha,\infty}, \end{aligned} \tag{2.3}$$

where $c_{\alpha,0}$ *and* $c_{\alpha,\infty}$ *are suitable positive constants.*

PROOF. We take $h(t) = \sum_{j\geq 1} h_j(t) w_j$, $u(t) = \sum_{j\geq 1} u_j(t) w_j$,

$$h_j(t) = \frac{1}{\sqrt{T}} \sum_{k\in\mathbb{Z}\setminus\{0\}} h_{j,k} e^{2\pi k t\imath/T}, \quad \bar{h}_{j,k} = h_{j,-k}$$

$$u_j(t) = \frac{1}{\sqrt{T}} \sum_{k\in\mathbb{Z}\setminus\{0\}} u_{j,k} e^{2\pi k t\imath/T}, \quad \bar{u}_{j,k} = u_{j,-k}.$$

Then

$$\|h\|_2^2 = \sum_{j\geq 1} \|h_j\|_2^2 = 2\sum_{j,k\geq 1} |h_{j,k}|^2,$$

$$\|u\|_{\alpha,2}^2 = \sum_{j\geq 1} \|u_j\|_2^2(\lambda_j^{2\alpha}+1) = 2\sum_{j,k\geq 1} |u_{j,k}|^2(\lambda_j^{2\alpha}+1).$$

So (2.2) gives

$$u_{j,k} = \frac{h_{j,k}}{\lambda_j - \frac{4\pi^2k^2}{T^2} + \delta\frac{2\pi k}{T}\imath}.$$

Hence

$$\|u\|_{\alpha,2}^2 \leq 2\sum_{j,k\geq 1} \frac{\lambda_j^{2\alpha}+1}{\left(\lambda_j - \frac{4\pi^2k^2}{T^2}\right)^2 + \delta^2\frac{4\pi^2k^2}{T^2}} |h_{j,k}|^2.$$

We evaluate

$$\frac{\lambda_j^{2\alpha}+1}{\left(\lambda_j - \frac{4\pi^2k^2}{T^2}\right)^2 + \delta^2\frac{4\pi^2k^2}{T^2}} \leq \frac{T^2}{4\pi^2\delta^2} + \frac{\lambda_j^{2\alpha}}{\left(\lambda_j - \frac{4\pi^2k^2}{T^2}\right)^2 + \delta^2\frac{4\pi^2k^2}{T^2}}.$$

Let us put $|\lambda_0| = \max\{|\lambda_i| \mid \lambda_i \leq 0\}$. Next, if $\lambda_j - \frac{\delta^2}{2} \geq \frac{4\pi^2}{T^2}$, then

$$\left(\lambda_j - \frac{4\pi^2k^2}{T^2}\right)^2 + \delta^2\frac{4\pi^2k^2}{T^2} \geq \delta^2\left(\lambda_j - \frac{\delta^2}{4}\right).$$

Hence, we have

$$\frac{\lambda_j^{2\alpha}}{\left(\lambda_j - \frac{4\pi^2k^2}{T^2}\right)^2 + \delta^2\frac{4\pi^2k^2}{T^2}} \leq \frac{\lambda_0^{2\alpha}T^2}{4\pi^2\delta^2}$$

for $\lambda_j \leq 0$,

$$\frac{\lambda_j^{2\alpha}}{\left(\lambda_j - \frac{4\pi^2 k^2}{T^2}\right)^2 + \delta^2 \frac{4\pi^2 k^2}{T^2}} \leq \left(\frac{\delta^2}{2} + \frac{4\pi^2}{T^2}\right)^{2\alpha} \frac{T^2}{4\pi^2 \delta^2}$$

for $0 \leq \lambda_j \leq \frac{\delta^2}{2} + \frac{4\pi^2}{T^2}$, and

$$\frac{\lambda_j^{2\alpha}}{\left(\lambda_j - \frac{4\pi^2 k^2}{T^2}\right)^2 + \delta^2 \frac{4\pi^2 k^2}{T^2}} \leq \frac{\lambda_i^{2\alpha}}{\delta^2 \left(\lambda_i - \frac{\delta^2}{4}\right)^2} \leq \frac{\left(\frac{\delta^2}{2} + \frac{4\pi^2}{T^2}\right)^{2\alpha}}{\delta^2 \left(\frac{\delta^2}{4} + \frac{4\pi^2}{T^2}\right)}$$

for $\lambda_j \geq \frac{\delta^2}{2} + \frac{4\pi^2}{T^2}$. Summarising, we get

$$\frac{\lambda_j^{2\alpha}}{\left(\lambda_j - \frac{4\pi^2 k^2}{T^2}\right)^2 + \delta^2 \frac{4\pi^2 k^2}{T^2}} \leq \Phi_{\delta,\alpha}(T),$$

where

$$\Phi_{\delta,\alpha}(T) := \max\left\{\frac{\lambda_0^{2\alpha} T^2}{4\pi^2\delta^2}, \left(\frac{\delta^2}{2} + \frac{4\pi^2}{T^2}\right)^{2\alpha} \frac{T^2}{4\pi^2\delta^2}, \frac{\left(\frac{\delta^2}{2} + \frac{4\pi^2}{T^2}\right)^{2\alpha}}{\delta^2\left(\frac{\delta^2}{4} + \frac{4\pi^2}{T^2}\right)}\right\}.$$

We see that $\Psi_{\delta,\alpha}$ defined by the equality

$$\Psi_{\delta,\alpha}(T) := \sqrt{\Phi_{\delta,\alpha}(T) + \frac{T^2}{4\pi^2\delta^2}}$$

satisfies the conditions of this lemma. So we obtain

$$\|u\|_{\alpha,2}^2 \leq 2 \sum_{j,k\geq 1} \Psi_{\delta,\alpha}^2(T)|h_{j,k}|^2 = \Psi_{\delta,\alpha}^2(T)\|h\|_2^2.$$

Consequently, we arrive at the estimate

$$\|u\|_{\alpha,2} \leq \Psi_{\delta,\alpha}(T)\|h\|_2.$$

The proof is complete. □

Now we return to (2.1) by splitting any $u \in L_T^2(\mathbb{R}, H^\alpha)$ as

$$u = u_1 + u_0$$

for $u_1 := \frac{1}{T}\int_0^T u(t)\,dt$ and $u_0 = u - u_1 \in L^2_{T,0}(\mathbb{R}, H^\alpha)$. Hence (2.1) has the form

$$\ddot{u}_0 + \delta\dot{u}_0 + Au_0 + f(u_1 + u_0) - \frac{1}{T}\int_0^T f(u_1 + u_0(t))\,dt = 0, \tag{2.4}$$

$$Au_1 + \frac{1}{T}\int_0^T f(u_1 + u_0(t))\,dt = 0. \tag{2.5}$$

We note that the linear operator $P : L^2_T(\mathbb{R}, H) \to L^2_T(\mathbb{R}, H)$ given by

$$Pu := u - \frac{1}{T}\int_0^T u(t)\,dt$$

is orthogonal and the Nemytskii operator $N : L^2_T(\mathbb{R}, H^\alpha) \to L^2_T(\mathbb{R}, H)$ given by the equality

$$N(u)(t) := f(u(t))$$

is globally Lipschitz continuous with a constant L. Then (2.4) gives

$$\begin{aligned} \|u_0\|_{\alpha,2} &\le \Psi_{\delta,\alpha}(T)\|PN(u_1 + u_0)\|_2 \\ &= \Psi_{\delta,\alpha}(T)\|P[N(u_1 + u_0) - N(u_1)]\|_2 \\ &\le \Psi_{\delta,\alpha}(T)L\|u_0\|_{\alpha,2}. \end{aligned}$$

Consequently, if

$$\Psi_{\delta,\alpha}(T)L < 1, \tag{2.6}$$

then $u_0 = 0$ and (2.5) becomes

$$Au_1 + f(u_1) = 0.$$

Summarising, we have the following result.

Theorem 2.3. *If* $0 \le \alpha \le \frac{1}{2}$ *and* (2.6) *holds, then any* T*-periodic weak solution of* (2.1) *is constant.*

Function $\Psi_{\delta,\alpha}(T)$ is depending also on λ_0. To avoid this, for $\lambda_j \le 0$ we compute

$$\frac{\lambda_j^{2\alpha}}{\left(\lambda_j - \frac{4\pi^2k^2}{T^2}\right)^2 + \delta^2\frac{4\pi^2k^2}{T^2}} \le \frac{\lambda_j^{2\alpha}}{\left(\lambda_j - \frac{4\pi^2k^2}{T^2}\right)^2} \le \left(\frac{1-\alpha}{4\pi^2}\right)^{2-2\alpha}\alpha^{2\alpha}T^{4(1-\alpha)}.$$

Then $\Phi_{\delta,\alpha}(T)$ is replaced by

$$\widetilde{\Phi}_{\delta,\alpha}(T) := \max\left\{\left(\frac{1-\alpha}{4\pi^2}\right)^{2-2\alpha}\alpha^{2\alpha}T^{4(1-\alpha)},\right.$$
$$\left.\left(\frac{\delta^2}{2} + \frac{4\pi^2}{T^2}\right)^{2\alpha}\frac{T^2}{4\pi^2\delta^2}, \frac{\left(\frac{\delta^2}{2} + \frac{4\pi^2}{T^2}\right)^{2\alpha}}{\delta^2\left(\frac{\delta^2}{4} + \frac{4\pi^2}{T^2}\right)}\right\}$$

and

$$\tilde{\Psi}_{\delta,\alpha}(T) := \sqrt{\tilde{\Phi}_{\delta,\alpha}(T) + \frac{T^2}{4\pi^2\delta^2}}.$$

So (2.3) is replaced by

$$\begin{aligned} \lim_{T\to 0_+} \tilde{\Psi}_{\delta,\alpha}(T)/T^{1-2\alpha} &= \tilde{c}_{\alpha,0}, \\ \lim_{T\to\infty} \tilde{\Psi}_{\delta,\alpha}(T)/T^{2(1-\alpha)} &= \tilde{c}_{\alpha,\infty} \end{aligned} \tag{2.7}$$

for suitable positive constants $\tilde{c}_{\alpha,0}$ and $\tilde{c}_{\alpha,\infty}$. Thus we have the following

Theorem 2.4. *If* $0 \le \alpha \le \frac{1}{2}$ *and the inequality*

$$\tilde{\Psi}_{\delta,\alpha}(T)L < 1$$

holds, then any T-periodic weak solution of (2.1) *is constant.*

Remark 2.5. Semilinear parabolic differential equations on Hilbert spaces are studied in [7, 12, 14].

Remark 2.6. We see that $\lim_{T\to 0_+} \Psi_{\delta,1/2}(T) \neq 0$ and $\lim_{T\to 0_+} \tilde{\Psi}_{\delta,1/2}(T) \neq 0$. So for $\alpha = 1/2$ we do not get a result on the minimal periods. But we think that it is not a handicap for our above estimates. Indeed, for $\alpha = 1/2$ and $\lambda_j \ge 0$, we consider the function

$$x \longmapsto \frac{x+1}{\left(x - \frac{4\pi^2k^2}{T^2}\right)^2 + \delta^2\frac{4\pi^2k^2}{T^2}}, \tag{2.8}$$

which has a global maximum over $[0,\infty)$:

$$\frac{4\pi^2k^2 + T^2 + \sqrt{16\pi^4k^4 + 4(2+\delta^2)\pi^2k^2T^2 + T^4}}{8\delta^2\pi^2k^2}. \tag{2.9}$$

We can check that (2.9) is decreasing with respect to $k \in \mathbb{N}$. So for $\lambda_j \ge 0$, we get

$$\frac{\lambda_j + 1}{\left(\lambda_j - \frac{4\pi^2k^2}{T^2}\right)^2 + \delta^2\frac{4\pi^2k^2}{T^2}} \le \frac{4\pi^2 + T^2 + \sqrt{16\pi^4 + 4(2+\delta^2)\pi^2T^2 + T^4}}{8\delta^2\pi^2}.$$

Consequently, when $\alpha = 1/2$ and $\lambda_j \ge 0\ \forall j \in \mathbb{N}$, the best estimate for $\Psi_{\delta,1/2}(T)$ seems to be

$$\Psi_{\delta,1/2}(T) = \sqrt{\frac{4\pi^2 + T^2 + \sqrt{16\pi^4 + 4(2+\delta^2)\pi^2T^2 + T^4}}{8\delta^2\pi^2}}. \tag{2.10}$$

Clearly, (2.10) is nonzero at $T = 0$.

Next, assuming $\lambda_j > 0\ \forall j \in \mathbb{N}$, we cannot improve much the above estimates. Now we can take

$$(x,y)_\alpha := \sum_{j\ge 1} \lambda_j^{2\alpha} x_j y_j$$

for $x = \sum_{j\ge 1} x_j w_j$ and $y = \sum_{j\ge 1} y_j w_j$. Then instead of the function (2.8), we consider the function

$$x \longmapsto \frac{x}{\left(x - \frac{4\pi^2 k^2}{T^2}\right)^2 + \delta^2 \frac{4\pi^2 k^2}{T^2}}. \tag{2.11}$$

Making the above analysis for (2.11), we obtain

$$\Psi_{\delta,1/2}(T) = \sqrt{\frac{2\pi + \sqrt{4\pi^2 + \delta^2 T^2}}{4\delta^2\pi}}. \tag{2.12}$$

Clearly (2.12) is again nonzero at $T = 0$. But (2.12) is simpler than (2.10). Then the lower bound estimate is

$$\frac{2\pi + \sqrt{4\pi^2 + \delta^2 T^2}}{4\delta^2\pi} \ge L^{-2}. \tag{2.13}$$

Analysing (2.13), we see that if $L \ge \delta$, then (2.13) holds for any $T > 0$, while for $\delta > L$, relation (2.13) yields

$$T \ge L^{-2} 4\pi \sqrt{\delta^2 - L^2}.$$

Consequently, we are able to estimate T from below only for small $L > 0$ with respect to δ.

Finally, the following semilinear parabolic equation is studied in [14]

$$\dot{u} + Au + f(u) = 0, \tag{2.14}$$

where A, f satisfy our assumptions with $\lambda_j > 0\ \forall j \ge 1$. Then we have

$$\frac{\lambda_j^{2\alpha}}{\frac{4\pi^2 k^2}{T^2} + \lambda_j^2} \le \frac{\lambda_j^{2\alpha}}{\frac{4\pi^2}{T^2} + \lambda_j^2}.$$

Analysing the function

$$x \longmapsto \frac{x^\alpha}{\frac{4\pi^2}{T^2} + x}$$

on $[0, \infty)$, we get its maximum $\alpha^\alpha (1-\alpha)^{1-\alpha} (2\pi)^{2(\alpha-1)} T^{2(1-\alpha)}$. So the minimal period estimate for (2.14) is

$$T \ge K_\alpha L^{-\frac{1}{1-\alpha}} \tag{2.15}$$

for any $0 \le \alpha < 1$ and

$$K_\alpha = \frac{2\pi}{\sqrt{1-\alpha}\,\alpha^{\frac{\alpha}{2(1-\alpha)}}}.$$

Inequality (2.15) is consistent with a similar one in [14], but here we allow $0 \le \alpha < 1$. We note that $K_0 = 2\pi$ and $\lim_{\alpha \to 1-} K_\alpha = \infty$.

Remark 2.7. Theorem 2.4 could help when the shifts

$$A \longleftrightarrow A - \lambda I, \quad f \longleftrightarrow f + \lambda I$$

can improve an estimate of the Lipschitz constant L.

We also note that it follows from the proof of Lemma 2.2 that $\dot{u} \in L^2_T(\mathbb{R}, H)$ and

$$\|\dot{u}\|_2 \leq \frac{1}{\delta}\|h\|_2$$

in Lemma 2.2. Then of course any T-periodic weak solution u of (2.1) satisfies $\dot{u} \in L^2_T(\mathbb{R}, H)$. Hence we get

$$\|u\|_{\alpha,2} + \|\dot{u}\|_2 \leq \left(\Psi_{\delta,\alpha}(T) + \frac{1}{\delta}\right)\|h\|_2. \tag{2.16}$$

Example 2.8. Let us consider the system

$$\begin{aligned} \ddot{u} + \delta\dot{u} - u'' + \sin(u - v) &= 0, \\ \ddot{v} + \delta\dot{v} - v'' + \sin(u + v) &= 0, \\ u(0, \cdot) = v(0, \cdot) = u(1, \cdot) = v(1, \cdot) &= 0. \end{aligned} \tag{2.17}$$

Now $H = L^2(0, 1)^2$, $\alpha = 0$ and

$$A(u, v) = -(u''(x), v''(x)), \quad f(u, v) = (\sin(u - v), \sin(u + v)).$$

We see that $\lambda_j > 0\ \forall j \in \mathbb{N}$ and $L = 2$. So we can choose a better function $\Psi_{\delta,0}(T) = \frac{T}{2\pi\delta}$ than in the proof of Lemma 2.2. Then for (2.17), the minimal period estimate is $T \geq \pi\delta$.

Now we present the following simple result which seems to be known but we prove it here for the reader's convenience.

Theorem 2.9. *Let us suppose that*

(1) $f : H^\alpha \to H$ *is locally Lipschitz continuous, i. e.* $\forall k > 0\ \exists L_k > 0$ *such that*

$$\|f(u_1) - f(u_2)\| \leq L_k\|u_1 - u_2\|_\alpha$$

for all $u_{1,2} \in H^\alpha$, $\|u_{1,2}\|_\alpha \leq k$.

(2) $\exists F \in C^1(H^\alpha, \mathbb{R}) : (f(v), w) = DF(v)w\ \forall v, w \in H^\alpha$.

Then any T*-periodic weak solution* $u \in L^\infty_T(\mathbb{R}, H^\alpha)$ *of* (2.1) *is constant for all* $T > 0$.

Proof. Let $\Gamma := \|u\|_{\alpha,\infty} + 1$ where $\|\cdot\|_{\alpha,\infty}$ is the usual sup-norm on $L^\infty_T(\mathbb{R}, H^\alpha)$. Assumption (1) gives

$$\|f(u_1) - f(u_2)\| \leq L_\Gamma\|u_1 - u_2\|_\alpha, \quad \forall u_{1,2} \in H^\alpha, \quad \|u_{1,2}\|_\alpha \leq \Gamma. \tag{2.18}$$

So we get

$$\|f(u(t))\| \leq L_\Gamma\|u(t)\|_\alpha + \|f(0)\| \leq \Gamma L_\Gamma + \|f(0)\|,$$

which implies $f(u) \in L^\infty_T(\mathbb{R}, H)$. Then the argument of Remark 2.7 shows that $\dot{u} \in L^2_T(\mathbb{R}, H)$. Moreover, (2.18) yields

$$\|f(u_1) - f(u_2)\|_2 \le L_\Gamma \|u_1 - u_2\|_{\alpha,2} \qquad \forall u_{1,2} \in L^\infty_T(\mathbb{R}, H^\alpha), \quad \|u_{1,2}\|_{\alpha,\infty} \le \Gamma. \tag{2.19}$$

Next using the notation of the proof of Lemma 2.2, from (2.1) we get the system

$$\ddot{u}_j + \delta \dot{u}_j + \lambda_j u_j + (f(u), w_j) = 0, \quad \forall j \in \mathbb{N}.$$

Since

$$\left\| \sum_{j=1}^n u_j(t) w_j \right\|_\alpha \le \|u(t)\|_\alpha \le \|u\|_{\alpha,\infty} \le \Gamma, \tag{2.20}$$

we obtain that $f\left(\sum_{j=1}^n u_j w_j\right) \in L^\infty_T(\mathbb{R}, H^\alpha)$ for any $n \in \mathbb{N}$. Then

$$\begin{aligned} \sum_{j=1}^n \delta \|\dot{u}_j\|_2^2 &= -\int_0^T \left(f(u(t)), \sum_{j=1}^n \dot{u}_j(t) w_j \right) dt \\ &= -\int_0^T \left(f\left(\sum_{j=1}^n u_j(t) w_j \right), \sum_{j=1}^n \dot{u}_j(t) w_j \right) dt \\ &\quad + \int_0^T \left(f\left(\sum_{j=1}^n u_j(t) w_j \right) - f(u(t)), \sum_{j=1}^n \dot{u}_j(t) w_j \right) dt \\ &\le \left(f\left(\sum_{j=1}^n u_j w_j \right) - f(u) \right)_2 \sqrt{\sum_{j=1}^n \|\dot{u}_j\|_2^2}. \end{aligned}$$

Hence (2.19) and (2.20) give

$$\delta^2 \sum_{j=1}^n \|\dot{u}_j\|_2^2 \le \left| f\left(\sum_{j=1}^n u_j w_j \right) - f(u) \right|_2^2 \le L_\Gamma^2 \left| \sum_{j=1}^n u_j w_j - u \right|_{\alpha,2}^2. \tag{2.21}$$

Since $\sum_{j=1}^n u_j w_j \to u$ in $L^2_T(\mathbb{R}, H^\alpha)$ and $\sum_{j=1}^n \dot{u}_j w_j \to \dot{u}$ in $L^2_T(\mathbb{R}, H)$ as $n \to \infty$, relation (2.21) implies that

$$\|\dot{u}\|_2^2 = \sum_{j=1}^\infty \|\dot{u}_j\|_2^2 = 0.$$

Hence, $\dot{u} = 0$. The proof is complete. ☐

Example 2.10. A standard p.d.e. which fits into the framework of Theorem 2.9 is an equation of a damped buckled beam [9]

$$
\begin{gathered}
\ddot{u} + \delta \dot{u} + u'''' + \left[\varkappa_1 - \varkappa_2 \left(\int_0^1 u'^2(\xi, \cdot)\, d\xi \right) \right] u'' = 0, \\
u(0,\cdot) = u(1,\cdot) = u''(0,\cdot) = u''(1,\cdot) = 0,
\end{gathered}
\tag{2.22}
$$

where $\varkappa_1$ and $\varkappa_2$ are constants. Now we take $H = L^2(0,1)$, $\alpha = 1/2$, so $H^{1/2} = H_0^2(0,1)$, and

$$
\begin{gathered}
Au = u''''(x), \quad f(u) = \left[\varkappa_1 - \varkappa_2 \left(\int_0^1 u'^2(\xi, \cdot)\, d\xi \right) \right] u'', \\
F(u) = -\frac{1}{2} \int_0^1 \varkappa_1 u'^2(\xi)\, d\xi + \varkappa_2 \frac{1}{4} \left(\int_0^1 u'^2(\xi)\, d\xi \right)^2 .
\end{gathered}
$$

Example 2.11. We finish this section with another partial differential equation different from (2.17) and (2.22) of the form

$$
\begin{gathered}
\ddot{u} + \delta \dot{u} + u'''' + \left[\varkappa_1 - \varkappa_2 \left(\int_0^1 u'^2(\xi, \cdot)\, d\xi \right) \right] u'' + \eta u' = 0, \\
u(0,\cdot) = u(1,\cdot) = u''(0,\cdot) = u''(1,\cdot) = 0,
\end{gathered}
\tag{2.23}
$$

where $\varkappa_1, \varkappa_2 > 0$, $\eta > 0$ are constants. Problem (2.23) models the oscillation of a buckled elastic panel excited by a fluid flow over its upper surface [3, 10]. Now we again take $H = L^2(0,1)$ with the usual integral norm $\|\cdot\|$, $H^{1/2} = H_0^2(0,1)$ and $\|u\|_{1/2} = \|u''\|$. Then any weak T-periodic solution $u \in L_T^\infty(\mathbb{R}, H^{1/2})$ of (2.23) satisfies the relations

$$
\begin{gathered}
\delta \|\dot{u}\|_2^2 + \eta (u', \dot{u})_2 = 0, \\
\|\dot{u}\|_2^2 = \|u''\|_2^2 - \varkappa_1 \|u'\|_2^2 + \varkappa_2 \int_0^T \|u'(\cdot, t)\|^4\, dt.
\end{gathered}
\tag{2.24}
$$

Since

$$
\|u'\|_2^4 \le T \int_0^T \|u'(\cdot, t)\|^4\, dt, \quad \pi \|u'\|_2 \le \|u''\|_2,
$$

from (2.24) we derive

$$
\begin{aligned}
\pi^2 \|u'\|_2^2 + \frac{\varkappa_2}{T} \|u'\|_2^4 &\le \|u''\|_2^2 + \varkappa_2 \int_0^T \|u'(\cdot, t)\|^4\, dt \\
&= \varkappa_1 \|u'\|_2^2 + \|\dot{u}\|_2^2 \le \varkappa_1 \|u'\|_2^2 + \frac{\eta^2}{\delta^2} \|u'\|_2^2 .
\end{aligned}
\tag{2.25}
$$

Hence

$$\|u'\|_2^2 \leq \frac{T}{\varkappa_2}\left(\varkappa_1 + \frac{\eta^2}{\delta^2} - \pi^2\right) = TK_1.$$

So we get

Theorem 2.12. *If* $\eta^2 \leq \delta^2(\pi^2 - \varkappa_1)$, *then the only periodic weak solution* $u \in L_T^\infty(\mathbb{R}, H^{1/2})$ *of* (2.23) *is the zero one,* $u = 0$.

Note that Theorem 2.12 is consistent with Proposition 2.1 of [10]. Now, let

$$\eta^2 > \delta^2(\pi^2 - \varkappa_1). \tag{2.26}$$

Then (2.25) implies

$$\|u''\|_2^2 \leq \left(\varkappa_1 + \frac{\eta^2}{\delta^2}\right)\|u'\|_2^2 \leq T\left(\varkappa_1 + \frac{\eta^2}{\delta^2}\right)K_1 = TK_2,$$

$$\|\dot{u}\|_2^2 \leq \frac{\eta^2}{\delta^2}\|u'\|_2^2 \leq T\frac{\eta^2}{\delta^2}K_1 = TK_3.$$

Hence

$$\|u''\|_2^2 \leq TK_2, \quad \|\dot{u}\|_2^2 \leq TK_3. \tag{2.27}$$

Since $u \in L_T^2(\mathbb{R}, H_0^2(0,1))$ and $\dot{u} \in L_T^2(\mathbb{R}, L^2(0,1))$, Theorem 4 from [4, p. 288] gives $u \in C_T^0(\mathbb{R}, H_0^1(0,1))$ along with

$$\max_{t\in\mathbb{R}} \|u'(\cdot,t)\|^2 \leq \frac{1}{T}\|u'\|_2^2 + \|u''\|_2^2 + \|\dot{u}\|_2^2 \leq K_1 + TK_2 + TK_3 = K_4. \tag{2.28}$$

Let us put

$$X := \left\{u \mid u \in L_T^2(\mathbb{R}, H_0^2(0,1)),\ \dot{u} \in L_T^2(\mathbb{R}, L^2(0,1))\right\}.$$

Then X is a Hilbert space with the norm

$$|u|_X := \|u''\|_2 + \|\dot{u}\|_2.$$

Again from Theorem 4 of [4, p. 288], we have $X \subset C_T^0(\mathbb{R}, H_0^1(0,1))$ along with

$$\max_{\mathbb{R}} \|v'(\cdot,t)\|^2 \leq \left(\frac{1}{\pi^2 T} + 1\right)\|v''\|_2^2 + \|\dot{v}\|_2^2 \leq \left(\frac{1}{\pi^2 T} + 1\right)\|v\|_X^2 \tag{2.29}$$

for any $v \in X$.

We put

$$Au = u''''(x) + \varkappa_1 u'', \quad f(u) = \varkappa_2 g(u) + \eta Bu,$$

$$g(u) := -\|u'\|^2 u'', \quad Bu := u'.$$

For any $v, w \in X$ satisfying (2.27), we derive

$$\|Bv - Bw\|_2 = \|v' - w'\|_2 \leq \|v'' - w''\|_2/\pi \leq |v - u|_X/\pi,$$

and by (2.28) and (2.29)

$$\begin{aligned}
\|g(v)-g(w)\|_2 &= \|\|v'\|^2 v'' - \|w'\|^2 w''\|_2 \\
&\le \|\|v'\|^2(v''-w'')\|_2 + \|w''(\|v'\|^2 - \|w'\|^2)\|_2 \\
&= \left(\int_0^T \int_0^1 \left[\int_0^1 v'^2(\xi,t)\,d\xi\right]^2 (v''(x,t)-w''(x,t))^2\,dx\,dt\right)^{1/2} \\
&\quad + \left(\int_0^T \int_0^1 w''^2(x,t)\left[\int_0^1 (v'^2(\xi,t)-w'^2(\xi,t))\,d\xi\right]^2 dx\,dt\right)^{1/2} \\
&\le K_4\|v''-w''\|_2 \\
&\quad + \left(\int_0^T \int_0^1 w''^2(x,t)\int_0^1 (v'(\xi,t)+w'(\xi,t))^2\,d\xi \right. \\
&\quad \left. \times \int_0^1 (v'(\xi,t)-w'(\xi,t))^2\,d\xi\,dx\,dt\right)^{1/2} \\
&\le K_4|v-w|_X \\
&\quad + 2\sqrt{K_4\left(\frac{1}{\pi^2}+T\right)}|v-w|_X \left(\frac{1}{T}\int_0^T \int_0^1 w''^2(x,t)\,dx\,dt\right)^{1/2} \\
&\le \left(K_4 + 2\sqrt{K_2K_4\left(\frac{1}{\pi^2}+T\right)}\right)|v-w|_X = K_5|v-w|_X.
\end{aligned}$$

So $f : X \to L^2_T(\mathbb{R}, H)$ has the Lipschitz constant

$$L = \varkappa_2 K_5 + \frac{\eta}{\pi}$$

on the subset $\mathcal{K} \subset X$ of all $u \in X$ satisfying (2.27).

It is clear that $\mathcal{K}$ is a closed and convex subset of X. Then it is well-known (see Lemma 3.5 in [2]) that there existsa retraction $R : X \to \mathcal{K} \subset X$ with a global Lipschitz constant 1. Using R, we modify f outside of $\mathcal{K}$ as follows

$$\tilde{f}(u) := f(Ru).$$

Clearly $\tilde{f} : X \to L^2_T(\mathbb{R}, H)$ is a globally Lipschitzian continuous with the constant L. According to the above construction, each T-periodic weak solution $u \in L^\infty_T(\mathbb{R}, H^{1/2})$ of (2.23) is also a T-periodic weak solution of the modified one with $\tilde{f}$ in place of f. Furthermore, now $\lambda_j = -j^2\pi^2(\varkappa_1 - j^2\pi^2)$ and let us assume for simplicity that $\varkappa_1 < \pi^2$ (see (2.26)). Then we can apply Remarks 2.6 and 2.7 (see

(2.12) and (2.16)). So if u is nonstationary, then

$$\left(\varkappa_2 K_5 + \frac{\eta}{\pi}\right)\left(\sqrt{\frac{2\pi + \sqrt{4\pi^2 + \delta^2 T^2}}{4\delta^2 \pi}} + \frac{1}{\delta}\right) \geq 1. \tag{2.30}$$

Clearly, relation (2.30) makes sense if

$$\left(\varkappa_2 \left(K_1 + \frac{2}{\pi}\sqrt{K_1 K_2}\right) + \frac{\eta}{\pi}\right)\frac{2}{\delta} < 1. \tag{2.31}$$

We do not express K_5 in terms of parameters $\varkappa_1, \varkappa_2, \eta$, δ and period T, since it is an awkward formula. But we note that for $\eta = \delta\sqrt{\pi^2 - \varkappa_1}$ (see (2.26)), we get $K_1 = K_2 = K_3 = K_4 = K_5 = 0$, and (2.31) becomes

$$3\pi^2/4 < \varkappa_1. \tag{2.32}$$

Hence if (2.32) holds, then there is a unique

$$\eta_0 = \eta(\varkappa_1, \varkappa_2, \delta) > \delta\sqrt{\pi^2 - \varkappa_1}$$

solving equation (see (2.31))

$$\left(\varkappa_2 \left(K_1 + \frac{2}{\pi}\sqrt{K_1 K_2}\right) + \frac{\eta_0}{\pi}\right)\frac{2}{\delta} = 1. \tag{2.33}$$

We note that the left-hand side of (2.33) is increasing with respect to η_0. Then by fixing η such that $\delta\sqrt{\pi^2 - \varkappa_1} < \eta < \eta_0$, there is a unique $T_0 > 0$ solving the equation (see (2.30))

$$\left(\varkappa_2 K_5 + \frac{\eta}{\pi}\right)\left(\sqrt{\frac{2\pi + \sqrt{4\pi^2 + \delta^2 T_0^2}}{4\delta^2 \pi}} + \frac{1}{\delta}\right) = 1. \tag{2.34}$$

We note that the left-hand side of (2.34) is increasing with respect to T_0.

Summarising, we have the following result.

Theorem 2.13. *If $3\pi^2/4 < \varkappa_1 < \pi^2$ and $\delta\sqrt{\pi^2 - \varkappa_1} < \eta < \eta_0$, then the period T of any weak nonstationary T-periodic solution $u \in L_T^\infty(\mathbb{R}, H^{1/2})$ of* (2.23) *satisfies inequality $T \geq T_0$. Here, η_0 and T_0 are the unique positive solutions of* (2.33) *and* (2.34)*, respectively.*

For instance if $\varkappa_1 = 8$, $\varkappa_2 = 5$ and $\delta = 100$, then equation (2.33) gives $\eta_0 = 152.667$. So we get $136.733 < \eta < 152.667$. Taking $\eta = 138$, we find from (2.34) that $T_0 = 0.122$. Consequently, we obtain $T \geq 0.122$ in Theorem 2.13.

3. Nonresonant $\mathbb{Z}_p$-symmetric autonomous ordinary differential equation

Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be an orthonormal matrix, i. e., $S^* = S^{-1}$, such that $S^p = I$ for some $p \in \mathbb{N}$. In this section, we deal with the ordinary differential equation

$$\dot{x} = g(x) \tag{3.1}$$

under the following assumptions:

(H1) $g : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz continuous, i. e., there is a constant $L > 0$ such that $\|g(x) - g(y)\| \leq L\|x - y\|$ $\forall x, y \in \mathbb{R}^n$.
(H2) g is S-symmetric, i. e., $g(Sx) = Sg(x)$ $\forall x \in \mathbb{R}^n$.
(H3) $1 \notin \sigma(S)$, the spectrum of S.

We call (3.1) under conditions (H1)-(H3) a nonresonant $\mathbb{Z}_p$-symmetric autonomous ordinary differential equation.

Remark 3.1. Assumptions (H1) and (H2) are reasonable, i. e., there are many g satisfying both (H1) and (H2). Indeed, let $\mathrm{Lip}\,(\mathbb{R}^n)$ be the space of all globally Lipschitz continuous mappings $h : \mathbb{R}^n \to \mathbb{R}^n$ endowed with the norm

$$\|h\|_{\mathrm{Lip}} := \inf\{L > 0 \mid L \text{ is the Lipschitz constant of } h\} + \|h(0)\|.$$

Then $\mathrm{Lip}\,(\mathbb{R}^n)$ becomes a Banach space. Moreover, we define a linear mapping $\mathcal{S}$: $\mathrm{Lip}\,(\mathbb{R}^n) \to \mathrm{Lip}\,(\mathbb{R}^n)$ by

$$(\mathcal{S}h)(x) := \frac{1}{p}\sum_{i=0}^{p-1} S^{-i}h(S^i x).$$

Clearly, $\mathcal{S}$ is a continuous projection of $\mathrm{Lip}\,(\mathbb{R}^n)$ onto $\mathrm{Lip}_S\,(\mathbb{R}^n)$, the space of all S-symmetric elements of $\mathrm{Lip}\,(\mathbb{R}^n)$. Furthermore, if h has a Lipschitz constant $L > 0$, then $\mathcal{S}h$ has also a Lipschitz constant L. Finally we note

$$a_0 I + a_1 S + \cdots + a_{p-1} S^{p-1} \in \mathrm{Lip}_S\,(\mathbb{R}^n)$$

for any $a_j \in \mathbb{R}$, $0 \leq j \leq p-1$. Hence $\mathrm{Lip}_S\,(\mathbb{R}^n) \neq \emptyset$.

Definition 3.2. Let $T > 0$. We call any $x \in C^1(\mathbb{R}, \mathbb{R}^n)$ satisfying (3.1) and

$$x(t + T) = Sx(t) \quad \forall t \in \mathbb{R} \tag{3.2}$$

the T-S-symmetric solution of (3.1).

We note that any $x(t)$ satisfying (3.2) is also pT-periodic. Moreover, (H2) and (H3) imply $g(0) = Sg(0)$, so $g(0) = 0$. Thus $x(t) = 0$ is a trivial T-S-symmetric solution of (3.1) for any $T > 0$, which is the only constant function satisfying (3.2).

We put

$$C^r_{T,S} := \left\{x \in C^r(\mathbb{R}, \mathbb{R}^n) \mid x(t) \text{ satisfies } (3.2)\right\}$$

with the usual maximum norm $\|x\|_r$ on the interval $[0, T]$. We note that (3.2) implies

$$\|x^{(r)}(t + T)\| = \|Sx^{(r)}(t)\| = \|x^{(r)}(t)\|.$$

So norms $\|\cdot\|_r$ are well-defined.

First we solve the linear boundary value problem

$$\dot{x} = h, \quad x \in C^1_{T,S}, \quad h \in C^0_{T,S}. \tag{3.3}$$

It is easy to find its solution

$$x(t) = (S - I)^{-1} \int_0^T h(s)\, ds + \int_0^t h(s)\, ds.$$

So we arrive at the following.

Lemma 3.3. *For any $h \in C^0_{T,S}$, problem* (3.3) *has a unique solution $x \in C^1_{T,S}$ with an estimate*

$$\|x\|_0 \le [\|(I - S)^{-1}\| + 1]\, \|h\|_0 T.$$

The Banach fixed point theorem together with Lemma 3.3 give

Theorem 3.4. *If $[\|(I-S)^{-1}\|+1]\, TL < 1$, then $x = 0$ is the only T-S-symmetric solution of* (3.1).

When $S = -I$, the T-S-symmetric solutions are called T-antiperiodic ones [1]. Then condition (H3) holds and g is just an odd mapping. Hence Theorem 3.4 implies

Theorem 3.5. *Let* (H1) *hold and g be odd. If x is a nonzero T-antiperiodic solution of* (3.1)*, i. e., $x(t + T) = -x(t)$ $\forall t \in \mathbb{R}$, then $T \ge \frac{2}{3L}$.*

Example 3.6. Now let $n = 2$, $\mathbb{R}^2 \simeq \mathbb{C}$, and $Sz = e^{\imath 2\pi/p} z$, $z \in \mathbb{C}$, $p \in \mathbb{N}$, $p \ge 3$. So $S : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation by the angle $2\pi/p$. Then $|z - Sz| = 2 \sin \frac{\pi}{p} |z|$, and

$$\|(I - S)^{-1}\| = \frac{1}{2 \sin(\pi/p)}.$$

Theorem 3.4 implies

Theorem 3.7. *Let $Sz = e^{\imath 2\pi/p} z$, $z \in \mathbb{C}$, $p \in \mathbb{N}$, $p \ge 3$, be a rotation in the plane by the angle $2\pi/p$. If x is a nonzero T-S-symmetric solution of* (3.1) *for g satisfying* (H1), (H2)*, then*

$$T \ge \frac{2L^{-1} \sin(\pi/p)}{1 + 2 \sin(\pi/p)}.$$

Such result as in Theorem 3.7 should hold for a general S, since from $S^p = I$ and (H3) it follows that $\sigma(S)$ consists of the unit roots, i. e.,

$$\sigma(S) \subset \left\{ e^{2\pi k \imath/p} \mid k = 1, 2, \dots, p - 1 \right\}.$$

Indeed, it is not difficult to show [11] that conditions $S^* = S^{-1}$, $S^p = I$, $S \ne -I$ and $1 \notin \sigma(S)$ imply both $p \ge 3$ and the existence of an orthonormal basis

$$\{e_{11}, e_{12}, e_{21}, e_{22}, \dots, e_{k1}, e_{k2}, e_{k+1}, \dots, e_n\}$$

of $\mathbb{R}^n$ such that on each invariant subspace $V_j := [e_{j1}, e_{j2}]$, $1 \leq j \leq k$, matrix S, $S : V_j \to V_j$ acts as a rotation by the angle $2\pi r_j/p$, $1 \leq r_j < p$, $r_j \neq p/2$. While $Se_j = -e_j, k+1 \leq j \leq n$, naturally, only for even p. We can suppose that

$$\sin(\pi r_1/p) = \min\left\{\sin(\pi r_j/p) \mid j = 1, 2, \ldots, k\right\}.$$

Then

$$\|(I - S)^{-1}\| = \frac{1}{2\sin(\pi r_1/p)}.$$

Summarising, we get

Theorem 3.8. *When assumptions* (H1)-(H3) *and $S \neq -I$ are satisfied, then*

$$T \geq \frac{2L^{-1}\sin(\pi r_1/p)}{1 + 2\sin(\pi r_1/p)}$$

for any nonzero T-S-symmetric solution of (3.1).

We note that any nonzero T-S-symmetric solution of (3.1) is nonconstant.

Remark 3.9. The existence and nonexistence of forced symmetric oscillations like (3.1) are studied in [8].

Remark 3.10. We note that if $S : \mathbb{R}^n \to \mathbb{R}^n$ is a matrix with $S^p = I$ for some $p \in \mathbb{N}$. Then there is a scalar product $(\cdot, \cdot)$ for which S is unitary, i. e., $(Su, Sv) = (u, v)$ $\forall u, v \in \mathbb{R}^n$. Indeed, let $\langle \cdot, \cdot \rangle$ be any scalar product on $\mathbb{R}^n$. Then we take [11]

$$(u, v) = \frac{1}{p}\sum_{j=0}^{p-1}\langle S^i u, S^i v\rangle.$$

4. Resonant $\mathbb{Z}_p$-symmetric autonomous ordinary differential equation

In this section, we proceed to study (3.1) only under assumptions (H1) and (H2), and we call (3.1) a resonant $\mathbb{Z}_p$-symmetric autonomous ordinary differential equation, since $1 \in \sigma(S)$. Then there is an orthogonal decomposition

$$\mathbb{R}^n = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$$

such that $S : \mathbb{R}^{n_{1,2}} \to \mathbb{R}^{n_{1,2}}$ and $S/\mathbb{R}^{n_1} = I$, $1 \notin \sigma(\tilde{S})$ for $\tilde{S} := S/\mathbb{R}^{n_2}$.

We split (3.1) as

$$\begin{aligned} \dot{x}_1 &= g_1(x_1 + x_2), \\ \dot{x}_2 &= g_2(x_1 + x_2), \end{aligned} \tag{4.1}$$

for $g_i(x) = P_i g(x)$, where $P_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ are orthogonal projections, and $x_i \in \mathbb{R}^{n_i}$, $i = 1, 2$. Clearly $g_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ are globally Lipschitz continuous with a constant

L. Assumption (H2) implies

$$g_1(x_1 + x_2) = g_1(x_1 + \tilde{S}x_2)\,,$$
$$\tilde{S}g_2(x_1 + x_2) = g_2(x_1 + \tilde{S}x_2).$$

Hence $g_2(x_1) = 0$, $\forall x_1 \in \mathbb{R}^{n_1}$. Now T-S-symmetric solutions of (4.1) are given by conditions

$$x_1(t + T) = x_1(t), \quad x_2(t + T) = \tilde{S}x_2(t). \tag{4.2}$$

We see that any constant functions satisfying (4.2) are $x_1(t) = \text{const}$ and $x_2(t) = 0$.

As in Section 2, we further split

$$x_1(t) = u(t) + c, \quad u \in L^2_{T,0}(\mathbb{R}, \mathbb{R}^{n_1}), \quad c \in \mathbb{R}^{n_1},$$

and decompose (4.1) as

$$\begin{aligned} \dot{u}(t) &= [g_1(u(t) + c + x_2(t)) - g_1(c)] \\ &\qquad - \frac{1}{T}\int_0^T [g_1(u(s) + c + x_2(s)) - g_1(c)]\, ds, \\ \dot{x}_2(t) &= g_2(u(t) + c + x_2(t)), \end{aligned} \tag{4.3}$$

and

$$0 = \int_0^T g_1(u(s) + c + x_2(s))\, ds. \tag{4.4}$$

Let us put

$$L^2_{T,\tilde{S}}(\mathbb{R}, \mathbb{R}^{n_2}) := \left\{h \in L^2_{\text{loc}}(\mathbb{R}, \mathbb{R}^{n_2}) \mid h(t + T) = \tilde{S}h(t)\right\},$$
$$W^{1,2}_{T,\tilde{S}}(\mathbb{R}, \mathbb{R}^{n_2}) := W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^{n_2}) \cap L^2_{T,\tilde{S}}(\mathbb{R}, \mathbb{R}^{n_2}).$$

We need the following result similarly to Lemmas 2.2 and 3.3:

Lemma 4.1. (a) *For any $h \in L^2_{T,\tilde{S}}(\mathbb{R}, \mathbb{R}^{n_2})$ there is a unique function $x_2 \in W^{1,2}_{T,\tilde{S}}(\mathbb{R}, \mathbb{R}^{n_2})$ satisfying the equation $\dot{x}_2 = h$ and the estimate*

$$\|x_2\|_2 \leq (\|(\tilde{S} - I)^{-1}\| + 1)T\|h\|_2,$$

where $\|\cdot\|_2$ is the usual integral norm in $L^2((0, T), \mathbb{R}^{n_2})$.

(b) *For any $h \in L^2_{T,0}(\mathbb{R}, \mathbb{R}^{n_1})$ there is a unique $u \in W^{1,2}_{T,0}(\mathbb{R}, \mathbb{R}^{n_1})$ satisfying $\dot{u} = h$ along with the estimate*

$$\|u\|_2 \leq \frac{T}{2\pi}\|h\|_2.$$

Then from (4.3) we obtain

$$\|u\|_2^2 + \|x_2\|_2^2 \leq \left(\frac{1}{4\pi^2} + (\|(\tilde{S} - I)^{-1}\| + 1)^2\right) T^2 L^2 \left(\|u\|_2^2 + \|x_2\|^2\right).$$

So if

$$\sqrt{\frac{1}{4\pi^2} + (\|(\tilde{S} - I)^{-1}\| + 1)^2} TL < 1,$$

then $u = 0$, $x_2 = 0$. Using results of Section 3, we arrive at the following

Theorem 4.2. *The number/period T of any nonconstant solution of* (4.1)*) satisfying* (4.2) *fulfils the inequality*

$$T \geq L^{-1} \frac{2\pi}{\sqrt{9\pi^2 + 1}}$$

for $\tilde{S} = -I$, and

$$T \geq L^{-1} \frac{2\pi \sin(\pi \tilde{r}_1/p)}{\sqrt{\sin^2(\pi \tilde{r}_1/p) + \pi^2(1 + 2\sin(\pi \tilde{r}_1/p))^2}}$$

for $\tilde{S} \neq -I$, where $\tilde{r}_1$ is defined for $\tilde{S}$ as in Theorem 3.8 in place of S.

Remark 4.3. Theorems 3.5, 3.8, 4.2 and Remark 3.10 completely solve the minimal period problem for $\mathbb{Z}_p$-symmetric autonomous ordinary differential equations, i. e., lower bounds for T of T-S-symmetric solutions are derived for (3.1) satisfying assumptions (H1) and (H2) for a matrix $S : \mathbb{R}^n \to \mathbb{R}^n$ with $S^p = I$ for some $p \in \mathbb{N}$.

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