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ON UNIQUE SOLVABILITY OF QUADRATIC INTEGRAL EQUATIONS WITH LINEAR MODIFICATION OF THE ARGUMENT

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Abstract. In this paper, we investigate the existence of a unique solution on a semi-infinite interval for quadratic integral equations with linear modification of the argument in Fréchet spaces using a nonlinear alternative of Leray-Schauder type for contraction maps in Fréchet spaces.

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Keywords: quadratic integral equations, linear modification of the argument, existence and uniqueness, fixed-point, contraction maps, nonlinear alternative of Leray-Schauder type, Fréchet space

1. INTRODUCTION

In this paper, we establish the existence of the unique solution, defined on a semi-infinite interval $J := [0, +\infty)$ for the following quadratic integral equations with a linear modification of the argument

$$x(t) = f(t) + (Ax)(t) \int_0^T u(t, s, x(s), x(\alpha s))ds, \quad t \in J, \tag{1.1}$$

and

$$x(t) = f(t) + g(t, x(t)) \int_0^T u(t, s, x(s), x(\alpha s))ds, \quad t \in J, \tag{1.2}$$

where $f : J \to \mathbb{R}$, $g : J \times \mathbb{R} \to \mathbb{R}$, $u : J \times J_T \times \mathbb{R}^2 \to \mathbb{R}$ are given functions, $0 < \alpha < 1$, $J_T := [0, T]$, and $A : C(J; \mathbb{R}) \to C(J; \mathbb{R})$ is an appropriate operator. Here $C(J; \mathbb{R})$ denotes the space of continuous functions $x : J \to \mathbb{R}$.

Integral equations arise naturally in many applications in describing numerous real world problems (see, for instance, the books [1, 2, 8, 10, 20] and references therein). Also quadratic integral equations have many useful applications in describing numerous events and problems of the real world. For example, quadratic integral equations are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. Especially, the so-called quadratic integral equation of Chandrasekher type can be very often encountered in many applications; see for instance the book by Chandrasekher [7] and the research papers by Banas *et al* [3], Benchohra and Darwish [4], Darwish [9], Hu *et al* [14], Kelley [15], Leggett [17] and Stuart [21] and references therein.

The study of differential equations with a modified argument is relatively new, it was initiated only in the past thirty years or so. These equations arise in the modelling of problems from the natural and social sciences such as biology, economics and physics. A special class is represented by the differential equations with affine modification of the argument which can be differential equations with linear modification of the argument or differential equation with delay. For more information and results concerning these equations, see [6, 11, 13, 16, 18, 19].

More recently, Caballero *et al* [5] investigated the so-called quadratic integral equation of the Volterra type with linear modification of the argument, Volterra counter part of equation (1.1), and proved the existence of monotonic solutions in $C([0,1],\mathbb{R})$. There equation can be considered with connection to the following Cauchy problem [5, 19]:

$$\begin{aligned} &x'(t) = u(t,s,x(t),x(\lambda t)), \quad t \in [0,1],\ 0<\lambda<1,\\ &x(0) = u_0. \end{aligned}$$

In this paper, we investigate the question of unique solvability of equation (1.1) and (1.2). Motivated by the previous papers considered for integral equations on a bounded interval, we extend here these results to semi-infinite intervals for a class of quadratic integral equations. The method we are going to use is to reduce the existence of the unique solution for the quadratic integral equation (1.1) to the search for the existence of the unique fixed-point of an appropriate operator on the Fréchet space $C(J;\mathbb{R})$ by applying a nonlinear alternative of Leray–Schauder type for contraction maps due to Frigon and Granas [12].

2. Preliminaries

We introduce notations, definitions, and theorems which are used throughout this paper.

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $M_n > 0$ such that

$$\|y\|_n \le M_n \quad \text{for every } y \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows: For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by the formula

$$x \sim_n y \quad \text{if and only if} \quad \|x-y\|_n = 0 \qquad \text{for all } x, y \in X.$$

We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: For every $x \in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we defined $Y^n = \{[x]_n : x \in Y\}$. We denote $\overline{Y^n}$, $\mathrm{Int}_n(Y^n)$, and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|$ in X^n. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies

$$\|x\|_1 \le \|x\|_2 \le \|x\|_3 \le \cdots \quad \text{for every } x \in X.$$

Definition 1 ([12]). A function $f : X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in (0,1)$ such that

$$\|f(x) - f(y)\|_n \le k_n \|x - y\|_n \quad \text{for all } x, y \in X.$$

Theorem 2.1 ([12]). *Suppose that U is an open subset of a Fréchet space X, $0 \in U$, and $F : \overline{U} \to X$ is a contraction such that $F(\overline{U})$ is bounded. Then either,*

(C1) *F has a unique fixed point in $\overline{U}$; or*
(C2) *there exist $\lambda \in (0,1)$ and $u \in \partial U$ with the property $u = \lambda F(u)$.*

3. Main theorem

In this section, we will study equation (1.1) assuming that the following assumptions are satisfied:

(a_1) $f : J \to \mathbb{R}$ is a continuous function.
(a_2) For each $n \in \mathbb{N}$ there exists $L_n > 0$ such that

$$|(Ax)(t) - (A\overline{x})(t)| \le L_n |x(t) - \overline{x}(t)| \quad \text{for each } x, \overline{x} \in C(J;\mathbb{R}) \text{ and } t \in [0,n].$$

(a_3) There exist nonnegative constants a and b such that

$$|(Ax)(t)| \le a + b|x(t)|$$

holds for each $x \in C(J;\mathbb{R})$ and $t \in J$.
(a_4) $u : J \times J_T \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and, for each $n \in \mathbb{N}$, there exists a constant $L_n^* > 0$ such that

$$|u(t,s,x,y) - u(t,s,\overline{x},\overline{y})| \le L_n^*(|x - \overline{x}| + |y - \overline{y}|)$$

holds for all $t \in [0,n]$, $s \in J_T$, and $x, y, \overline{x}, \overline{y} \in \mathbb{R}$.
(a_5) There exists a continuous nondecreasing function $\psi : J^2 \to (0,\infty)$ and $p \in C(J;\mathbb{R}_+)$ such that

$$|u(t,s,x,y)| \le p(s)\psi(|x|,|y|)$$

for each $(t,s) \in J \times J_T$ and $x, y \in \mathbb{R}$, and, moreover, there exist constants $M_n \in J$, $n \in \mathbb{N}$, such that

$$\frac{M_n}{\|f\|_n + T(a + bM_n)\psi(M_n, M_n)p^*} > 1 \tag{3.1}$$

holds for all $n \in \mathbb{N}$, where $p^* = \sup\{p(s) : s \in J_T\}$.

Theorem 3.1. *Let the assumptions* (a_1)–(a_5) *be satisfied. If, in addition, the inequality*

$$2\,(a+b\,M_n)\,L_n^*\,T+TL_n\,\psi(M_n,M_n)p^*<1 \tag{3.2}$$

holds for all $n\in\mathbb{N}$*, then the equation* (1.1) *has a unique solution.*

Proof. For every $n\in\mathbb{N}$, we define in $C(J;\mathbb{R})$ the semi-norms by the formula

$$\|y\|_n:=\sup\{|y(t)|:t\in[0,n]\}.$$

Then $C(J;\mathbb{R})$ is a Fréchet space with the family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$.

Transform the problem (1.1) into a fixed-point problem. Consider the operator $\mathcal{F}:C(J;\mathbb{R})\to C(J;\mathbb{R})$ defined by the relation

$$(\mathcal{F}y)(t)=f(t)+(Ay)(t)\int_0^T u(t,s,y(s),y(\alpha s))ds,\quad t\in J.$$

Let y be a possible solution of the problem (1.1). For given $n\in\mathbb{N}$ and $t\le n$, in view of (a_1), (a_2), and (a_5), we have

$$\begin{aligned}|y(t)|&\le|f(t)|+|(Ay)(t)|\int_0^T|u(t,s,y(s),y(\alpha s))|ds\\&\le|f(t)|+(a+b|y(t)|)\int_0^T p(s)\psi(|y(s)|,|y(\alpha s)|)ds\\&\le\|f\|_n+T(a+b\|y\|_n)\psi(\|y\|_n,\|y\|_n)p^*\end{aligned}$$

and thus

$$\frac{\|y\|_n}{\|f\|_n+T(a+b\|y\|_n)\psi(\|y\|_n,\|y\|_n)p^*}\le 1.$$

From (3.1) it follows that $\|y\|_n\neq M_n$ for each $n\in\mathbb{N}$. Now, set

$$\Omega=\{y\in(J;\mathbb{R}):\|y\|_n<M_n\text{ for every }n\in\mathbb{N}\}.$$

Clearly, Ω is an open subset of $C(J;\mathbb{R})$. We shall show that $\mathcal{F}:\overline{\Omega}\to C(J;\mathbb{R})$ is a contraction operator. Indeed, consider $y,\overline{y}\in C(J;\mathbb{R})$, for each $t\in[0,n]$ and $n\in\mathbb{N}$, from (a_2)–(a_5) we get

$$\begin{aligned}&|(\mathcal{F}y)(t)-(\mathcal{F}\overline{y})(t)|\\&\quad\le\left|(Ay)(t)\int_0^T u(t,s,y(s),y(\alpha s))ds-(A\overline{y})(t)\int_0^T u(t,s,\overline{y}(s),\overline{y}(\alpha s))ds\right|\\&\quad\le\left|(Ay)(t)\int_0^T u(t,s,y(s),y(\alpha s))ds-(Ay)(t)\int_0^T u(t,s,\overline{y}(s),\overline{y}(\alpha s))ds\right|\\&\qquad+\left|(Ay)(t)\int_0^T u(t,s,\overline{y}(s),\overline{y}(\alpha s))ds-(A\overline{y})(t)\int_0^T u(t,s,\overline{y}(s),\overline{y}(\alpha s))ds\right|\end{aligned}$$

$$\begin{aligned}
&\le |(Ay)(t)| \int_0^T |u(t,s,y(s),y(\alpha s)) - u(t,s,\overline{y}(s),\overline{y}(\alpha s))|ds \\
&\quad + |(Ay)(t)-(A\overline{y})(t)| \int_0^T |u(t,s,\overline{y}(s),\overline{y}(\alpha s))|ds \\
&\le (a+b|y(t)|)\, L_n^* \int_0^T (|y(s)-\overline{y}(s)| + |y(\alpha s)-\overline{y}(\alpha s)|)ds \\
&\quad + L_n|y(t)-\overline{y}(t)| \int_0^T p(s)\psi(|\overline{y}(s)|,|\overline{y}(\alpha s)|)ds \\
&\le \left[2(a+bM_n)L_n^*T + TL_n\psi(M_n,M_n)p^*\right] \|y-\overline{y}\|_n.
\end{aligned}$$

Therefore,

$$\|\mathcal{F}y - \mathcal{F}\overline{y}\|_n \le \left[2(a+bM_n)L_n^*T + TL_n\psi(M_n,M_n)p^*\right] \|y-\overline{y}\|_n.$$

So by (3.2) the operator $\mathcal{F}$ is a contraction for all $n \in \mathbb{N}$. From the choice of Ω there is no $y \in \partial\Omega$ such that $y = \lambda\mathcal{F}(y)$ for some $\lambda \in (0,1)$. Then the statement (C2) in Theorem 2.1 does not hold. A consequence of the Leray–Schauder type nonlinear alternative from [12] yields that (C1) holds and thus we deduce that the operator $\mathcal{F}$ has a unique fixed-point y in $\overline{\Omega}$, which is a solution to Equation (1.1). This completes the proof. □

Theorem 3.2. *Let the following assumptions be satisfied:*

($\hat{a}_1$) $f\colon J \to \mathbb{R}$ *is a continuous function.*

($\hat{a}_2$) $g\colon J \times \mathbb{R} \to \mathbb{R}$ *is continuous and, for each* $n \in \mathbb{N}$, *there exists* $L_n > 0$ *such that*

$$|g(t,x) - g(t,\overline{x})| \le L_n|x-\overline{x}| \quad \text{for all } x,\overline{x} \in \mathbb{R} \text{ and } t \in [0,n].$$

($\hat{a}_3$) $u\colon J \times J_T \times \mathbb{R}^2 \to \mathbb{R}$ *is a continuous function and, for each* $n \in \mathbb{N}$, *there exists a constant* $L_n^* > 0$ *such that*

$$|u(t,s,x,y) - u(t,s,\overline{x},\overline{y})| \le L_n^*(|x-\overline{x}| + |y-\overline{y}|)$$

holds for all $(t,s) \in [0,n] \times J_T$ *and* $x,y,\overline{x},\overline{y} \in \mathbb{R}$.

($\hat{a}_4$) *There exists a continuous nondecreasing function* $\psi\colon J^2 \to (0,\infty)$ *and* $p \in C(J;\mathbb{R}_+)$ *such that*

$$|u(t,s,x,y)| \le p(s)\psi(|x|,|y|)$$

for each $(t,s) \in J \times J_T$ *and* $x,y \in \mathbb{R}$ *and, moreover, there exists constants* $M_n \in J$, $n \in \mathbb{N}$, *such that*

$$\frac{M_n}{\|f\|_n + T(L_nM_n + m_n)\psi(M_n,M_n)p^*} > 1$$

holds for all $n \in \mathbb{N}$, *where* $p^* = \sup\{p(s) : s \in J_T\}$ *and* $m_n = \sup\{g(t,0) : t \in [0,n]\}$.

If, in addition, the inequality

$$2(L_nM_n+m_n)TL_n^*+TL_n\psi(M_n,M_n)p^*<1 \tag{3.3}$$

holds, then Equation (1.2) *has a unique solution.*

Proof. The proof is similar to those of Theorem 3.1. □

Example 1. Consider the quadratic integral equation of the Urysohn type

$$x(t)=1+\frac{|x(t)|}{1+|x(t)|}\int_0^T\frac{ts}{t^3+1}\left(x(s)+x\left(\frac{s}{2}\right)\right)ds,\quad t\in J:=[0,+\infty). \tag{3.4}$$

Set $f(t)=1$ for each $t\in J$, $\psi(x,y)=x+y$ for all $x,y\geq 0$,

$$(Ax)(t)=\frac{|x(t)|}{1+|x(t)|},\qquad t\in J \text{ and } x\in C(J;\mathbb{R}),$$

and

$$u(t,s,x,y)=\frac{ts}{t^3+1}(x+y)$$

for all $(t,s)\in J\times J_T$ and $x,y\in\mathbb{R}$. It is clear that (3.4) is a particular case of equation (1.1). Let us show that conditions (a_1)–(a_5) hold.

For each $n\in\mathbb{N}$, $t\in[0,n]$, and $x,\overline{x}\in C(J;\mathbb{R}_+)$, we have

$$\begin{aligned}|(Ax)(t)-(A\overline{x})(t)| &= \left|\frac{|x(t)|}{1+|x(t)|}-\frac{|\overline{x}(t)|}{1+|\overline{x}(t)|}\right|=\left|\frac{|x(t)|-|\overline{x}(t)|}{(1+|x(t)|)(1+|\overline{x}(t)|)}\right| \\ &\leq \frac{|x(t)-\overline{x}(t)|}{(1+|x(t)|)(1+|\overline{x}(t)|)}\leq|x(t)-\overline{x}(t)|.\end{aligned}$$

Hence (a_2) is satisfied with $L_n=1$.

For each $t\in J$ and $x\in C(J;\mathbb{R})$, we have

$$|(Ax)(t)|=\left|\frac{|x(t)|}{1+|x(t)|}\right|\leq|x(t)|.$$

Hence (a_3) holds with $a=0$ and $b=1$.

For each $n\in\mathbb{N}$, $(t,s)\in[0,n]\times J_T$, and $x,y,\overline{x},\overline{y}\in\mathbb{R}$, we have

$$\begin{aligned}\left|u(t,s,x,\overline{x})-u(t,s,y,\overline{y})\right| &= \left|\frac{ts}{t^3+1}[(x+\overline{x})-(y+\overline{y})]\right| \\ &\leq \left|\frac{ts}{t^3+1}[(x-y)+(\overline{x}-\overline{y})]\right| \\ &\leq \frac{nT}{n^3+1}[|x-y|+|\overline{x}-\overline{y}|] \\ &\leq T\left[|x-y|+|\overline{x}-\overline{y}|\right].\end{aligned}$$

Hence (a_4) is satisfied with $L_n^*=T$.

For each $n \in \mathbb{N}$, $(t,s) \in [0,n] \times J_T$, and $x, y \in \mathbb{R}$, we have

$$\begin{aligned} |u(t,s,x,y)| &= \left| \frac{ts}{t^3+1}(x+y) \right| \\ &\le s(|x|+|y|) = s\psi(|x|,|y|). \end{aligned}$$

To conclude that (a_5) holds we shall show that (3.1) is satisfied. Indeed

$$\begin{aligned} \frac{M_n}{\|f\|_n + T(a+bM_n)\psi(M_n,M_n)p^*} > 1 &\iff \frac{M_n}{1+2T^2M_n^2} > 1 \\ &\iff 2T^2M_n^2 - M_n + 1 < 0. \end{aligned}$$

Notice that the last inequality holds for T such that $1 - 8T^2 > 0$, i. e.,

$$T < \frac{1}{2\sqrt{2}}. \tag{3.5}$$

Hence for $T > 0$ satisfying (3.5), there exists $M_n > 0$ satisfying (3.1).

Finally let us show that (3.2) is satisfied.

$$\begin{aligned} 2(a+bM_n)L_n^*T + TL_n\psi(M_n,M_n)p^* - 1 &= 2M_nT^2 + 2T^2M_n - 1 \\ &= 4M_nT^2 - 1. \end{aligned}$$

Hence (3.2) is satisfied for T or M_n satisfying $4M_nT^2 - 1 < 0$, i. e., for

$$0 < T < \frac{1}{2\sqrt{M_n}}$$

or $0 < M_n < \frac{1}{4}T^{-2}$. Consequently, if T satisfies the inequalities

$$0 < T < \min\left\{ \frac{1}{2\sqrt{M_n}}, \frac{1}{2\sqrt{2}} \right\},$$

then it follows from Theorem 3.1 that equation (3.4) has a unique solution.

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