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# Common fixed point theorems via implicit relations

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# COMMON FIXED POINT THEOREMS VIA IMPLICIT RELATIONS

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*Abstract.* We prove common fixed point theorems for four mappings satisfying implicit relations without decreasing assumption which improve theorems of [1, 13, 14]. We also prove common fixed point theorems for four mappings satisfying implicit relations which generalize theorems of [3, 4].

2000 *Mathematics Subject Classification:* 54H25, 47H10

*Keywords:* weakly compatible, common fixed point, metric space

## 1. INTRODUCTION

Let $S$ and $T$ be self-mappings of a metric space $(X,d)$. $S$ and $T$ are *commuting* if $STx = TSx$ for all $x \in X$. Sessa [15] defined $S$ and $T$ to be *weakly commuting* if, for all $x \in X$,

$$d(STx,TSx) \leq d(Tx,Sx). \tag{1.1}$$

Jungck [7] defined $S$ and $T$ to be *compatible*, as a generalization of the weakly commuting property, if

$$\lim_{n\to\infty} d(STx_n,TSx_n) = 0 \tag{1.2}$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

It is easy to show that "commuting" implies "weakly commuting," which, in turn, implies "compatible," and there are examples in the literature justifying that the inclusions are proper (see [7, 15]).

Jungck et al. [6] defined $S$ and $T$ to be *compatible* mappings *of type (A)* if

$$\lim_{n\to\infty} d(STx_n,T^2x_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(TSx_n,S^2x_n) = 0 \tag{1.3}$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Clearly, "weakly commuting" implies "compatible of type (A)". By [6], the converse is not true, and the two concepts of compatibility are independent.

Recently, Pathak and Khan [12] defined $S$ and $T$ to be *compatible* mappings *of type (B)*, as a generalization of compatible mappings of type (A), if

$$\begin{aligned} \lim_{n\to\infty} d(TSx_n, S^2x_n) &\le \frac{1}{2}\Big[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d(Tt, T^2x_n)\Big], \\ \lim_{n\to\infty} d(STx_n, T^2x_n) &\le \frac{1}{2}\Big[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d(St, S^2x_n)\Big] \end{aligned} \tag{1.4}$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Clearly, compatible mappings of type (A) are compatible mappings of type (B), but the converse is not true [9]. However, the notions of compatibility, compatibility of type (A), and compatibility of type (B) are equivalent to one another if $S$ and $T$ are continuous [12].

Pathak et al. [10] defined $S$ and $T$ to be *compatible* mappings *of type (P)* if

$$\lim_{n\to\infty} d(S^2x_n, T^2x_n) = 0 \tag{1.5}$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

The notions of compatibility, compatibility of type (A), and compatibility of type (P) are mutually equivalent if $S$ and $T$ are continuous [10].

Pathak et al. [11] defined $S$ and $T$ to be *compatible* mappings *of type (C)* as a generalization of compatible mappings of type (A) if

$$\begin{aligned} \lim_{n\to\infty} d(TSx_n, S^2x_n) &\le \frac{1}{3}\Big(\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d(Tt, S^2x_n) \\ &\quad + \lim_{n\to\infty} d(Tt, T^2x_n)\Big), \\ \lim_{n\to\infty} d(STx_n, T^2x_n) &\le \frac{1}{3}\Big(\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d(St, T^2x_n) \\ &\quad + \lim_{n\to\infty} d(St, S^2x_n)\Big) \end{aligned} \tag{1.6}$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Clearly, compatible mappings of type (A) are also compatible mappings of type (C), but the converse implication is not true [11]. However, the properties of compatibility of type (A) and compatibility of type (C) are mutually equivalent if $S$ and $T$ are continuous (see [11]).

## 2. PRELIMINARIES

**Definition 1** ([5])**.** $S$ and $T$ are said to be *weakly compatible* if they commute at their coincidence points, i. e., the equality $Su = Tu$ for some $u \in X$ implies that $STu = TSu$.

**Lemma 1** ([6, 7, 10–12])**.** *If $S$ and $T$ are compatible, or compatible of type* (A)*, or compatible of type* (P)*, or compatible of type* (B)*, or compatible of type* (C)*, then they are weakly compatible.*

The converse is not true in general, see [2].

**Definition 2** ([8])**.** $S$ and $T$ are said to be *$R$-weakly commuting* at a point $x \in X$ if for some $R > 0$

$$d(STx, TSx) \leq Rd(Tx, Sx). \tag{2.1}$$

**Definition 3** ([9])**.** $S$ and $T$ are said to be *pointwise $R$-weakly commuting* on $X$ if, given an $x \in X$, there exists an $R > 0$ such that (2.1) holds.

It was proved in [9] that the $R$-weak commutativity is equivalent to commutativity at coincidence points, i. e., $S$ and $T$ are pointwise $R$-weakly commuting if and only if they are weakly compatible.

Let $\mathbb{R}_+$ be the set of all non-negative reals numbers and $K_6$ the family of all continuous mappings $K(t_1, t_2, t_3, t_4, t_5, t_6)\colon \mathbb{R}_+^6 \to \mathbb{R}$ with $t_3 + t_4 \neq 0$ satisfying the following conditions:

$(K_1)$ $K$ is decreasing in variables $t_5$and $t_6$.

$(K_2)$ there exists $0 \leq h < 1$ such that for all $u, v \geq 0$ with

$(K_a)$ $K(u, v, v, u, u+v, 0) \leq 0$ or

$(K_b)$ $K(u, v, u, v, 0, u+v) \leq 0$

we have $u \leq hv$.

The following theorem was proved in [13].

**Theorem 1.** *Let $S$, $T$, $I$ and $J$ be self-mappings of a metric space $(X, d)$ satisfying*

$(a)$ *$S(X) \subset J(X)$ and $T(X) \subset I(X)$,*

$(b)$ *the inequality*

$$F\big(d(Sx,Ty), d(Ix,Jy), d(Ix,Sx), d(Jy,Ty), d(Ix,Ty), d(Sx,Jy)\big) \leq 0$$

*holds for all $x, y \in X$ and $F \in K_6$ satisfies conditions $(K_1)$ and $(K_2)$ if $d(Ix,Sx) + d(Jy,Ty) \neq 0$, or*

$$d(Sx,Ty) = 0 \quad \text{if} \quad (Ix,Sx) + d(Jy,Ty) = 0,$$

$(c)$ *the pairs $(S, I)$ and $(T, J)$ are weakly compatible.*

*If one of $S(X)$, $T(X)$, $I(X)$ and $J(X)$ is a complete subspace of $X$, then, $S$, $T$, $I$ and $J$ have a unique common fixed point $z$ in $X$. Further, $z$ is the unique common fixed point of $S$ and $I$ and $T$ and $J$.*

It is our purpose in this paper to prove common fixed point theorems for weakly compatible mappings satisfying implicit relations without condition $(K_1)$, which improve results of Ali and Imdad [1] and Popa [13, 14]. We also prove a common fixed point theorem for weakly compatible mappings satisfying implicit relations which generalizes results of Jeong and Rhoades [4] and Jeong [3].

## 3. Implicit relations

Let $F_6$ be the family of all continuous mappings $F(t_1,t_2,t_3,t_4,t_5,t_6)\colon \mathbb{R}^6_+ \to \mathbb{R}$ with $t_3+t_4 \neq 0$ satisfying the following condition:

$(F_1)$ there exists $0 \leq h < 1$ such that for all $u,v,w \geq 0$ with
 $(F_a)$ $F(u,v,v,u,w,0) \leq 0$ or
 $(F_b)$ $F(u,v,u,v,0,w) \leq 0$
 we have $u \leq hv$.

*Example* 1. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a\frac{t_2t_3}{t_3+t_4} - b\frac{t_4t_5}{t_5+t_6+1}$, where $0 < a,b < 1$ and $a+b < 1$.

$(F_1)$ : Let $u,v,w \geq 0$ and $F(u,v,v,u,w,0) = u - a\frac{v^2}{u+v} - b\frac{uw}{w+1} \leq 0$. Then, $u \leq h_1v$, where $h_1 = \frac{a}{1-b} < 1$. Similarly, if $F(u,v,u,v,0,w) \leq 0$ then $u \leq h_2v$, where $h_2 = a < 1$. We take $h = \max\{h_1,h_2\} < 1$.

*Example* 2. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a\frac{t_2t_4}{t_3+t_4} - b\frac{t_3t_6}{t_5+t_6+1}$, where $0 < a,b < 1$ and $a+b < 1$. The condition $(F_1)$ can be verified as in Example 1.

Let $H_6$ be the family of all continuous mappings $H(t_1,t_2,t_3,t_4,t_5,t_6)\colon \mathbb{R}^6_+ \to \mathbb{R}$ with $t_5+t_6 \neq 0$ satisfying the following conditions:

$(H_1)$ there exists $0 \leq h < 1$ such that for all $u,v,w \geq 0$ with
 $(H_a)$ $H(u,v,v,u,w,0) \leq 0$ or
 $(H_b)$ $H(u,v,u,v,0,w) \leq 0$
 we have $u \leq hv$.

$(H_2)$ $H(u,u,0,0,u,u) > 0$ for all $u > 0$.

*Example* 3. $H(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a\frac{t_3t_6+t_4t_5}{t_5+t_6} - bt_2$, where $a,b > 0$ and $a+b < 1$.

$(H_1)$ : Let $u,v,w \geq 0$ and $H(u,v,v,u,w,0) = u - au - bv \leq 0$. Then, $u \leq hv$, where $h = \frac{b}{1-a} < 1$. Similarly, if $H(u,v,u,v,0,w) \leq 0$ then $u \leq hv$.

$(H_2)$ : $H(u,u,0,0,u,u) = (1-b)u > 0$ for all $u > 0$.

*Example* 4. $H(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \frac{at_3t_5+bt_4t_6}{t_5+t_6} - ct_2$, where $a,b,c > 0$ and $a+b+c < 1$.

$(H_1)$ : Let $u,v,w \geq 0$ and $H(u,v,v,u,w,0) = u - av - cv \leq 0$. Then, $u \leq h_1v$, where $h_1 = a+c < 1$. Similarly, if $H(u,v,u,v,0,w) \leq 0$ then $u \leq h_2v$, where $h_2 = b+c < 1$. We take $h = \max\{h_1,h_2\} < 1$.

$(H_2)$ : $H(u,u,0,0,u,u) = (1-c)u > 0$ for all $u > 0$.

Let $G_6$ be the family of all continuous mappings $G(t_1,t_2,t_3,t_4,t_5,t_6)\colon \mathbb{R}^6_+ \to \mathbb{R}$ with $t_2+t_4 \neq 0$ satisfying the following conditions:

($G_1$) $G$ is decreasing in variables $t_5$ and $t_6$.
($G_2$) there exists $0 \le h < 1$ such that for all $u, v \ge 0$ with
($G_a$) $G(u,v,v,u,u+v,0) \le 0$ or
($G_b$) $G(u,v,u,v,0,u+v) \le 0$
we have $u \le hv$.
($G_3$) $G(u,u,0,0,u,u) > 0$ for all $u > 0$.

*Example* 5. $G(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a_1\frac{t_2t_5}{t_2+t_4} - a_2(t_3+t_4) - a_3(t_5+t_6) - a_4t_2$, where $a_1,a_2,a_3,a_4 > 0$ and $a_1+2a_2+2a_3+a_4 < 1$.

($G_1$) : It is clear.

($G_2$) : Let $u,v \ge 0$ and $G(u,v,v,u,u+v,0) = u - a_1\dfrac{v(u+v)}{u+v} - a_2(u+v) - a_3(u+v) - a_4v \le 0$. Then $u \le h_1v$, where $h_1 = \frac{a_1+a_2+a_3+a_4}{1-a_2-a_3} < 1$. Similarly, if $G(u,v,u,v,0,u+v) \le 0$ then $u \le h_2v$, where $h_2 = \dfrac{a_2+a_3+a_4}{1-a_2-a_3} < 1$. We take $h = \max\{h_1,h_2\} < 1$.

($G_3$) : $G(u,u,0,0,u,u) = (1-a_1-2a_3-a_4)u > 0$ for all $u > 0$.

Let $C_6$ be the family of all continuous mappings $C(t_1,t_2,t_3,t_4,t_5,t_6)\colon \mathbb{R}_+^6 \to \mathbb{R}$ with $t_2+t_4 \ne 0$ satisfying the following conditions:
($C_1$) there exists $0 \le h < 1$ such that for all $u,v,w \ge 0$ with
($C_a$) $C(u,v,v,u,w,0) \le 0$ or
($C_b$) $C(u,v,u,v,0,w) \le 0$
we have $u \le hv$.
($C_2$) $C(u,u,0,0,u,u) > 0$ for all $u > 0$.

*Example* 6. $C(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a\frac{t_2t_4}{t_2+t_4} - b\frac{t_3t_5}{t_5+t_6+1}$, where $0 < a,b < 1$ and $a+b < 1$.

($C_1$) : Let $u,v,w \ge 0$ and $C(u,v,v,u,w,0) = u - a\frac{uv}{u+v} - b\dfrac{vw}{w+1} \le 0$. Then $u \le h_1v$, where $h_1 = a+b < 1$. Similarly, if $C(u,v,u,v,0,w) \le 0$ then $u \le h_2v$, $h_2 = \frac{a}{2} < 1$. We take $h = \max\{h_1,h_2\} < 1$.

($C_2$) : $C(u,u,0,0,u,u) = u > 0$ for all $u > 0$.

*Example* 7. $C(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a\frac{t_2t_4}{t_2+t_4} - b\frac{t_3t_6}{t_5+t_6+1}$, where $0 < a,b < 1$ and $a+2b < 2$. ($C_1$) and ($C_2$) follow as in Example 6.

## 4. Main results

**Theorem 2.** *Let $f$, $g$, $S$ and $T$ be self-mappings of a metric space $(X,d)$ satisfying the following conditions:*

$$S(X) \subset g(X), \qquad T(X) \subset f(X), \tag{4.1}$$

$$F\big(d(Sx,Ty), d(fx,gy), d(fx,Sx), d(gy,Ty), d(fx,Ty), d(Sx,gy)\big) \le 0 \tag{4.2}$$

*for all* $x, y \in X$ *and* $F \in F_6$ *satisfies* $(F_1)$ *if* $d(fx, Sx) + d(gy, Ty) \neq 0$*, or*

$$d(Sx, Ty) = 0 \quad \text{if} \quad d(fx, Sx) + d(gy, Ty) = 0. \tag{4.3}$$

*Suppose that one of* $S(X)$*,* $T(X)$*,* $f(X)$*, and* $g(X)$ *is a complete subspace of* $X$ *and the pairs* $(S, f)$ *and* $(T, g)$ *are weakly compatible.*

*Then,* $f$*,* $g$ *,*$S$*, and* $T$ *have a unique common fixed point* $z$ *in* $X$*. Further,* $z$ *is the unique common fixed point of* $S$ *and* $f$ *and* $T$ *and* $g$*.*

*Proof.* Let $x_0$ be an arbitrary point in $X$. By (4.1), we can define inductively a sequence $\{y_n\}$ in $X$ such that

$$y_{2n} = Sx_{2n} = gx_{2n+1}, \; y_{2n+1} = fx_{2n+2} = Tx_{2n+1} \quad \text{for } n = 0, 1, 2, \ldots. \tag{4.4}$$

If

$$d(fx_{2n}, Sx_{2n}) + d(gx_{2n+1}, Tx_{2n+1}) \neq 0,$$

then, using (4.2) and (4.4), we have

$$\begin{aligned} &F\big(d(Sx_{2n}, Tx_{2n+1}), d(fx_{2n}, gx_{2n+1}), d(fx_{2n}, Sx_{2n}), d(gx_{2n+1}, Tx_{2n+1}),\\ &\qquad d(fx_{2n}, Tx_{2n+1}), d(Sx_{2n}, gx_{2n+1})\big)\\ &= F\big(d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}),\\ &\qquad d(y_{2n-1}, y_{2n+1}), 0\big) \leq 0 \end{aligned}$$

By $(F_a)$, we get

$$d(y_{2n}, y_{2n+1}) \leq h d(y_{2n-1}, y_{2n}).$$

Similarly, if

$$d(fx_{2n+2}, Sx_{2n+2}) + d(gx_{2n+1}, Tx_{2n+1}) \neq 0.$$

we obtain

$$d(y_{2n+1}, y_{2n+2}) \leq h d(y_{2n}, y_{2n+1}).$$

Therefore,

$$d(y_n, y_{n+1}) \leq h d(y_{n-1}, y_n).$$

Thus, $\{y_n\}$ is a Cauchy sequence in $X$ and, therefore, the subsequence $\{y_{2n}\} = \{gx_{2n+1}\} \subset g(X)$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, it converges to a point $z = gv$ for some $v \in X$.

Therefore, the sequence $\{y_n\}$ converges also to $z$ and the subsequences $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$, and $\{fx_{2n+2}\}$ converge to $z$.

If $z \neq Tv$, using (4.2), we obtain

$$\begin{aligned} &F\big((d(Sx_{2n}, Tv), d(fx_{2n}, gv), d(fx_{2n}, Sx_{2n}), d(gv, Tv), d(fx_{2n}, Tv),\\ &\quad d(Sx_{2n}, gv)\big) \leq 0 \end{aligned}$$

Letting $n \to \infty$ and using the continuity of $F$, we obtain

$$F(d(z, Tv), 0, 0, d(z, Tv), d(z, Tv), 0) \leq 0.$$

By $(F_a)$, we get $z = Tv = gv$. Since $T(X) \subset f(X)$, there exists an $u \in X$ such that $z = fu = Tv$.

If $z \neq Su$, using (4.2) we have

$$F\big(d(Su,Tv),d(fu,gv),d(fu,Su),d(gv,Tv),d(fu,Tv),d(Su,gv)\big)$$
$$= F\big(d(Su,z),0,d(z,Su),0,0,d(Su,z)\big) \leq 0.$$

By $(F_b)$, we get $z = Su = fu$. Since the pairs $(S, f)$ and $(T, g)$ are weakly compatible, we get $fz = Sz$ and $gz = Tz$. Since $d(fz,Sz)+d(gv,Tv) = 0$, in view of (4.3), it follows that $d(Sz,Tv) = 0$, i. e., $z = Sz = fz$.

Similarly, we can prove that $z = gz = Tz$. Hence, $z$ is a common fixed point of $f$, $g$, $S$, and $T$.

The proof is similar if we suppose that, instead of $g(X)$, one of $S(X)$, $T(X)$, and $f(X)$ is complete. The uniqueness of $z$ follows from (4.3). □

*Remark* 1. As the function $F$ in Theorem 2 is not decreasing in variables $t_5$ and $t_6$, theorems of [1, 13, 14] are not applicable in this case.

Considering Examples 1 and 2, we obtain two corollaries of Theorem 2.

**Theorem 3.** *Let $f$, $g$, $S$, and $T$ be self-mappings of a metric space $(X,d)$ satisfying (4.1),*

$$H\big(d(Sx,Ty),d(fx,gy),d(fx,Sx),d(gy,Ty),d(fx,Ty),d(Sx,gy)\big) \leq 0 \tag{4.5}$$

*for all $x, y \in X$ and $H \in H_6$ satisfies $(H_1)$ and $(H_2)$ if $d(fx,Ty)+d(Sx,gy) \neq 0$, or*

$$d(Sx,Ty) = 0 \quad \text{if} \quad d(fx,Ty)+d(Sx,gy) = 0.$$

*Suppose that one of $S(X)$, $T(X)$, $f(X)$, and $g(X)$ is a complete subspace of $X$ and the pairs $(S, f)$ and $(T, g)$ are weakly compatible.*

*Then $f$, $g$, $S$, and $T$ have a unique common fixed point $z$ in $X$. Further, $z$ is the unique common fixed point of $S$ and $f$ and $T$ and $g$.*

*Proof.* Let $x_0$ be an arbitrary point in $X$. By (4.1), we can define inductively a sequence $\{y_n\}$ in $X$ by (4.4).

If

$$d(fx_{2n},Tx_{2n+1})+d(Sx_{2n},gx_{2n+1}) = 0$$

or

$$d(fx_{2n+2},Tx_{2n+1})+d(gx_{2n+1},Sx_{2n+2}) = 0$$

for some $n$, the proof is similar to [14].

If

$$d(fx_{2n},Tx_{2n+1})+d(Sx_{2n},gx_{2n+1}) \neq 0$$

and

$$d(fx_{2n+2},Tx_{2n+1})+d(gx_{2n+1},Sx_{2n+2}) \neq 0,$$

as in the proof of Theorem 2, we obtain $z = fu = Su = gv = Tv$.

Since the pairs $(S, f)$ and $(T, g)$ are weakly compatible, we get $fz = Sz$ and $gz = Tz$.

If $z \neq Sz$, using (4.2) we have

$$H\big(d(Sz,Tv),d(fz,gv),d(fz,Sz),d(gv,Tv),d(fz,Tv),d(Sz,gv)\big) = H\big(d(Sz,z),d(Sz,z),0,0,d(Sz,z),d(Sz,z)\big) \leq 0.$$

which is a contradiction to $(H_2)$. Therefore, $z = Sz = fz$.

Similarly, we can prove that $z = Tz = gz$. Hence, $z$ is a common fixed point of $f$, $g$, $S$, and $T$. The uniqueness of $z$ follows from (4.5) and $(H_2)$. □

Theorem 3 generalizes [4, Theorem 5] and [3, Theorem 2].

**Theorem 4.** *Let $f$, $g$, $S$, and $T$ be self-mappings of a metric space $(X,d)$ satisfying (4.1),*

$$G\big(d(Sx,Ty),d(fx,gy),d(fx,Sx),d(gy,Ty),d(fx,Ty),d(Sx,gy)\big) \leq 0 \tag{4.6}$$

*for all $x, y \in X$ and $G \in G_6$ satisfies conditions $(G_1)$, $(G_2)$, and $(G_3)$ if $d(fx,gy)+d(gy,Ty) \neq 0$, or*

$$d(Sx,Ty) = 0 \quad \text{if} \quad d(fx,gy)+d(gy,Ty) = 0.$$

*Suppose that one of $S(X)$, $T(X)$, $f(X)$, and $g(X)$ is a complete subspace of $X$ and the pairs $(S, f)$ and $(T, g)$ are weakly compatible.*

*Then, $f$, $g$, $S$, and $T$ have a unique common fixed point $z$ in $X$. Further, $z$ is the unique common fixed point of $S$ and $f$ and $T$ and $g$.*

*Proof.* Let $x_0$ be an arbitrary point in $X$. By (4.1), we can define inductively a sequence $\{y_n\}$ in $X$ by (4.4).

If

$$d(fx_{2n},Tx_{2n+1})+d(Sx_{2n},gx_{2n+1}) \neq 0$$

using (4.2) and (4.4) we have

$$\begin{aligned}&G\big(d(Sx_{2n},Tx_{2n+1}),d(fx_{2n},gx_{2n+1}),d(fx_{2n},Sx_{2n}),d(gx_{2n+1},Tx_{2n+1}),\\&\quad d(fx_{2n},Tx_{2n+1}),d(Sx_{2n},gx_{2n+1})\big)\\&= G\big(d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n}),d(y_{2n-1},y_{2n}),d(y_{2n},y_{2n+1}),\\&\quad d(y_{2n-1},y_{2n+1}),0\big) \leq 0.\end{aligned}$$

By $(G_1)$ we have

$$\begin{aligned}&G\big(d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n}),d(y_{2n-1},y_{2n}),\\&\quad d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1}),0\big) \leq 0.\end{aligned}$$

By virtue of $(G_a)$, we get

$$d(y_{2n},y_{2n+1}) \leq hd(y_{2n-1},y_{2n}).$$

In the same manner, if

$$d(fx_{2n+2},Sx_{2n+2}) + d(gx_{2n+1},Tx_{2n+1}) \neq 0$$

we obtain

$$d(y_{2n+1},y_{2n+2}) \leq hd(y_{2n},y_{2n+1}).$$

Therefore,

$$d(y_n,y_{n+1}) \leq hd(y_{n-1},y_n).$$

As in the proofs of Theorems 2 and 3, $z$ is a common fixed point of $f$, $g$, $S$, and $T$. The uniqueness of $z$ follows from (4.6) and $(G_3)$. □

*Remark* 2. Theorem 4 generalizes [4, Theorem 9] and [3, Theorem 5].

Similarly to Theorem 3, we can prove the following statement which improves Theorem 4.

**Theorem 5.** *Let $f$, $g$, $S$, and $T$ be self-mappings of a metric space $(X,d)$ satisfying* (4.1),

$$C\big(d(Sx,Ty),d(fx,gy),d(fx,Sx),d(gy,Ty),d(fx,Ty),d(Sx,gy)\big) \leq 0$$

*for all $x,y \in X$ and $C \in C_6$ satisfies $(C_1)$ and $(C_2)$ if $d(fx,gy)+d(gy,Ty) \neq 0$, or*

$$d(Sx,Ty) = 0 \quad \text{if} \quad d(fx,gy)+d(gy,Ty) = 0.$$

*Suppose that one of $S(X)$ $T(X)$, $f(X)$, and $g(X)$ is a complete subspace of $X$ and the pairs $(S,f)$ and $(T,g)$ are weakly compatible.*

*Then, $f$, $g$, $S$, and $T$ have a unique common fixed point $z$ in $X$. Further, $z$ is the unique common fixed point of $S$ and $f$ and $T$ and $g$.*

If $S=T$ or $f=g$ or $S=T$ and $f=g$, or $f=g=I$, where $I$ denotes the identity mapping, or $S=T$ and $f=g=I$ in Theorems 2, 3, 4, and 5, one can obtain several corollaries.

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