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A CLASS OF CONCAVE YOUNG FUNCTIONS POSSESSING A POSITIVE FIXED POINT

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Abstract. We obtained the class of all concave Young functions which possess a positive fixed point.

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1. Introduction

Let $\varphi : (0,\infty) \to (0,\infty)$ be a right-continuous and decreasing function such that it is integrable on every finite interval $(0,x)$. It is easily seen that the function $\Phi : [0,\infty) \to [0,\infty)$, defined by the equality

$$\Phi(x) = \int_0^x \varphi(t)\,dt, \tag{1.1}$$

is a nonnegative, increasing and concave function with $\Phi(0) = 0$. We further assume that $\Phi(\infty) = \infty$. Function Φ is thus referred to as a *concave Young function* in the literature, and the set of all such functions will be denoted by $\mathcal{Y}_{\text{conc}}$. For more about these functions see, e. g, [1–3, 5].

In [3], we obtained the following results.

Proposition 1.1. *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ and $s \in (0,\infty)$ be arbitrary. Then*

$$|\Phi(x) - \Phi(y)| \le \varphi(s)\,|x - y|$$

for all numbers $x, y \in (s,\infty)$.

We sought for all those positive numbers that can be a fixed point for a given concave Young function.

Theorem 1.1. *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ and c^* be any positive number. In order that the equality $\Phi(c^*) = c^*$ hold, it is necessary and sufficient that the range of the function $\Phi|_{[c^*,\infty)} : [c^*,\infty) \to [0,\infty)$, defined by $\Phi|_{[c^*,\infty)}(x) = \Phi(x)$, should equal the interval $[c^*,\infty)$.*

Let $\Phi \in \mathcal{Y}_{\mathrm{conc}}$ with

$$\int_1^\infty \frac{\varphi(t)}{t} dt < \infty. \tag{1.2}$$

In [3], a number $c \in (0,\infty)$ was called the degree of contraction of Φ if

$$\int_c^\infty \frac{\varphi(t)}{t} dt = 1$$

and

$$\int_c^{bc} \frac{\varphi(t)}{t} dt = \varphi(c)$$

for some $b \in (1,\infty)$. We intend to extend this notion to other concave Young functions which do not possess property (1.2).

2. Main result

Theorem 2.1. *Let $\Phi \in \mathcal{Y}_{\mathrm{conc}}$ be arbitrary with φ denoting its derivative. In order that there be a constant $s > 0$ for which $\varphi(s) < 1$, it is necessary and sufficient that Φ admit a positive fixed point, i. e., $\Phi(x) = x$ for some number $x > 0$.*

Proof. To prove the sufficiency, assume that there is a number $s > 0$ such that $\varphi(s) < 1$. Then by recalling Proposition 1.1 one can easily observe that Φ is a contraction in the interval (s,∞). Consequently, the Contraction Principle [6] yields $\Phi(x) = x$ for some $x \geq s$. Next, let us show the necessity. Assume that there exists some $x_0 > 0$ for which $\Phi(x_0) = x_0$, but in the contrary $\varphi(t) \geq 1$ for all $t > 0$. Then it is easy to check that $\Phi(x) \geq x$ for all $x > 0$. Since Φ is a strictly concave and increasing function, the graph of Φ must lie below that of the line $y = x$ on the interval (x_0,∞). This fact, however, contradicts the inequality $\Phi(x) \geq x$ for all $x > 0$. □

Proposition 2.1. *Let $\Phi \in \mathcal{Y}_{\mathrm{conc}}$ be arbitrary with φ denoting its derivative. If $x_0 \in (0,\infty)$ is such that $\Phi(x_0) = x_0$, then $\varphi(x_0) < 1$.*

Proof. It is not difficult to see that $\Phi(t) \geq t\varphi(t)$ whenever $t \in (0,\infty)$. Assume the existence of some $x_0 \in (0,\infty)$ for which $\Phi(x_0) = x_0$. Then, as noted above,

$$x_0 = \Phi(x_0) \geq x_0 \varphi(x_0),$$

and hence $\varphi(x_0) \leq 1$. Now, suppose that $\varphi(x_0) = 1$. Since φ is a decreasing function on $(0,\infty)$, there must be some $\varepsilon \in (0,1)$ such that $\varphi(x_0+\varepsilon) < 1$, making Φ be a contraction on $(x_0+\varepsilon,\infty)$, via Proposition 1.1. But then it would mean that there must be some $x^* \in (x_0+\varepsilon,\infty)$ with $\Phi(x^*) = x^*$. Necessarily, it would ensue that Φ is not a concave function on the interval $(x_0, x^*]$, a contradiction. Therefore, $\varphi(x_0) < 1$. □

Now, we are in a position to reformulate the definition of the degree of contraction to cover a broader class of concave Young functions.

Definition 2.1. A number $s > 0$ is called the degree of contraction of a function $\Phi \in \mathcal{Y}_{\text{conc}}$ if $\varphi(s) = 1$, where φ is the derivative of Φ.

We note in this case that $\varphi(s+\delta) < 1$ for any positive number δ, which makes Φ be a contraction on the interval $(s+\delta, \infty)$ for some suitable δ.

Example 1. The degree of contraction of $\Phi(x) = 4\sqrt{x+1} - 4$, $x \in [0,\infty)$, equals 3.

Example 2. For any fixed number $p \in (0,1)$, the degree of contraction of the function $\Phi_p(x) = x^p$, $x \in [0,\infty)$ is equal to $p^{1/(1-p)}$.

Example 3. The function $\Phi(x) = \log(x+1)$, $x \in [0,\infty)$, has no degree of contraction.

Example 4. The degree of contraction of function $\Phi(x) = 2\log(x+1)$ exists and equals 1.

Example 5. The concave Young function Φ defined by $\Phi(x) = \dfrac{x}{2} + \sqrt{x}$ does not meet condition (1.2). Yet its degree of contraction exists and equals 1.

An algorithm for finding positive fixed points for concave Young functions:

Step 1: Input $\Phi(x)$ a concave Young function, c_0 a positive number.
Step 2: Compute the derivative $\varphi(x)$ of $\Phi(x)$.
Step 3: Starting from c_0 find an approximate root of the equation $\varphi(x) - 1 = 0$ and put the result into c.
Step 4: If $c = 0$ then STOP else GOTO *Step 5*.
Step 5: Starting from c apply the Fixed Point algorithm, i. e., $x_0 := c$; $x_{k+1} := \Phi(x_k)$; $k = k+1$.

3. Concluding remarks

In dynamic models, stationary equilibrium is typically described as a solution of the equation $x = f(x)$, where f is a mapping which determines the current state as a function of the previous state, or as a function of the expected future state. In many cases x is a finite dimensional vector, and in general positive solutions (i. e., fixed points of f) are rather sought for. Problems of this kind have been investigated for decades, and often for concave functions. Alfred Tarski in [7] obtained, in particular, the following result.

Theorem 3.1 (Tarski)**.** *Suppose f is an increasing function from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $f(a) > a$ for some positive vector a, and $f(b) < b$ for some vector $b > a$. Then f has a positive fixed point.*

For the proof we refer the reader, e. g., to [4]. In [4], J. Kennan obtained the result stated below by using Tarski's theorem and [4, Theorem 3.1]. He observed that it gave simple sufficient conditions for the existence and uniqueness of a positive fixed point.

Theorem 3.2 ([4, Theorem 3.3]). *Suppose that f is an increasing and stricly concave function from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $f(0) \geq 0$, $f(a) > a$ for some positive vector a, and $f(b) < b$ for some vector $b > a$. Then f has a unique positive fixed point.*

We note that the concavity and increasing property of f mean that every component f_k $(k = 1, \ldots, n)$ of f, considered as a function from $\mathbb{R}^n$ to $\mathbb{R}$, is increasing and strictly concave in every argument $x_j \in \mathbb{R}$, $j = 1, \ldots, n$.

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