Miskolc Mathematical Notes
Vol. 5 (2004), No 1, pp. 3-18

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2004.20

Linear boundary value problems for differential algebraic equations

Katalin Balla and Roswitha März

Miskolc Mathematical Notes HU ISSN 1586-8850
Vol. 5 (2004), No. 1, pp. 3–18

LINEAR BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL ALGEBRAIC EQUATIONS

KATALIN BALLA AND ROSWITHA MÄRZ

[Received: September 15, 2003]

ABSTRACT. By the use of the corresponding shift matrix, the paper gives a criterion for the unique solvability of linear boundary value problems posed for linear differential algebraic equations up to index 2 with well-matched leading coefficients. The solution is constructed by a proper Green function. Another characterisation of the solutions is based upon the description of arbitrary affine linear subspaces of solutions to linear differential algebraic equations in terms of solutions to the adjoint equation. When applied to boundary value problems, the result provides a constructive criterion for unique solvability and allows one to reduce the problem to initial value problems and linear algebraic equations.

Mathematics Subject Classification: 65L80, 34A09

Keywords: Differential algebraic equations, boundary value problems, adjoint equations

1. INTRODUCTION

FOR THE linear differential algebraic equations (DAEs for the sake of brevity in what follows) of the form

$$A(t)\,(D(t)x(t))' + B(t)x(t) = q(t) \tag{1.1}$$

with continuous, quadratic matrix-valued functions A, D and B with complex entries, the "index-1" and "index-2" notion was introduced in [2]. A theorem on the unique solvability of the properly formulated initial value problems (IVPs) for (1.1) equipped with these indices was proven. It was shown that, under the same conditions, the adjoint equation

$$-D^*(t)\,(A^*(t)y(t))' + B^*(t)y(t) = p(t) \tag{1.2}$$

is of the same index, and the proper IVP for (1.2) is solvable simultaneously with that for (1.1). Meanwhile, some properties of the inherent ordinary differential equation

Supported by Hungarian National Scientific Foundation Grants (OTKA) ## T029572, T031807 and by DFG Research Center "Mathematics for Key Technologies" (FZT 86) in Berlin.

(ODE) of (1.1) were investigated. The fundamental matrices for (1.1) and a specific one called the normalised fundamental matrix were also introduced.

The main goal of this paper is to study the boundary value problems (BVPs) for (1.1) up to index 2. The assertion on the simultaneous solvability of (1.1) and (1.2) turns out to be the keystone in the analysis of BVPs.

The paper is organised as follows. In Section 2, we recall the basic definitions and some propositions concerning equation (1.1). For the sake of completeness, we also define "index-0" equations. The results of [2] can be extended to "index-0" equations in a obvious way. The solvability theorem for IVPs posed for the pair (1.1) and (1.2) is cited in this section. Existence results for two-point BVPs in terms of Green's function and shift matrix will be stated in Section 3. In Section 4, we describe affine linear subspaces of solutions to (1.1) by the help of solutions to (1.2). The transfer of boundary conditions (BCs) for BVPs both with separated and non-separated BCs and the related constructive existence theorem will be the topic of Section 5. The paper is concluded with some remarks on numerical implementation in the final Section 6.

2. Preliminaries

We consider equations of the form (1.1), where A, D and B are continuous $m \times m$ matrix functions with complex entries on closed interval $\mathcal{I} = [a, b]$, and q is a continuous vector-valued function with complex components on $\mathcal{I}$. Parallel to (1.1), equation (1.2) is involved in our study, p also being a continuous vector-valued function with complex components on $\mathcal{I}$. The pair of leading terms in (1.1) is assumed to be *well-matched* in the following sense:

Condition C1 ([2])**.** For every $t \in \mathcal{I}$, the equality

$$\ker A(t) \oplus \operatorname{im} D(t) = \mathbb{C}^m \tag{2.1}$$

is true, and there exist continuously differentiable functions $a_1, \dots, a_{m-r}$ and $d_1, \dots, d_r$ such that

$$\ker A(t) = \operatorname{span}\{a_1(t), \dots, a_{m-r}(t)\}, \quad \operatorname{im} D(t) = \{d_1(t), \dots, d_r(t)\}, \quad t \in \mathcal{I}. \tag{2.2}$$

We proved

Lemma 1 ([2, Lemma 2.1])**.** *Equation* (1.1) *has well-matched leading coefficients A and D if and only if the leading coefficients A^* and D^* of equation* (1.2) *do so.*

If R is the continuously differentiable projector function realizing the decomposition (2.1), i. e., $\ker R(t) = \ker A(t)$ and $\operatorname{im} R(t) = \operatorname{im} D(t)$, $t \in \mathcal{I}$, then R^* is the projector function corresponding to the decomposition induced by A^* and D^*.

Remark 1. We are mainly interested in considering singular well-matched leading pairs $A(t)$ and $D(t)$. Assumption (2.1), however, includes the case where both matrices $A(t)$ and $D(t)$ are nonsingular over the entire interval $\mathcal{I}$. Then, $r = m$ and $R(t) \equiv I$ where I is the $m \times m$ identity matrix. The considerations of [2] can be easily extended to the case of nonsingular well-matched leading terms. Equation (1.1) turns into a standard explicit ODE if $A(t) \equiv D(t) \equiv I$.

Definition 1 ([2, Definition 2.1])**.** A vector function $x : \mathcal{I} \to \mathbb{C}^m$ is called a solution of (1.1) if $x \in C^1_D(\mathcal{I}) := \{x \in C(\mathcal{I}) : Dx \in C^1(\mathcal{I})\}$ and (1.1) is satisfied pointwise.

A solution of (1.2) is defined similarly. A kind of Lagrange identity is stated.

Lemma 2. *Let the matrix functions A and D be well-matched. Then, for every pair of solutions $x \in C^1_D$, $y \in C^1_{A^*}$ of the homogeneous equations* (1.1) *and* (1.2)*, respectively, the identity*

$$y^*(t)A(t)D(t)x(t) = \text{const} \quad \textit{for } t \in \mathcal{I}. \tag{2.3}$$

holds.

A key tool in the investigation of problems carried out in [2] is a chain of matrix-valued and subspace-valued functions associated with (1.1), namely,

$$\begin{aligned} &G_0 := AD, \quad B_0 := B; \\ &\text{for } i = 0, 1, \quad Q_i, P_i, W_i \text{ are projector functions: } Q_i^2 = Q_i, \quad W_i^2 = W_i, \\ &\qquad N_i := \ker G_i = \operatorname{im} Q_i, \quad P_i = I - Q_i, \\ &\qquad \ker W_i = \operatorname{im} G_i, \\ &\qquad G_{i+1} := G_i + B_i Q_i, \quad B_{i+1} = B_i P_i, \\ &\qquad S_i := \{z \in \mathbb{C}^m : B_i z \in \operatorname{im} G_i\} = \ker W_i B_i. \end{aligned} \tag{2.4}$$

In the sequel, D^- denotes the reflexive generalised inverse (RGI) function of D such that $DD^- = R$ and $D^-D = P_0$; A^- is an RGI function of A such that $A^-A = R$ and $AA^- = I - W_0$; G_1^- stands for the RGI function of G_1 such that $G_1G_1^- = I - W_1$ and $G_1^-G_1 = P_1$. We recall [5] that a matrix $T^- \in L(\mathbb{C}^k, \mathbb{C}^l)$ is an RGI of a matrix $T \in L(\mathbb{C}^l, \mathbb{C}^k)$ if it satisfies the equalities $T^-TT^- = T^-$ and $TT^-T = T$. The products $P_{RGI1} := TT^-$ and $P_{RGI2} := T^-T$ are projectors. If P_{RGI1}, P_{RGI2} are given projectors such that $\operatorname{im} P_{RGI1} = \operatorname{im} T$ and $\ker P_{RGI2} = \ker T$, then they define an RGI T^- uniquely.

Due to condition C1, $\dim \operatorname{im} G_0(t) \equiv r$. Let $\dim \operatorname{im} G_1(t) = r_1(t)$. Based on the properties of terms in the chain, an index may be assigned to some equations of the form (1.1) if, in addition to condition C1, another requirement is also fulfilled.

Condition C2 ([2])**.** The dimensions of $D(t)S_1(t)$ and $D(t)N_1(t)$ are constant,

$$\dim D(t)S_1(t) =: \varrho \quad \text{and} \quad \dim D(t)N_1(t) =: \nu, \tag{2.5}$$

and there exist continuously differentiable functions $s_1^D, \dots, s_\varrho^D$ and $n_1^D, \dots, n_\nu^D$ such that for all $t \in \mathcal{I}$,

$$D(t)S_1(t) = \operatorname{span}\{s_1^D(t), \dots, s_\varrho^D(t)\}, \quad D(t)N_1(t) = \operatorname{span}\{n_1^D(t), \dots, n_\nu^D(t)\}.$$

Here, we extend Definition 2.2 from [2] as follows:

Definition 2. Let conditions C1 and C2 be valid. Equation (1.1) is said to be

(0) an *"index-0" tractable* DAE if

$$N_0(t) = \{0\}, \qquad t \in \mathcal{I}, \tag{2.6}$$

(1) an *"index-1" tractable* DAE if

$$N_0(t) \neq \{0\}, \tag{2.7}$$

$$N_0(t) \cap S_0(t) = \{0\}, \qquad t \in \mathcal{I}, \tag{2.8}$$

(2) an *"index-2" tractable* DAE if

$$\dim N_0(t) \cap S_0(t) = \text{const} > 0, \tag{2.9}$$

$$N_1(t) \cap S_1(t) = \{0\}, \qquad t \in \mathcal{I}. \tag{2.10}$$

When $r < m$, the chain associated with an equation is not uniquely defined due to the freedom in the choices of the projectors. The index, however, does not depend on these choices. Thus, the index value, if it exists, is an inherent property of the equation. One may choose a specific projector $\hat{Q}_1$ so that $\ker \hat{Q}_1(t) = S_1(t)$. The related terms in the chain will be marked by "ˆ" (a hat). For equations equipped with an index, the assumptions ensure $r_1(t) \equiv \text{const} =: r_1$ and $\varrho = r + r_1 - m$, $\nu = m - r_1$. In the "index-0" and "index-1" cases, $r_1 = m$. The function $D\hat{P}_1D^-$ is a continuously differentiable projector function: for every t, it projects onto DS_1 along $DN_1 \oplus \ker A$.

Setting $A_* = -D^*$, $D_* = A^*$, and $B_* = B^*$, one can construct a chain similar to (2.4) starting with A_*, D_*, B_*, i. e., for equation (1.2). The terms derived in this chain will be marked by an additional first subscript "$_*$" (a star).

With the inclusion of the "index-0" equations, Theorem 5.1 of [2] reads as follows:

Theorem 1. *Equation* (1.1) *is of index μ, $\mu = 0, 1, 2$, if and only if equation* (1.2) *possesses this property.*

The main point in the proof [2] of this theorem consists in showing that

$$\begin{aligned} DS_1 &= R(A^*N_{*1})^\perp = (A^*N_{*1} \oplus \ker D^*)^\perp, \\ A^*S_{*1} &= R^*(DN_1)^\perp = (DN_1 \oplus \ker A)^\perp. \end{aligned} \tag{2.11}$$

The so-called inherent regular ODE for DAE (1.1) has the form

$$u' + DG_0^{-1}BD^{-1}u = A^{-1}q \tag{2.12}$$

in the "index-0" case, and it has the form

$$u' - R'u + DG_1^{-1}BD^-u = DG_1^{-1}q \tag{2.13}$$

in the "index-1" case. If the DAE is of index 2, then the inherent ODE is

$$u' - (D\hat{P}_1D^-)'u + D\hat{P}_1\hat{G}_2^{-1}BD^-u = \mathcal{N}_0 q, \tag{2.14}$$

where

$$\mathcal{N}_0 q := D\hat{P}_1\hat{G}_2^{-1}q + (D\hat{P}_1D^-)'D\hat{Q}_1\hat{G}_2^{-1}q, \quad \mathcal{N}_0 q = D\hat{P}_1D^-\mathcal{N}_0 q.$$

For the inherent ODEs derived from an "index-1" DAE it was shown that if $u(\tilde{t}) \in \operatorname{im} D(\tilde{t})$ for some $\tilde{t} \in \mathcal{I}$, then $u(t) \in \operatorname{im} D(t)$ for all $t \in \mathcal{I}$. Similarly, in the "index-2" case $u(\tilde{t}) \in \operatorname{im} D(\tilde{t})\hat{P}_1(\tilde{t})$ involves $u(t) \in \operatorname{im} D(t)\hat{P}_1(t)$. Equations (2.13) and (2.14) are independent of the choice of P_0 and P_0, P_1, respectively.

Let x be a solution of equation (1.1). If (1.1) is a DAE of index 0, then Dx is a solution of (2.12). In the "index-1" case, Dx is a solution of (2.13). In the "index-2" case, function $D\hat{P}_1 x$ is a solution of (2.14).

Finally, we recall the solvability statement for IVPs.

Theorem 2 ([2, Theorems 3.1 and 3.2]). *Let $t_0 \in \mathcal{I}$. Assume that one of the following conditions is satisfied:*

(i) (1.1) *is an "index-0" or "index-1" DAE, the inclusion $q \in C(\mathcal{I})$ holds, and the initial condition has the form*

$$D(t_0)x(t_0) = d_0 \quad \textit{with} \quad d_0 \in \operatorname{im} D(t_0);$$

(ii) (1.1) *is an "index-2" DAE, the inclusion $q \in C^1_{DQ_1G_2^{-1}}(\mathcal{I})$ holds, and the initial condition has the form*

$$D(t_0)\hat{P}_1(t_0)x(t_0) = d_0 \quad \textit{with} \quad d_0 \in \operatorname{im} D(t_0)\hat{P}_1(t_0). \tag{2.15}$$

Then there exists a unique solution x of the IVP.

Now, the assertion on simultaneous solvability of DAEs (1.1) and (1.2) with proper right-hand sides and initial conditions appears to be a direct consequence of Theorems 1 and 2.

Note that for the "index-0" equations the initial condition is equivalent simply to condition $x_0 \in \mathbb{C}^m$ and the equation may be considered formally a particular case of "index-1" equations with $Q_0 = W_0 = 0$, $r = m$. In turn, an "index-1" DAE may be considered formally a particular case of "index-2" equations with $Q_1 = W_1 = 0$; then $\varrho = r$, $D\hat{P}_1D^- = R$, $D(t)N_1(t) \equiv \{0\}$, $G_2 = G_1$. Thus, in the next sections it is sufficient to prove the statements only for the "index-2" DAEs.

3. The approach based on Green's function

Let equation (1.1) be tractable with index μ, $\mu \in \{0, 1, 2\}$. Denote the maximal fundamental solution matrix normalised at $t_0 \in \mathcal{I}$ by $X(t, t_0)$, i. e., $X(t, t_0) \in L(\mathbb{C}^m)$ and $X(\cdot, t_0)$ is the matrix-valued solution of the IVP

$$A(DX)' + BX = 0, \quad D(t_0)\hat{P}_1(t_0)(X(t_0) - I) = 0. \tag{3.1}$$

We recall from [2] the following properties of the maximal fundamental solutions:

$$\operatorname{im} X(t, t_0) = \operatorname{im} \Pi_{\mathrm{can}\,\mu}(t) \quad \ker X(t, t_0) = \ker \Pi_{\mathrm{can}\,\mu}(t_0), \qquad t \in \mathcal{I},$$

where $\Pi_{\mathrm{can}\,\mu}$ is a projector function onto the geometric solution space of the homogeneous DAE (1.1) ($q = 0$), $S_{\mathrm{ind}\,\mu}(t) = \operatorname{im} \Pi_{\mathrm{can}\,\mu}(t)$,

$$\Pi_{\mathrm{can}\,\mu} := KP_0\hat{P}_1, \quad K := I - Q_0\hat{P}_1\hat{G}_2^{-1}BP_0 - Q_0\hat{Q}_1D^-(D\hat{Q}_1D^-)'D, \tag{3.2}$$

U is nonsingular.

Let the RGI $X(t,t_0)^- \in L(\mathbb{C}^m)$ of $X(t,t_0)$ be defined by the relations

$$X(t,t_0)X(t,t_0)^- = \Pi_{\mathrm{can}\,\mu}(t),$$
$$X(t,t_0)^- X(t,t_0) = \Pi_{\mathrm{can}\,\mu}(t_0).$$

(See Section 2 for the definition of RGI). The usual group properties

$$X(t_1,t_2)X(t_2,t_3) = X(t_1,t_3), \qquad X(t_1,t_2)^- = X(t_2,t_1).$$

hold. It follows from Theorem 2 that, for all $q \in C_{DQ_1G_2^{-1}}(\mathcal{I})$ and $x^0 \in \mathbb{C}^m$, the IVP

$$A(Dx)' + Bx = q, \quad D(t_0)\hat{P}_1(t_0)(x(t_0) - x^0) = 0, \tag{3.3}$$

is uniquely solvable. Due to the linearity, the solution can be split into two terms,

$$x(t) = X(t,t_0)x^0 + \tilde{x}(t), \quad t \in \mathcal{I}, \tag{3.4}$$

where $\tilde{x}$ denotes the solution of the IVP

$$A(Dx)' + Bx = q, \qquad D(t_0)\hat{P}_1(t_0)x(t_0) = 0. \tag{3.5}$$

In [2], it was shown that every solution of (1.1) can be represented in the form $x = \Pi_{\mathrm{can}\,\mu}x + \mathcal{N}_1 q$, where

$$\mathcal{N}_1 q := (P_0\hat{Q}_1 + Q_0\hat{P}_1)\hat{G}_2^{-1}q + Q_0\hat{Q}_1 D^-(DQ_1G_2^{-1}q)'.$$

Thus, one can obtain the relation

$$\tilde{x}(t) = \int_{t_0}^{t} X(t,s)(\mathcal{N}_0 q)(s)ds + (\mathcal{N}_1 q)(t), \quad t \in \mathcal{I}. \tag{3.6}$$

Now we turn to the BVP for (1.1) with the boundary condition

$$K_a x(a) + K_b x(b) = d, \tag{3.7}$$

$d \in L_{BC}$, where $L_{BC} := \operatorname{im}(K_a \mid K_b) \subseteq \mathbb{C}^m$ is the linear subspace associated with the boundary condition. The values $x^0 \in \mathbb{C}^m$ in (3.4) that yield solutions of the BVP (1.1), (3.7) must satisfy the linear system

$$Mx^0 = d - K_a\tilde{x}(a) - K_b\tilde{x}(b) \tag{3.8}$$

with the "shift matrix" M,

$$M := K_a X(a,t_0) + K_b X(b,t_0). \tag{3.9}$$

Theorem 3. *Let DAE* (1.1) *be tractable with index μ, $\mu \in \{0,1,2\}$. Then, for arbitrary $d \in L_{BC}$ and $q \in C^1_{DQ_1G_2^{-1}}(\mathcal{I})$, the BVP* (1.1), (3.7) *is uniquely solvable if and only if the shift matrix M satisfies the conditions*

$$\ker M = \ker \Pi_{\mathrm{can}\,\mu}(t_0), \tag{3.10}$$
$$\operatorname{im} M = L_{BC}. \tag{3.11}$$

Proof. By construction, the relations $\ker \Pi_{\mathrm{can}\,\mu}(t_0) \subseteq \ker M$, $\mathrm{im}\, M \subseteq L_{BC}$ are true.

Let the BVP (1.1), (3.7) be uniquely solvable for all $d \in L_{BC}$ and $q \in C^1_{DQ_1G_2^{-1}}(\mathcal{I})$. Put $q = 0$. For every $d \in L_{BC}$, there is an $x^0 \in \mathbb{C}^m$ such that $Mx^0 = d$. Hence, $L_{BC} \subseteq \mathrm{im}\, M$, i. e., (3.11) holds.

Moreover, since the homogeneous BVP (1.1), (3.7) with $d = 0$ and $q = 0$ has only the trivial solution, the IVP

$$A(Dx)' + Bx = 0, \quad D(t_0)\hat{P}_1(t_0)(x(t_0) - x^0) = 0, \quad x^0 \in \ker M$$

may have only the identically vanishing solution. This means that

$$\ker M \subseteq \ker D(t_0)\hat{P}_1(t_0) = \ker \Pi_{\mathrm{can}\,\mu}(t_0)$$

must be true, and consequently, (3.10) holds.

Conversely, let (3.10) and (3.11) be satisfied. Then, for every $d \in L_{BC}$ and $q \in C^1_{DQ_1G_2^{-1}}(\mathcal{I})$, a solution of the BVP is determined by (3.4) and (3.8). The relations $d = 0$ and $q = 0$ imply $\tilde{x} = 0$ and $Mx^0 = 0$. Thus, $x^0 \in \ker M = \ker X(t, t_0)$. Now (3.4) leads us to the solution x which equal identically to zero. □

Remark 2. The conditions (3.10), (3.11) ensure that $\mathrm{rank}\, M = \varrho = r + r_1 - m$.

When (3.10) and (3.11) are true, we can introduce an RGI $M^- \in L(\mathbb{C}^m)$ of M such that $M^-M = \Pi_{\mathrm{can}\,\mu}(t_0)$ holds.

Theorem 4. *Let DAE* (1.1) *have tractability index* μ, $\mu \in \{0, 1, 2\}$, *and let conditions* (3.10) *and* (3.11) *be satisfied. Then the solution of BVP* (1.1), (3.7) *with* $d \in L_{BC}$, $q \in C^1_{DQ_1G_2^{-1}}(\mathcal{I})$ *is given by the formula*

$$\begin{aligned} x(t) = X(t, t_0)M^-d + \int_a^b \mathcal{G}(t, s)(\mathcal{N}_0 q)(s)ds \\ + (\mathcal{N}_1 q)(t) - X(t, t_0)M^- \{K_a(\mathcal{N}_1 q)(a) + K_b(\mathcal{N}_1 q)(b)\}, \end{aligned} \tag{3.12}$$

where Green's function $\mathcal{G}$ *is defined as follows:*

$$\mathcal{G}(t, s) = \begin{cases} X(t, t_0)M^-K_aX(a, t_0)X(s, t_0)^-, & s \le t, \\ -X(t, t_0)M^-K_bX(b, t_0)X(s, t_0)^-, & s > t. \end{cases}$$

Proof. It follows from (3.8) that $\Pi_{\mathrm{can}\,\mu}(t_0)x^0 = M^-(d - K_a\tilde{x}(a) - K_b\tilde{x}(b))$, whereas from (3.4), one obtains (3.12) by standard calculations. Note that (3.12) is defined in a unique manner, while there is freedom in the choice of M^-. □

Remark 3. The map $\mathcal{L} : C^1_D(\mathcal{I}) \to C(\mathcal{I}) \times L_{BC}$ defined by the relation

$$\mathcal{L}x := (A(Dx)' + Bx, K_ax(a) + K_bx(b)), \; x \in C^1_D(\mathcal{I})$$

is linear and bounded. It acts bijectively between $C^1_D(\mathcal{I})$ and $C^1_{DQ_1G_2^1}(\mathcal{I}) \times L_{BC}$. Recall that, in the case where $\mu = 2$, the set $C^1_{DQ_1G_2^{-1}}(\mathcal{I})$ is a proper dense subset of $C(\mathcal{I})$.

Hence, when $\mu = 2$, $\mathcal{L}$ has a densely defined unbounded inverse $\mathcal{L}^{-1}$. However, if we equip $C^1_{DQ_1G_2^{-1}}(\mathcal{I})$ with a natural norm and consider $\mathcal{L}$ as a mapping $\mathcal{L} : C^1_D(\mathcal{I}) \to C^1_{DQ_1G_2^{-1}}(\mathcal{I}) \times L_{BC}$, in this setting, $\mathcal{L}$ has a bounded inverse.

4. Affine linear subspaces of solutions

In [1], for a subclass of homogeneous "index-1" DAEs (1.1) with $D = P_0$ and continuously differentiable coefficients A, P_0 and B, we described the linear subspaces of solutions in terms of the adjoint equation. In this section, we formulate and prove an analogous theorem for the affine linear subspaces of the solutions for DAE (1.1) with an arbitrary function D well-matched with A. The DAE is assumed to be of index μ, $\mu \in \{0, 1, 2\}$, and it is not necessarily homogeneous. The smoothness conditions on the coefficients A, D and B are exactly as in Section 2, i. e., they must allow for assignment of an index only. The function q is assumed to be of the class required by Theorem 2; in the "index-0" and "index-1" cases, q is only continuous, whereas $q \in C^1_{DQ_1G_2^{-1}}(\mathcal{I})$ in the "index-2" case.

A set of functions $\mathcal{M} \subset C^1_D(\mathcal{I})$ is called an affine linear subspace of functions $x \in C^1_D(\mathcal{I})$ if $\mathcal{M} = \tilde{x} + \mathcal{L}_{\mathcal{M}}$, where $\tilde{x} \in C^1_D(\mathcal{I})$ and $\mathcal{L}_{\mathcal{M}} \subset C^1_D(\mathcal{I})$ is a linear subspace.

Let us put

$$M(t) = \{v \in \mathbb{C}^m : v = x(t),\ x \in \mathcal{M}\}$$

and

$$L_M(t) = \{w \in \mathbb{C}^m : w = z(t),\ z \in \mathcal{L}_{\mathcal{M}}\}.$$

If $\dim L_M(t) \equiv \text{const} =: l$, then $\dim \mathcal{M} := \dim \mathcal{L} := l$.

The set $\mathcal{M}_{\mathrm{ind}\,\mu}$ of all solutions of the DAE (1.1) is an affine linear subspace of dimension $\varrho = r + r_1 - m$ in $C^1_D(\mathcal{I})$. This fact follows immediately from the representation (3.4). The linear subspace $L_{M_{\mathrm{ind}\,\mu}}(t) \in \mathbb{C}^m$ corresponding to the affine linear subspace $M_{\mathrm{ind}\,\mu}(t)$ describes the geometric constraint to which every solution of the homogeneous equation is subjected. It reads as follows:

$$L_{M_{\mathrm{ind}\,\mu}}(t) = S_{\,\mathrm{ind}\,\mu}(t) = \mathrm{im}\,\Pi_{\mathrm{can}\,\mu}(t).$$

Lemma 3. *The set $\mathcal{M}_{\mathrm{ind}\,\mu}$ admits an equivalent description in the form*

$$\left\{x \in C^1_D(\mathcal{I}) : W_0Bx = W_0q,\ Hx = \mathcal{H}(q)\right\} \tag{4.1}$$

where the matrix function H is defined by the relation

$$H = D\hat{Q}_1D^-[A^-B - (D\hat{Q}_1D^-)'D] \tag{4.2}$$

and the linear map $\mathcal{H} : C^1_{DQ_1G_2^{-1}}(\mathcal{I}) \to C(\mathcal{I})$ is given by the formula

$$\mathcal{H}(q) = D\hat{Q}_1D^-\left[A^-q - (DQ_1G_2^{-1}q)'\right]. \tag{4.3}$$

Proof. Let us denote the set (4.1) by $\tilde{\mathcal{M}}$. Multiplying equation (1.1) by W_0, we get the first required relation, while the second one, $Hx = \mathcal{H}(q)$, is the so-called hidden constraint derived in [2]. Thus, $\mathcal{M}_{\mathrm{ind}\,\mu} \subset \tilde{\mathcal{M}}$. Now it is enough to study the kernels of W_0B and H (argument t is omitted). Instead of showing that $\dim(\ker W_0B \cap \ker H) = \varrho$, we check the intersection of kernels of W_0BK and HK with invertible matrix function K from (3.2), noting that the identities $W_0BK = W_0\hat{G}_2$ and

$$HK = D\hat{Q}_1D^-A^-\hat{G}_2(I - \hat{P}_1P_0)$$

can be verified by direct computation (we omit the details for the sake of brevity). If $W_0\hat{G}_2x = 0$, then there exist $y :\ y = P_0y,\ x = \hat{G}_2^{-1}ADy = \hat{P}_1P_0y = \hat{P}_1y$, i. e., $\hat{Q}_1x = 0$. If, additionally, $0 = HKx = D\hat{Q}_1D^-A^-\hat{G}_2(I - \hat{P}_1P_0)x$, then $0 = D\hat{Q}_1P_0y$, i. e., $\hat{G}_2y = ADy$. Therefore, $\hat{G}_2y = \hat{G}_2x$, i. e., $x = y$. Finally, $x = y = P_0y = P_0x = P_0\hat{P}_1x$. This yields $\ker H \cap \ker W_0B = \operatorname{im} KP_0\hat{P}_1 = \operatorname{im}\Pi_{\mathrm{can}\,\mu}$. □

Remark 4. Observe that $M_{\mathrm{ind}\,0}(t)$ coincides with $\mathbb{C}^m$ because $W_0 = 0$ and $\hat{Q}_1 = 0$. For $\mu = 1$, W_0 is non-trivial while $\hat{Q}_1$ vanishes. For $\mu = 2$, both W_0 and $\hat{Q}_1$ are non-trivial, and the hidden constraint $H(t)z = \mathcal{H}(q)(t)$ is active.

For the purposes of the following assertions, we decompose equation (1.1) using the identity

$$I = Q_{*0}^* + \hat{Q}_{*1}^*P_{*0}^* + \hat{P}_{*1}^*P_{*0}^*. \tag{4.4}$$

We obtain

$$Q_{*0}^*Bx = Q_{*0}^*q, \tag{4.5}$$

$$\hat{Q}_{*1}^*A(Dx)' + \hat{Q}_{*1}^*P_{*0}Bx = \hat{Q}_{*1}^*P_{*0}^*q, \tag{4.6}$$

$$\hat{P}_{*1}^*A(Dx)' + \hat{P}_{*1}^*P_{*0}^*Bx = \hat{P}_{*1}^*P_{*0}^*q. \tag{4.7}$$

Equations (4.6) and (4.7) are equivalent to

$$A^{*-*}\hat{Q}_{*1}^*A(Dx)' + A^{*-*}\hat{Q}_{*1}^*P_{*0}^*Bx = A^{*-*}\hat{Q}_{*1}^*P_{*0}^*q, \tag{4.8}$$

$$A^{*-*}\hat{P}_{*1}^*A(Dx)' + A^{*-*}\hat{P}_{*1}^*P_{*0}^*Bx = A^{*-*}\hat{P}_{*1}^*P_{*0}^*q. \tag{4.9}$$

Since $\hat{Q}_{*1}^*Q_{*0}^*B = \hat{Q}_{*1}^*AD$, it follows from (4.5) that

$$A^{*-*}\hat{Q}_{*1}^*ADx = A^{*-*}\hat{Q}_{*1}^*Q_{*0}^*q. \tag{4.10}$$

Due to Condition C2 and Theorem 1, the projectors $A^*\hat{Q}_{*1}A^{*-}$ and $A^*\hat{P}_{*1}A^{*-}$ are differentiable functions, therefore so are the functions $A^{*-*}\hat{Q}_{*1}^*A$ and $A^{*-*}\hat{P}_{*1}^*A$. It immediately gives that $A^{*-*}\hat{Q}_{*1}^*Q_{*0}^*q \in C^1$ is a necessary condition for a function x to be a solution. One can check, however, that

$$D\hat{Q}_1\hat{G}_2^{-1} = A^{*-*}\hat{Q}_{*1}^*Q_{*0}^*. \tag{4.11}$$

Further, due to $AD = \hat{G}_2P_1P_0$ and $AD = -P_{*0}^*P_{*1}^*\hat{G}_{*2}^*$, the identities $D\hat{P}_1 = D\hat{G}_2^{-1}AD = -D\hat{G}_{*2}^{-*}AD = A^{*-*}\hat{P}_{*1}^*AD$ hold. Thus, in Theorem 2, we could use matrix functions associated with equation (1.2), i. e., we could suppose

$$A^{*-*}\hat{Q}^*_{*1}Q^*_{*0}q \in C^1,$$

and replace (2.15) by

$$P^*_{*1}(t_0)A(t_0)D(t_0)x(t_0) = d, \quad d \in \operatorname{im} P^*_{*1}(t_0)A(t_0)D(t_0).$$

If $x \in C^1_D$, then the first term in (4.8) rewrites as

$$A^{*-*}\hat{Q}^*_{*1}A[(A^{*-*}\hat{Q}^*_{*1}ADx)' - (A^{*-*}\hat{Q}^*_{*1}A)'Dx].$$

Thus, combined with (4.10), i. e., with (4.5), equation (4.8) rewrites as

$$A^{*-*}\hat{Q}^*_{*1}A[(A^{*-*}\hat{Q}^*_{*1}Q^*_{*0}q)' - (A^{*-*}\hat{Q}^*_{*1}A)'Dx] = A^{*-*}\hat{Q}^*_{*1}P^*_{*0}(q - Bx). \tag{4.12}$$

This is exactly the so-called hidden constraint $Hx = \mathcal{H}(q)$ in a different form. Indeed, since $\ker W_0 = \ker Q^*_{*0}$ and (4.5) hold, the multiplier $I-W_0$ may be inserted before the term $q - Bx$ in (4.12). On the other hand, the relation $\ker W_0 = \ker Q^*_{*0}$ involves that (4.5) and $W_0Bx = W_0q$ are equivalent. In fact, we checked the following statement:

Lemma 4. *A function $x \in C^1_D$ satisfies (4.5) and (4.6) if and only if $x \in \mathcal{M}_{\mathrm{ind}\,\mu}$.*

Using (4.5) and (4.12), we can rewrite (4.9) as a regular ODE

$$\begin{aligned} &u' - (A^{*-*}\hat{P}^*_{*1}A)'u - (A^{*-*}\hat{P}^*_{*1}A)A^{*-*}BG^{-*}_{*2}Au \\ &\quad = [(A^{*-*}\hat{P}^*_{*1}A)'D - A^{*-*}\hat{P}^*_{*1}P^*_{*0}B]G^{-*}_{*2}Q^*_{*0}q + A^{*-*}\hat{P}^*_{*1}P^*_{*0}q, \end{aligned} \tag{4.13}$$

for $u := A^{*-*}\hat{P}^*_{*1}A(Dx)$. Equation (4.13) is nothing else but the inherent ODE (2.14) in terms connected with (1.2). Indeed, the term by term coincidence can be verified by direct computation. The forms (4.12) and (4.13) show that both the hidden constraint and the inherent ODE are independent of the chosen projectors P_0, P_1 since so are P_{*0}, P_{*1}; this assertion was proven in [2] in a different way.

Remark 5. Since $Q^*_{*0}G^*_{*2} = Q^*_{*0}B$, (4.5) defines the projection of the function x onto $\operatorname{im} G^{-*}_{*2}Q^*_{*0}G^*_{*2}$:

$$G^{-*}_{*2}Q^*_{*0}G^*_{*2}x = G^{-*}_{*2}Q^*_{*0}q. \tag{4.14}$$

This is an equivalent of the first equation in formula (4.1) of Lemma 3. A combination of (4.14) with the second equation $Hx = \mathcal{H}(q)$ in (4.1) defines another projection of the function x. Namely, we can obtain the relations

$$G^{-*}_{*2}V^*G^*_{*2}x = G^{-*}_{*2}\left[\hat{Q}^*_{*1}(P^*_{*0} - Q^*_{*0})q - \hat{Q}^*_{*1}A(A^{*-*}\hat{Q}^*_{*1}Q^*_{*0}q)'\right], \tag{4.15}$$

$$V = \left[P_{*0} + A^{*-}(A^*\hat{P}_{*1}A^{*-})'A^*\right]\hat{Q}_{*1}, \quad V^2 = V, \quad VQ_{*0} = Q_{*0}V = 0,$$

and, therefore, the system of equations in (4.1) becomes equivalent to system (4.14)–(4.15) which defines two projections of the function x. This observation will be used in Section 5.

Theorem 5. *Let* (1.1) *be tractable with index* μ, $\mu \in \{0,1,2\}$ *and* $q \in C^1_{DQ_1G_2^{-1}}(\mathcal{I})$. *Then a set* $\mathcal{K} \subset C^1_D(\mathcal{I})$ *is a k-dimensional affine linear subspace of solutions of the DAE* (1.1) *if and only if, for all* $t \in \mathcal{I}$,

$$K(t) = \left\{ w \in \mathbb{C}^m : y^*(t)A(t)D(t)w + v^*(t) = 0, \quad w \in M_{\operatorname{ind}\mu}(t) \right\} \tag{4.16}$$

where $y : \mathcal{I} \to L(\mathbb{C}^s, \mathbb{C}^m)$, $\dim \operatorname{im} y(t) \equiv s$, $v : \mathcal{I} \to \mathbb{C}^s$, $s = \varrho - k$, *and*

$$-D^*(A^*y)' + B^*y = 0, \tag{4.17}$$

$$v' + q^*y = 0. \tag{4.18}$$

Theorem 5 states that any affine linear subspace within the whole solution set can be segregated by the help of functions that are solutions of the homogeneous adjoint DAE and solutions of an explicit ODE.

Proof. We denote the set on the right-hand side of (4.16) by $\tilde{K}(t)$ and provide the proof for $\mu = 2$.

Let $\mathcal{K} \in C^1_D(\mathcal{I})$ be a k–dimensional affine linear subspace of solutions of the DAE (1.1) of index 2 and choose an arbitrary $x_a \in \mathcal{K}$. Let

$$\mathcal{L}_{\mathcal{K}} := \left\{ \ell \in C^1_D(\mathcal{I}) : \ell = x - x_a,\ x \in \mathcal{K} \right\}$$

be the linear subspace of functions corresponding to $\mathcal{K}$ and $L_K(t) \in \mathbb{C}^m$, $t \in \mathcal{I}$, $t \in \mathcal{I}$, be the corresponding subspaces. Note that, for every t, we have $D(t)\hat{Q}_1(t)L_K(t) \equiv \{0\}$ and $\dim D(t)L_K(t) = \dim L_K(t)$. Fix $t_0 \in \mathcal{I}$. Let

$$L^c_K(t_0) := (D(t_0)L_{\mathcal{K}}(t_0) \oplus DN_1(t_0) \oplus \ker A(t_0))^{\perp}.$$

One has

$$\dim L^c_{\mathcal{K}}(t_0) = m - [k + \nu + (m - r)] = r - k - \nu = s.$$

Thus, there exist s linearly independent vectors $z^0_1, \ldots, z^0_s$ spanning $L^c_{\mathcal{K}}(t_0)$. Since

$$L^c_{\mathcal{K}}(t_0) \subset (DN_1(t_0) \oplus \ker A(t_0))^{\perp} = A^*(t_0)S_{*1}(t_0),$$

the IVPs for the homogeneous equation (1.2) ($p = 0$) with the initial conditions

$$A^*(t_0)\hat{P}_{*1}(t_0)y(t_0) = z_i, \qquad i = 1, \ldots, s,$$

have unique solutions y_i.

The above solutions $y_1, \ldots, y_s$ of the homogeneous equation (1.2) are linearly independent. Indeed, assume the contrary, that is for the solution

$$\xi(t) = \sum_{i=1}^{s} c_i y_i(t)$$

with at least one non-zero c_i, the equality $\xi(\tilde{t}) = 0$ holds for some $\tilde{t}$. The IVP for homogeneous equation (1.2) with initial condition $A^*(\tilde{t})\hat{P}_{*1}(\tilde{t})y(\tilde{t}) = 0$ has the unique solution $y = 0$. This is in contradiction to the fact that

$$A^*(t_0)\hat{P}_{*1}(t_0)\xi(t_0) = \sum_{i=1}^{s} c_i z_i \neq 0.$$

Set $v_i^0 = -z_i^{0*} D(t_0)x_a(t_0)$ and let $v_i : \mathcal{I} \to \mathbb{C}$ be the solution of the IVP for the ODE $v_i' + q^* y_i = 0$ satisfying $v_i(t_0) = v_i^0$.

Let $x \in \mathcal{K}$. Then

$$\begin{aligned}(y_i^*(t)A(t)D(t)x(t) + v_i^*(t))' &= \\ \left[(y_i^*(t)A(t))'D(t)\right] x(t) &+ y_i^*(t)\left[A(t)(D(t)x(t))'\right] - y_i^*(t)q(t) = \\ &y_i^*(t)B(t)x(t) + y_i^*(t)[q(t) - B(t)x(t)] - y_i^*(t)q(t) = 0.\end{aligned}$$

Thus,

$$y_i^*(t)A(t)D(t)x(t) + v_i^*(t) \equiv y_i^*(t_0)A(t_0)D(t_0)x(t_0) + v_i^*(t_0). \tag{4.19}$$

Note that

$$A^*(t_0)\hat{Q}_{*1}(t_0)y_i(t_0) = 0.$$

Thus, the expression in right-hand side of (4.19) can be brought to the form

$$y_i^*(t_0)\hat{P}_{*1}^*(t_0)A(t_0)D(t_0)x(t_0) - z_i^{0*}D(t_0)x_a(t_0) = z_i^* D(t_0)[x(t_0) - x_a(t_0)].$$

Since $x(t_0) - x_a(t_0) \in L_{\mathcal{K}}(t_0)$, the latter expression vanishes by construction.

Let $y : \mathcal{I} \to L(\mathbb{C}^s, \mathbb{C}^m)$ be defined "columnwise" by vector-valued functions y_i, $i = 1, \dots, s$, as $y(t) := (y_1(t), \dots, y_s(t))$. Clearly, y is a solution of DAE (4.3). Similarly, $v : \mathcal{I} \to \mathbb{C}^s$, $v(t) := (v_1(t), \dots, v_s(t))$ satisfies ODE (4.4).

Thus, we checked that for $x \in \mathcal{K} \subset \mathcal{M}_{\mathrm{ind}\,\mu}$ and every fixed $t \in \mathcal{I}$, $w := x(t)$ belongs to the set $\tilde{K}(t)$.

For the second part, for each t let the set $\tilde{K}(t)$ be given. We have

$$y^*AD = y^*\Pi_{*\,\mathrm{can}\,2}^* AD = y^*AD\Pi_{\mathrm{can}\,2} = y^*AD\hat{P}_1.$$

On the other hand, by construction, the equality

$$s = \dim \operatorname{im} y = \dim \operatorname{im} A^* y$$

holds. Thus,

$$\dim(\ker y^*A \cap DS_1) = (m - s) - [\nu - (m - r)] = r - s - \nu = r - (\varrho - k) - \nu = k,$$

i. e., $\dim \tilde{K}(t) \equiv k$.

Fix a $\tilde{t} \in \mathcal{I}$. Due to the above considerations, there exists $w_0 \in \mathbb{C}^m$ such that $w_0 \in \tilde{K}(\tilde{t})$ and there exist k linearly independent vectors $w_1, \dots, w_k \in \mathbb{C}^m$ such that

$$y^*(\tilde{t})A(\tilde{t})D(\tilde{t})w_i = 0$$

and

$$w_i = \Pi_{\mathrm{can}\,2} w_i.$$

Let us consider the solution x_0 of (1.1) with initial value $x_0(\tilde{t}) = w_0$ and solutions $x_1, \ldots, x_k$ of homogeneous equations (1.1) with initial values $x_i(\tilde{t}) = w_i$, $i = 1, \ldots, k$, respectively. For every t, we have

$$x_0(t) + \operatorname{span}\{x_1(t), \ldots, x_k\} \subset \tilde{K}(t).$$

A similar reasoning that we applied when showing the linear independence of the solutions of the homogeneous adjoint equation with linearly independent initial values yields that for each t, the affine linear set $x_0(t) + \operatorname{span}\{x_1(t), \ldots, x_k(t)\}$ is of dimension k. Thus,

$$\tilde{K}(t) = x_0(t) + \operatorname{span}\{x_1(t), \ldots, x_k(t)\}.$$

On the other hand, $x_0 + \operatorname{span}\{x_1, \ldots, x_k\}$ is an affine linear solution set as it was claimed. □

5. Transfer of boundary conditions

5.1. Separated boundary conditions. First let us consider the BVP for (1.1) with separated boundary condition (3.7), i. e., $K_a^* = (K_{a1}^* \mid 0)$, $K_b^* = (0 \mid K_{b2}^*)$, $d^* = (d_1^* \mid d_2^*)$, where $K_{a1} \in L(\mathbb{C}^m, \mathbb{C}^{m_a})$, $K_{b2} \in L(\mathbb{C}^m, \mathbb{C}^{m_b})$, $d_1 \in \mathbb{C}^{m_a}$, $d_2 \in \mathbb{C}^{m_b}$, and the symbol 0 stands for the zero matrix of the appropriate dimension.

Both sets of solutions defined by one and the other boundary conditions, i. e.,

$$\mathcal{K}_a := \{x \in \mathcal{M}_{\operatorname{ind}2} : K_{a1}x(a) = d_1\}$$

and

$$\mathcal{K}_b := \{x \in \mathcal{M}_{\operatorname{ind}2} : K_{b2}x(b) = d_2\}$$

are affine linear solution sets and so is $\mathcal{K} := \mathcal{K}_a \bigcap \mathcal{K}_b$, the solution set of BVP. Due to Remark 5, $\mathcal{K}_a$ and $\mathcal{K}_a$ admit the equivalent representation

$$\mathcal{K}_a := \{x \in \mathcal{M}_{\operatorname{ind}2} : \tilde{K}_{a1}x(a) = \tilde{d}_1\},$$
$$\mathcal{K}_b := \{x \in \mathcal{M}_{\operatorname{ind}2} : \tilde{K}_{b2}x(b) = \tilde{d}_2\},$$

where

$$\tilde{K}_{a1} = K_{a1}G_{*2}^{-*}(a)(I - Q_{*0}^*(a) - V^*(a))G_{*2}^*(a), \tag{5.1}$$
$$\tilde{K}_{b2} = K_{b2}G_{*2}^{-*}(b)(I - Q_{*0}^*(b) - V^*(b))G_{*2}^*(b), \tag{5.2}$$
$$\tilde{d}_1 = d_1 - G_{*2}^{-*}(a)[(P_{*1}^*(a)Q_{*0}^*(a) + Q_{*1}^*(a)P_{*0}^*(a))q(a) \tag{5.3}$$
$$- Q_{*1}^*(a)A(a)(A^{*-*}Q_{*1}^*Q_{*0}^*q)'(a)], \tag{5.4}$$
$$\tilde{d}_2 = d_2 - G_{*2}^{-*}(b)[(P_{*1}^*(b)Q_{*0}^*(b) + Q_{*1}^*(b)P_{*0}^*(b))q(b) \tag{5.5}$$
$$- Q_{*1}^*(b)A(b)(A^{*-*}Q_{*1}^*Q_{*0}^*q)'(b)].$$

Note that

$$(I - Q_{*0}^*(t) - V^*(t))G_{*2}^*(t) = -[I + A(t)(A^{*-*}Q_{*1}^*A)'(t)A^{*-*}(t)]P_{*1}^*(t)A(t)D(t).$$

We can always assume that the boundary conditions are given in their modified form and the matrices $\tilde{K}_{a1}, \tilde{K}_{a1}$ are of full rank. Let us set

$$y_{aa} := K_*(a)A^{*-}(a)D^{*-}(a)\tilde{K}_{a1}^*$$

and

$$y_{bb} := K_*(b)A^{*-}(b)D^{*-}(b)\tilde{K}_{b2}^*,$$

where K_* is the counterpart of K from (3.2), i. e.,

$$K_* := I - Q_{*0}\hat{P}_{*1}\hat{G}_{*2}^{-1}B^*P_{*0} - Q_{*0}\hat{Q}_{*1}A^{*-}(A^*\hat{Q}_{*1}A^{*-})'A^*$$

Clearly, the conditions $\tilde{K}_{a1}x(a) = \tilde{d}_1$ and $y_{aa}^*A(a)D(a)x(a) = \tilde{d}_1$ are identical, and the same is true for the pair of conditions $\tilde{K}_{b2}x(b) = \tilde{d}_2$ and $y_{bb}^*A(a)D(a)x(a)x(b) = \tilde{d}_2$.

Let y_a and y_b be the solutions of (4.17) with initial values $y_a(a) = y_{aa}$ and $y_b(a) = y_{bb}$. In parallel, let v_a and v_b be the solutions of (4.18) constructed with the corresponding y_a and y_b and initial values $v(a) = \tilde{d}_1$ and $v(b) = \tilde{d}_2$, respectively. Due to Theorem 5, a function $x \in C_D^1$ is a solution of BVP (1.1) if and only if, for every t, $x(t)$ satisfies the system

$$y_a^*(t)A(t)D(t)x(t) = -v_a^*(t) \tag{5.6}$$

$$y_b^*(t)A(t)D(t)x(t) = -v_b^*(t), \tag{5.7}$$

$$\begin{aligned} V^*(t)G_{*2}^*(t)x(t) = {} & \hat{Q}_{*1}^*(t)(P_{*0}^*(t) - Q_{*0}^*(t))q(t) \\ & - \hat{Q}_{*1}^*(t)A(t)(A^{*-*}\hat{Q}_{*1}^*Q_{*0}^*q)'(t), \end{aligned} \tag{5.8}$$

$$Q_{*0}^*(t)B(t)x(t) = Q_{*0}^*(t)q(t). \tag{5.9}$$

By construction, the first pair of equations is linearly independent of the second one. Let $\tau := \dim \operatorname{im}(y_a \mid y_b)$. Also by construction, the equalities

$$\dim \operatorname{im} D^*A^*\,(y_a \mid y_b) \equiv \dim \operatorname{im}(y_a \mid y_b) \equiv \dim \operatorname{im}(y_{aa} \mid y_{bb})$$

hold. We have proved above that $\dim M(t) \equiv r + r_1 - m$. The Fredholm alternative for (5.6)–(5.9) now yields the following statement.

Theorem 6. *For any $q \in C^1_{D\hat{Q}_1G_2^{-1}}$ and $\tilde{d}_1 \in \operatorname{im}\tilde{K}_{a1}$, $\tilde{d}_2 \in \operatorname{im}\tilde{K}_{b2}$, a unique solution x exists if and only if*

$$r + r_1 - m - \tau = 0.$$

5.2. Non-separated boundary conditions. By using Moszyǹski's trick [4], we transform the problem into an equivalent one with separated boundary conditions. For

$t \in [a, (a+b)/2]$, set

$$\bar{x}(t) := \begin{pmatrix} x(t) \\ x(b+a-t) \end{pmatrix}, \quad \bar{q}(t) := \begin{pmatrix} q(t) \\ q(b+a-t) \end{pmatrix},$$

$$\bar{A}(t) := \operatorname{diag}(A(t), A(b+a-t)), \quad \bar{D}(t) := \operatorname{diag}(D(t), D(b+a-t)),$$

$$\bar{B}(t) := \operatorname{diag}(B(t), -B(b+a-t)),$$

$$\bar{K}_a = \begin{pmatrix} K_a & K_b \\ 0 & 0 \end{pmatrix}, \quad \bar{K}_{\frac{a+b}{2}} = \begin{pmatrix} 0 & 0 \\ I_m & -I_m \end{pmatrix}, \quad \bar{d} = \begin{pmatrix} d \\ 0 \end{pmatrix},$$

where I_m is the $m\times m$ identity matrix. Obviously, the original boundary value problem is equivalent to the BVP of doubled dimension on the halved interval $[a, (a+b)/2]$ with the above data. This latter problem for $\bar{x}$ is a BVP with separated boundary conditions, and all considerations of the previous subsection apply.

6. Final remarks

Remark 6. The homogeneous IVPs for (1.2) with our initial data $y(a) = y_{aa}$ and $y(b) = y_{bb}$ in Section 5 are always solvable. Thus, integrating system (1.2) from each of the interval ends up to an arbitrary common point t_0, one obtains $y_a(t_0)$ and $y_b(t_0)$. In parallel, the IVPs for equation (4.18) are to be solved. One should compute (preserve) the values only at points $\hat{t}$ where the solution x is needed. At these points, the other two matrices, $G_{*2}(\hat{t})V(\hat{t})$ and $B^*(\hat{t})Q_{*0}(\hat{t})$, should also be calculated. If the linear system (5.6)–(5.9) is nonsingular at an arbitrary $\hat{t} = t_0$, then so is it for all $\hat{t}$, and one can establish the solvability and uniqueness of the solution and get the solution at all $\hat{t}$.

Remark 7. To go in line with this program, a reliable integrator for (1.2) is needed and all of the other coefficients occurring in (5.6)–(5.9) must be available. It is worth noting that in this system we need only $A^*(t)y(t) = A^*(t)\hat{P}_{*1}y(t)$, i. e., the solution of the inherent ODE of the adjoint equation. One may prefer solving this homogeneous inherent ODE instead of the homogeneous DAE (1.2). Practically, there is no difference in computational complexity. A reliable integrator for any DAE would use its inherent ODE to keep the numerical solution in the corresponding subspace at least implicitly [3].

Remark 8. Theoretically, a properly discretised version of the transfer method would yield an algorithm for the numerical solution of the BVPs for (1.1). However, the resulting procedure may be very sensitive to the accumulation of numerical errors. This phenomenon may appear even when the BVP is well-conditioned and the relevant subspaces vary slowly. Therefore, a modification of the transfer algorithm seems reasonable. That modification would rely on the orthonormalisation of basis vectors of the subspaces in question at the meshpoints or it would build a smoothly varying basis on the whole interval. These issues will be reported in another publication.

Remark 9. There is no gain in the complexity if one avoids using the adjoint equation and the method relies upon any kind of shooting. In that case the computational effort is spent on keeping either a fundamental matrix (not necessarily the maximal normalised fundamental matrix) of the homogeneous DAE (1.1) in the corresponding subspace or some solutions of the DAE (1.1) in the corresponding affine subspace. To achieve this aim, one must use projectors not simpler than those in our analysis.

References

[1] Balla, K.: *Linear subspaces for linear DAEs of index* 1, Computers and Mathematics with Applications, **31** (1996), No. 4/5, 81–86.

[2] Balla, K. and März, R.: *A unified approach to linear differential algebraic equations and their adjoints*, Zeitschrift für Analysis und ihre Anwendungen, **21** (2002), No. 3, 783–802.

[3] Higueras, I. and März, R.: *Differential-algebraic equations with properly stated leading terms*, Computers and Mathematics with Applications, to appear.

[4] Moszyǹski, K.: *A method of solving the boundary value problem for a system of linear ordinary differential equations*, Algoritmy, **11** (1964), No. 3, 25–43.

[5] Zielke, G.: *Motivation und Darstellung von verallgemeinerten Matrixinversen*, Beitr. Num. Math., **7** (1979), 177–218.

Authors' Addresses

Katalin Balla:

Computer and Automation Research Institute, Hungarian Academy of Sciences, H-1518 Budapest, P. O. Box 63, Hungary

E-mail address: balla@sztaki.hu

Roswitha März:

Humboldt University, Institute of Mathematics, D-10099 Berlin, Germany

E-mail address: iam@mathematik.hu-berlin.de