On the uniqueness, stability and hyperbolicity of symmetric and periodic solutions of weakly nonlinear ordinary differential equations

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ON THE UNIQUENESS, STABILITY AND HYPERBOLICITY OF SYMMETRIC AND PERIODIC SOLUTIONS OF WEAKLY NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. We show the existence of unique periodic and symmetric solutions of weakly nonlinear ordinary differential equations. Conditions of stability and hyperbolicity of these solutions are established as well. Concrete examples are presented to show advantages of our method to the classical averaging theory.

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1. INTRODUCTION

In this paper we consider systems of ordinary differential equations under periodic and symmetric assumptions. More concretely, we mostly consider a weakly nonlinear ordinary differential equation of the form

$$x'(t) = \varepsilon f(x(t), t), \qquad t \in \mathbb{R}, \tag{1.1}$$

with a small parameter $\varepsilon \in \mathbb{R}$, where $x \colon \mathbb{R} \to \mathbb{R}^n$. We suppose that the function $f \colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ in (1.1) is sufficiently smooth, pT-periodic in the second variable, and symmetric in the first one, i. e.,

$$Af(x, t) = f(Ax, t + T), \qquad t \in \mathbb{R}, \tag{1.2}$$

where $A \colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear map such that $A^p = \mathbb{1}$ for some $p \in \mathbb{N}$. Due to this property, we have the following useful construction. Let $\langle \cdot, \cdot \rangle$ be a scalar product on $\mathbb{R}^n$. By setting

$$(x, y) := \frac{1}{p} \sum_{i=0}^{p-1} \langle A^i x, A^i y \rangle, \qquad \{x, y\} \subset \mathbb{R}^n,$$

as a new scalar product on $\mathbb{R}^n$, we obtain $(Ax, Ay) = (x, y)$ and $|Ax| = |x|$ for $|x| := \sqrt{(x, x)}$, $x \in \mathbb{R}^n$, and thus A is unitary with respect to $(\cdot, \cdot)$. We mostly suppose $1 \notin \sigma(A)$, where $\sigma(A)$ is the spectrum of A. The norm[*] on $\mathcal{L}(\mathbb{R}^n)$ generated by $|\cdot|$ is denoted by $\|\cdot\|$.

Remark 1.1. We study (1.1) always in an arbitrary but fixed open bounded subset of $\mathbb{R}^n$. So we do not study bifurcations from infinity of (1.1) for ε small.

Next, for $T > 0$ we introduce the Banach space

$$X := \left\{ x \in C^0(\mathbb{R}, \mathbb{R}^n) \mid x(t + T) = Ax(t), \, t \in \mathbb{R} \right\}. \tag{1.3}$$

Note $|x(\cdot)|$ is a T-periodic function because $|x(t + T)| = |Ax(t)| = |x(t)|$ for every t. So we take the norm $\|x\| = \max_{t \in [0,T]} |x(t)| = \max_{t \in \mathbb{R}} |x(t)|$ on X. It is easy to see that if a function x belongs to the space X then it is a pT-periodic function.

Definition 1.1. By a *symmetric and periodic solution x* of equation (1.1) we understand a solution $x \in X$, where X is defined by (1.3).

The main goal of this paper is to find a unique symmetric and periodic solution (see Sections 2 and 3) for (1.1) and to establish conditions, under which this solution is either stable (see Section 4) or hyperbolic (see Sections 5 and 6). We also present examples to illustrate the theory in Sections 7 and 8. We show that our symmetric averaging method could help when the classical averaging theory fails [18, 27].

The results presented here are generalizations of achievements for anti-periodic problems with $A = -\mathbb{1}$ [1,2], and continuations of [7]. Doubly symmetric solutions of reversible systems are studied in [20]. Symmetric properties of periodic solutions of nonlinear nonautonomous ordinary differential equations are also studied in [23,24].

2. UNIQUENESS FOR GENERAL NONLINEARITIES

In spite of the fact that we are interested in (1.1), we start with its general form with $\varepsilon = 1$, i. e., we first consider the equation

$$x'(t) = f(x(t), t), \qquad t \in \mathbb{R}. \tag{2.1}$$

Looking for a symmetric and periodic solution x of equation (2.1) is equivalent to solve (2.1) on $t \in [0, T]$ under the boundary condition

$$x(T) = Ax(0). \tag{2.2}$$

The following lemma guarantees the existence and uniqueness of a solution of problem (2.1), (2.2).

Lemma 2.1. *Assume that* $1 \notin \sigma(A)$ *and, moreover, there exists a constant* $K > 0$ *such that*

$$|f(x_1, t) - f(x_2, t)| \leq K |x_1 - x_2| \tag{2.3}$$

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[*]As usual, the symbol $\mathcal{L}(\mathbb{R}^n)$ stands for the algebra of square matrices of dimension n.

for all x_1, x_2 from $\mathbb{R}^n$. If

$$\left(\|(A - \mathbb{1})^{-1}\| + 1\right) KT < 1, \tag{2.4}$$

then equation (2.1) has a unique solution $x \in X$.

Proof. Let us first solve the boundary value problem

$$x' = \widetilde{h}, \qquad x(T) = Ax(0), \tag{2.5}$$

where $\widetilde{h} \in X$. Obviously, $x(t) = x(0) + \int_0^t \widetilde{h}(s)ds$ and $Ax(0) = x(T) = x(0) + \int_0^T \widetilde{h}(s)ds$, whence

$$x(0) = (A - \mathbb{1})^{-1} \int_0^T \widetilde{h}(s)ds.$$

and, therefore, the solution of (2.5) has the form

$$x(t) = (A - \mathbb{1})^{-1} \int_0^T \widetilde{h}(s)ds + \int_0^t \widetilde{h}(s)ds, \quad t \in [0, T].$$

Let us put

$$B(x)(t) := (A - \mathbb{1})^{-1} \int_0^T f(x(s), s)ds + \int_0^t f(x(s), s)ds, \quad t \in [0, T],$$

for an arbitrary x from X. For any $x_1, x_2 \in X$, using the Lagrange theorem and (2.3), we get

$$\|B(x_1) - B(x_2)\| = \left\|(A - \mathbb{1})^{-1} \int_0^T (f(x_1(s), s) - f(x_2(s), s))ds \right.$$
$$\left. + \int_0^t (f(x_1(s), s) - f(x_2(s), s))ds \right\|$$
$$\leq \left(\|(A - \mathbb{1})^{-1}\| + 1\right) KT \|x_1(t) - x_2(t)\|.$$

Taking condition (2.4) into account and applying the Banach fixed point theorem we get that problem (2.1), (2.2) has a unique solution $x \in X$. The proof is complete. □

Now we replace Lipschitz condition (2.3) as follows. Let $|\cdot|_0$ be a norm on $\mathbb{R}^n$. Then for any $x, y \in \mathbb{R}^n$ the limit [3, 4]

$$N(x, y) := \lim_{\lambda \to +0} \frac{|x + \lambda y|_0 - |x|_0}{\lambda}$$

exists and has the following elementary properties:

$$\left. \begin{aligned} N(cx, cy) &= cN(x, y), \\ |N(x, y_1) - N(x, y_1)| &\leq |y_1 - y_2|_0, \\ N\left(x, \frac{1}{m}\sum_{k=1}^{m} y_k\right) &\leq \frac{1}{m}\sum_{k=1}^{m} N(x, y_k) \end{aligned} \right\} \tag{2.6}$$

for any $c > 0$ and x, y, y_1, y_2, ..., y_m from $\mathbb{R}^n$. So the function $y \mapsto N(x, y)$ is globally Lipschitz continuous and convex. Next, for any $B \in \mathcal{L}(\mathbb{R}^n)$, there is its Lozinskii logarithmic norm [16]

$$\mu(B) := \lim_{\lambda \to +0} \frac{\|\mathbb{1} + \lambda B\|_0 - 1}{\lambda},$$

where $\|B\|_0 := \max_{|x|_0=1} |Bx|_0$. Note that

$$\sup_{|x|_0=1} N(x, Bx) = \mu(B). \tag{2.7}$$

Now we are ready to prove the following results motivated by [6, 14, 15] and [30, Theorem 9.6].

Lemma 2.2. *Let* $f \in C^1(\mathbb{R}^{n+1}, \mathbb{R}^n)$. *If there is a constant* $\alpha > 0$ *such that*

$$\mu(D_x f(x, t)) \leq -\alpha \quad \forall (x, t) \in \mathbb{R}^{n+1}$$

then (2.1) *possesses a unique symmetric and periodic solution which is globally asymptotically stable.*

Proof. Let $x(t)$ be a solution of (2.1) defined on $[0, \delta)$ for some $\delta > 0$. Then by (2.6), (2.7) and the Jensen inequality [25], for almost each $t \in [0, \delta)$, we derive

$$\begin{aligned} \frac{d}{dt}|x(t)|_0 &= N(x(t), x'(t)) = N(x(t), f(x(t), t)) \\ &\leq N(x(t), f(x(t), t) - f(0, t)) + |f(0, t)|_0 \\ &\leq N\left(x(t), \int_0^1 D_x f(\theta x(t), t) x(t) d\theta\right) + \|f(0, \cdot)\|_0 \\ &\leq \int_0^1 N(x(t), D_x f(\theta x(t), t) x(t)) d\theta + \|f(0, \cdot)\|_0 \\ &\leq \int_0^1 \mu(D_x f(\theta x(t), t)) d\theta |x(t)|_0 + \|f(0, \cdot)\|_0 \\ &\leq -\alpha|x(t)|_0 + \|f(0, \cdot)\|_0, \end{aligned} \tag{2.8}$$

where $\|f(0,\cdot)\|_0 = \max_{t\in\mathbb{R}}|f(0,t)|$. Note $|x(t)|_0$ is locally absolute continuous. The Gronwall inequality implies

$$|x(t)|_0 \le \mathrm{e}^{-\alpha t}|x(0)|_0 + \frac{\|f(0,\cdot)\|_0}{\alpha}\left(1 - \mathrm{e}^{-\alpha t}\right),$$

so $x(t)$ is defined for any $t \ge 0$ and it terminates in the ball $B\left(0, 2\|f(0,\cdot)\|_0/\alpha\right)$ centered at 0 with the radius $2\|f(0,\cdot)\|_0/\alpha$, i.e., (2.1) is dissipative possessing a pT-periodic solution $x_0(t)$ inside $B\left(0, 2\|f(0,\cdot)\|_0/\alpha\right)$ (see [4]).

Next, let $x_1(t)$ and $x_2(t)$ be two solutions of (2.1) on $\mathbb{R}_+$. Then similarly to the consideration above, for almost every $t \in \mathbb{R}_+$, we obtain

$$\frac{d}{dt}|x_1(t)-x_2(t)|_0 = N\left(x_1(t)-x_2(t), \frac{d}{dt}(x_1(t)-x_2(t))\right)$$

$$= N(x_1(t)-x_2(t), f(x_1(t),t)-f(x_2(t),t))$$

$$= N\left(x_1(t)-x_2(t), \int_0^1 D_x f(\theta x_1(t)+(1-\theta)x_2(t),t)(x_1(t)-x_2(t))d\theta\right)$$

$$\le \int_0^1 N\left(x_1(t)-x_2(t), D_x f(\theta x_1(t)+(1-\theta)x_2(t),t)(x_1(t)-x_2(t))\right)d\theta$$

$$\le \int_0^1 \mu\left(D_x f(\theta x_1(t)+(1-\theta)x_2(t),t)\right)d\theta|x_1(t)-x_2(t)|_0$$

$$\le -\alpha|x_1(t)-x_2(t)|_0,$$

i.e., the Gronwall inequality again implies

$$|x_1(t)-x_2(t)|_0 \le \mathrm{e}^{-\alpha t}|x_1(0)-x_2(0)|_0. \tag{2.9}$$

So (2.9) gives the uniqueness of $x_0(t)$ and it is globally asymptotically stable. It is easy to verify that $x(t) := A^{-1}x_0(t+T)$ is also a pT-periodic solution of (2.1). The uniqueness gives $x(t) = x_0(t)$, i.e., $x_0 \in X$. The proof is finished. $\square$

Lemma 2.2 is extended in Corollary 4.1 of Section 4 in the context of averaging theory. The above result can be slightly modified as follows.

Lemma 2.3. *Let $f \in C^1(\mathbb{R}^{n+1},\mathbb{R}^n)$. Assume that*

$$\mu\left(D_x f(x,t)\right) \le 0 \quad \forall(x,t) \in \mathbb{R}^{n+1}$$

and, moreover, that (2.1) possesses a bounded uniformly asymptotically stable solution x_0 on $\mathbb{R}_+$. Then x_0 is symmetric, periodic, and globally asymptotically stable.

Proof. Let $x(t)$ be a solution of (2.1) defined on $[0,\delta)$ for some $\delta > 0$. Then for $\alpha = 0$, (2.8) implies that $|x(t)|_0 \le |x(0)|_0 + \|f(0,\cdot)\|_0 t$ holds for all $t \ge 0$. So $x(t)$ is defined on $[0,\infty)$.

Next, let $x_1(t)$ and $x_2(t)$ be two solutions of (2.1) on $\mathbb{R}_+$. Then for $\alpha = 0$, (2.9) implies

$$|x_1(t)-x_2(t)|_0 \le |x_1(0)-x_2(0)|_0 \quad \forall t \ge 0. \tag{2.10}$$

Furthermore, from the uniform asymptotic stability of $x_0(t)$ on $\mathbb{R}_+$, there is an $\delta_0 > 0$ such that

$$\{x \in \mathbb{R}^n \mid \mathrm{dist}\,(x, \{x_0(t) \mid t \in \mathbb{R}_+\}) < \delta_0\} \subset B_A\,,$$

where B_A is the domain of attraction of $x_0(t)$. Then (2.10) implies that $B(z, \delta_0/2) \subset B_A$ holds for any $z \in B_A$. Consequently, $B_A = \mathbb{R}^n$. Since $x_0(t)$ is bounded on $\mathbb{R}_+$ by assumption, all solutions of (2.1) terminates in a large ball. So (2.1) has a pT-periodic solution which now must be $x_0(t)$. Then like above it is unique and symmetric. The proof is finished. $\square$

We recall the following well-known results:

(1) If $|x|_0 = \sqrt{\sum_{k=1}^{m} x_k^2}$ then $\mu(B) = \Lambda\left(\frac{1}{2}(B + B^*)\right)$, where Λ means the largest eigenvalue.

(2) If $|x|_0 = \max_{k=1,2,\ldots,m} |x_k|$ then

$$\mu(B) = \max_{k=1,2,\ldots,m} \left(b_{kk} + \sum_{j \neq k} |b_{kj}| \right),$$

where b_{kj}, $k, j = 1, 2, \ldots, m$, mean the entries of matrix B.

(3) If $|x|_0 = \sum_{k=1}^{m} |x_k|$ then $\mu(B) = \max_{k=1,2,\ldots,m} \left(b_{kk} + \sum_{j \neq k} |b_{jk}| \right)$.

Applying Lemmas 2.2 and 2.3, we obtain the following particular results used in the sequel [6].

Lemma 2.4. *Let us put*

$$J(x,t) := \frac{1}{2}\left(D_x f(x,t) + D_x f(x,t)^* \right) \tag{2.11}$$

and let $\Lambda(x,t)$ be the largest eigenvalue of $J(x,t)$. If there is a constant $\alpha > 0$ such that

$$\Lambda(x,t) \leq -\alpha \quad \forall (x,t) \in \mathbb{R}^{n+1} \tag{2.12}$$

then (2.1) possesses a unique symmetric and periodic solution which is globally asymptotically stable.

Lemma 2.5. *Assume*

$$\Lambda(x,t) \leq 0 \quad \forall (x,t) \in \mathbb{R}^{n+1} \tag{2.13}$$

and (2.1) possesses a bounded, uniformly asymptotically stable solution $x_0(t)$ on $\mathbb{R}_+$. Then $x_0(t)$ is symmetric, periodic and globally asymptotically stable.

Remark 2.1. We do not need to suppose that $1 \notin \sigma(A)$ in Lemmas 2.2–2.5.

Remark 2.2. By reversing the time $t \leftrightarrow -t$ in (2.1), we get similar results like above ensuring that $x_0(t)$ is a global repeller, namely, (2.12) and (2.13) are replaced by the inequalities $\lambda(x,t) \geq \alpha$ and $\lambda(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^{n+1}$, respectively, where $\lambda(x,t)$ is the smallest eigenvalue of $J(x,t)$.

3. UNIQUENESS FOR WEAK NONLINEARITIES

First, we consider the Cauchy problem

$$x'(t) = \varepsilon f(x(t), t), \quad t \in [0, T], \tag{3.1}$$

$$x(0) = x. \tag{3.2}$$

Let us denote by $\varphi_\varepsilon(x, t)$ the unique solution of problem (3.1), (3.2). Note that

$$\varphi_\varepsilon(x, 0) = x. \tag{3.3}$$

Obviously, the equation

$$\varphi_\varepsilon(x, pT) = x \tag{3.4}$$

determines pT-periodic solutions of (1.1). For this reason, we consider the diffeomorphism given by the formula

$$P_\varepsilon(x) := \varphi_\varepsilon(x, pT). \tag{3.5}$$

Taking (3.4) and (3.5) into account, we get

$$P_\varepsilon(x) = x. \tag{3.6}$$

On the other hand, the T-periodic and symmetric solutions of (1.1), i. e., the solutions belonging to X, are given by the equation

$$Ax = \varphi_\varepsilon(x, T).$$

In this connection, we introduce the new mapping

$$g_\varepsilon(x) := A^{-1} \varphi_\varepsilon(x, T). \tag{3.7}$$

Summarizing the considerations above, we arrive at the following simple lemma.

Lemma 3.1. *A fixed point x of P_ε, respectively of g_ε, determines a pT-periodic, respectively a periodic and symmetric, solution of equation* (1.1) *satisfying* (3.2).

The basic relationship between mappings P_ε and g_ε is expressed by the following lemma.

Lemma 3.2. *The relation*

$$P_\varepsilon(x) = [g_\varepsilon(x)]^p \tag{3.8}$$

holds, where P_ε and g_ε are defined, respectively, by (3.5) *and* (3.7).

In (3.8), $[g_\varepsilon(x)]^p$ is defined as

$$[g_\varepsilon(x)]^p = \underbrace{g_\varepsilon(g_\varepsilon(g_\varepsilon(g_\varepsilon \ldots (x)))))}_{p}, \quad x \in \mathbb{R}^n.$$

Proof of Lemma 3.2. It is easy to see that

$$\varphi_\varepsilon(x, t + T) = A\varphi_\varepsilon(A^{-1}\varphi_\varepsilon(x, T), t),$$

which implies

$$\varphi_\varepsilon(x, 2T) = A\varphi_\varepsilon(A^{-1}\varphi_\varepsilon(x, T), T) = AAA^{-1}\varphi_\varepsilon(g_\varepsilon(x), T).$$

Taking into account (3.7), the last relation is equal to

$$\varphi_\varepsilon(x, 2T) = A^2[g_\varepsilon(x)]^2.$$

Let us apply inductive method. Assume that relation

$$\varphi_\varepsilon(x, iT) = A^i[g_\varepsilon(x)]^i$$

is true. Now we show that this relation is true also for $i + 1$:

$$\varphi_\varepsilon(x, (i + 1)T) = \varphi_\varepsilon(x, iT + T) = A\varphi_\varepsilon(A^{-1}\varphi_\varepsilon(x, T), iT)$$
$$= A\varphi_\varepsilon(g_\varepsilon(x), iT) = A \cdot A^i g_\varepsilon[g_\varepsilon(x)]^i = A^{i+1}[g_\varepsilon(x)]^{i+1}.$$

So, finally we obtain

$$P_\varepsilon(x) = \varphi_\varepsilon(x, pT) = A^p[g_\varepsilon(x)]^p = [g_\varepsilon(x)]^p.$$

The proof is finished. $\square$

In the sequel we assume for simplicity that f is C^∞-smooth. Asymptotic expansion can be obtained by the methods of averaging and multiple scales. So there exist the Taylor series of P_ε and g_ε with respect to the small parameter ε:

$$P_\varepsilon(x) = P_0(x) + \varepsilon P_1(x) + \dots \tag{3.9}$$

and

$$g_\varepsilon(x) = g_0(x) + \varepsilon g_1(x) + \dots.$$

Remark 3.1. It follows from the definition of the mapping P_ε given by formula (3.5) that $P_0 = \mathbb{1}$.

The next lemma follows from Lemma 3.2.

Lemma 3.3. *If x is a solution of equation*

$$g_\varepsilon(x) = x, \tag{3.10}$$

then it is also a solution of equation (3.6).

Let us solve equation (3.10), which, by (3.7), has the form

$$H(x, \varepsilon) := A^{-1}\varphi_\varepsilon(x, T) - x = 0.$$

Since

$$g_0(x) = A^{-1}\varphi_0(x, T) = A^{-1}x, \tag{3.11}$$

we obtain that $H(0, 0) = 0$ and $D_x H(0, 0) = A^{-1} - \mathbb{1}$. Using the assumption $1 \notin \sigma(A)$ and applying the Implicit Function Theorem, we get the following result.

Theorem 3.1. *If* $1 \notin \sigma(A)$ *then equation* (3.10) *has a unique* C^∞*-smooth solution* $x_\varepsilon = O(\varepsilon)$ *for* $\varepsilon \neq 0$ *small. Note that*

$$P_\varepsilon(x_\varepsilon) = x_\varepsilon.$$

Theorem 3.1 implies the following one.

Theorem 3.2. *If* $1 \notin \sigma(A)$ *and* $\varepsilon \neq 0$ *is small, then equation* (1.1) *has a unique* pT*-periodic solution* $x_\varepsilon(t) \in X$ *such that* $x_\varepsilon(0) = x_\varepsilon$ *and* $x_\varepsilon(t) = O(\varepsilon)$.

Proof. Lemmas 3.1, 3.2 and Theorem 3.1 immediately give the result. □

4. Stability of periodic and symmetric solutions

To study stability of the T-periodic and symmetric solution $x_\varepsilon(\cdot) \in X$ of equation (1.1), we consider the linearization

$$M(\varepsilon) := D_x g_\varepsilon(x_\varepsilon) \tag{4.1}$$

of mapping (3.7) at the fixed point $x_\varepsilon = O(\varepsilon)$. Note that, by (3.8), we have

$$[D_x g_\varepsilon(x_\varepsilon)]^p = D_x P_\varepsilon(x_\varepsilon),$$

which, by the Dunford spectral mapping theorem [11], yields

$$\sigma(D_x P_\varepsilon(x_\varepsilon)) = [\sigma(D_x g_\varepsilon(x_\varepsilon))]^p. \tag{4.2}$$

Taking (3.7) into account, we get

$$D_x g_\varepsilon(x) = A^{-1} D_x \varphi_\varepsilon(x, T).$$

So, using (3.11), we obtain

$$D_x g_0(x) = A^{-1} D_x \varphi_0(x, T) = A^{-1}. \tag{4.3}$$

From (4.1), (4.3) we have that

$$M(0) = A^{-1}, \tag{4.4}$$

but A is a unitary linear mapping with all eigenvalues on the unit circle.

Let us consider the decomposition of the mapping $M(\varepsilon)$ into the Taylor series on a small parameter ε using (4.4):

$$M(\varepsilon) = A^{-1} + \varepsilon M_1 + \ldots. \tag{4.5}$$

By definition,

$$M_1 = \frac{d}{d\varepsilon} M(0),$$

it is easy to see, taking into account (4.1), $x_0 = 0$ and $D_x^2 g_0(0) = 0$, that

$$M_1 = \frac{d}{d\varepsilon} [D_x g_\varepsilon(x_\varepsilon)] \Big|_{\varepsilon=0} = D_x^2 g_0(0) \frac{d}{d\varepsilon} x_0 + \frac{d}{d\varepsilon} D_x g_0(x_0)$$

$$= A^{-1} \frac{d}{d\varepsilon} D_x \varphi_0(0, T). \tag{4.6}$$

Let us consider problem (3.1), (3.2). By (3.3), we get that

$$\dot{\varphi}_\varepsilon(x,t) = \varepsilon f(\varphi_\varepsilon(x,t),t), \tag{4.7}$$

$$\varphi_\varepsilon(x,0) = x. \tag{4.8}$$

We differentiate (4.7), (4.8) with respect to x and get

$$\frac{d}{dt}(D_x\varphi_\varepsilon(x,t)) = \varepsilon D_x f(\varphi_\varepsilon(x,t),t) D_x\varphi_\varepsilon(x,t), \tag{4.9}$$

$$D_x\varphi_\varepsilon(x,0) = \mathbb{1}. \tag{4.10}$$

Now differentiating (4.9), (4.10) with respect to ε and putting $\varepsilon = 0$, we get

$$\frac{d}{dt}\Big(\frac{d}{d\varepsilon}D_x\varphi_0(x,t)\Big) = D_x f(x,t) D_x\varphi_0(x,t),$$

$$\frac{d}{d\varepsilon}D_x\varphi_0(x,0) = 0.$$

Thus,

$$\frac{d}{d\varepsilon}D_x\varphi_0(x,T) = \int_0^T D_x f(x,s)ds. \tag{4.11}$$

Using (4.6) and (4.11) we arrive at the equality

$$M_1 = A^{-1}\int_0^T D_x f(0,s)ds.$$

Now we return to the Taylor series (4.5). Considering previous calculations and using that $x_\varepsilon = O(\varepsilon)$ from Theorem 3.1, we derive

$$M(\varepsilon) = A^{-1} + \varepsilon A^{-1}\int_0^T D_x f(0,s)ds + \cdots$$

$$= A^{-1}\Big(\mathbb{1} + \varepsilon \int_0^T D_x f(0,s)ds + \cdots\Big) = A^{-1}G(\varepsilon), \quad (4.12)$$

where $G(\varepsilon) = AM(\varepsilon) = \mathbb{1} + \varepsilon\int_0^T D_x f(0,s)ds + \cdots$. Since A^{-1} is unitary, we get that $M(\varepsilon)$ is stable if $G(\varepsilon)$ is stable. Let us now prove a generalization of a classical result [5, 6, 22].

Theorem 4.1. *Let $M(\varepsilon) \in \mathcal{L}(\mathbb{R}^n)$ be given by (4.12). If*

$$\mathrm{Re}\left\{\sigma\left(\int_0^T D_x f(0,s)ds\right)\right\} \subset (-\infty,0), \tag{4.13}$$

then there exists an $\varepsilon_0 > 0$ such that $r(M(\varepsilon)) < 1$ for any $0 < \varepsilon \le \varepsilon_0$, where $r(M(\varepsilon))$ is the spectral radius of $M(\varepsilon)$.

In order to prove Theorem 4.1, we recall the following well-known results [10, 11, 17].

Theorem 4.2. *Let $K \in \mathcal{L}(\mathbb{R}^n)$. Then $r(K) < 1$ if and only if there exists a norm on $\mathbb{R}^n$ such that $\|K\| < 1$.*

Theorem 4.3. *Assume that* (4.13) *is fulfilled then there exists a scalar product $\langle \cdot, \cdot \rangle_1$ and constant α, $\alpha > 0$ such that*

$$\left\langle \left(\int_0^T D_x f(0,s) ds \right) x, x \right\rangle_1 \leq -\alpha \|x\|_1^2 \quad \text{for any } x \in \mathbb{R}^n, \tag{4.14}$$

where $\|x\|_1 = \sqrt{\langle x, x \rangle_1}$, $x \in \mathbb{R}^n$.

Proof of Theorem 4.1. Consider the expression $\|G(\varepsilon)x\|_1^2 = \langle G(\varepsilon)x, G(\varepsilon)x \rangle_1$. Differentiating $\|G(\varepsilon)\|_1^2$ with respect to ε, we get

$$\frac{d}{d\varepsilon} \|G(\varepsilon)x\|_1^2 = 2\langle G'(\varepsilon)x, G(\varepsilon)x \rangle_1.$$

Using the Lagrange theorem, we obtain

$$\begin{aligned}
\|G(\varepsilon)x\|_1^2 &= \|G(0)x\|_1^2 + \frac{d}{d\varepsilon} \|G(\theta)x\|_1^2 \varepsilon \\
&= \|x\|_1^2 + 2\langle G'(\theta)x, G(\theta)x \rangle_1 \varepsilon, \quad 0 \leq \theta \leq \varepsilon.
\end{aligned} \tag{4.15}$$

By virtue of (4.14), we arrive at the equality

$$\langle G'(0)x, G(0)x \rangle_1 = \left\langle \left(\int_0^T D_x f(0,s) ds \right) x, x \right\rangle_1 \leq -\alpha \|x\|_1^2.$$

Thus,

$$2\langle G'(\theta)x, G(\theta)x \rangle_1 \leq -\alpha \|x\|_1^2$$

for $0 < \varepsilon \leq \varepsilon_0$ with ε_0 small enough. Returning to (4.15), we obtain

$$\|G(\varepsilon)x\|_1^2 \leq \|x\|_1^2 - \alpha \|x\|_1^2 \varepsilon = (1 - \alpha\varepsilon)\|x\|_1^2.$$

Supposing that $\varepsilon_0 \alpha < 1$, we get

$$\|G(\varepsilon)x\|_1 \leq \sqrt{1 - \alpha\varepsilon} \|x\|_1 \quad \forall x \in \mathbb{R}^n.$$

Consequently, we have that $\|G(\varepsilon)\|_1 \leq \sqrt{1 - \alpha\varepsilon} < 1$. Then Theorem 4.2 implies that $r(G(\varepsilon)) < 1$. Finally, since A is unitary with respect to the norm $|\cdot|$, we have [11]

$$r(M(\varepsilon)) = \lim_{k \to \infty} \sqrt[k]{\|M(\varepsilon)^k\|} = \lim_{k \to \infty} \sqrt[k]{\|G(\varepsilon)^k\|} = r(G(\varepsilon)) < 1,$$

which completes the proof of Theorem 4.1. $\qquad \square$

Summarizing, we obtain the following result.

Theorem 4.4. *Suppose $1 \notin \sigma(A)$. Then for any $\varepsilon > 0$ small, the unique symmetric and T-periodic solution $x(t) = O(\varepsilon)$ of equation* (1.1) *is asymptotically stable if condition* (4.13) *is fulfilled.*

Proof. We know from Theorem 4.1 that $r\left(D_x g_\varepsilon(x_\varepsilon)\right) < 1$. Then, using (4.2), we have

$$r\left(D_x P_\varepsilon(x_\varepsilon)\right) = r\left(D_x g_\varepsilon(x_\varepsilon)\right)^p < 1.$$

The proof is complete. □

Finally, we generalize Lemma 2.2 as follows to get a practical global stability of symmetric and periodic solutions of (1.1) [12]. By using the change of variables [8]

$$x \longleftrightarrow x + \varepsilon\left(\int_0^t f(x,s)\,ds - \bar{f}(x)t\right),$$

equation (1.1) is transformed to the form

$$x'(t) = \varepsilon\,\bar{f}(x(t)) + \varepsilon^2 g(x(t),t,\varepsilon) \tag{4.16}$$

with a C^∞-smooth and pT-periodic g. Here $\bar{f}(x) := \frac{1}{pT}\int_0^{pT} f(x,s)\,ds$. Note that $A\bar{f}(x) = \bar{f}(Ax)$. Therefore, $A\bar{f}(0) = \bar{f}(0)$ and, thus, $\bar{f}(0) = 0$ because $1 \notin \sigma(A)$. Suppose the existence of a norm $|\cdot|_0$ on $\mathbb{R}^n$ and a constant $\alpha > 0$ such that

$$\mu(D\bar{f}(x)) < -\alpha \quad \text{for any } x \in \mathbb{R}^n, \tag{4.17}$$

where μ is the corresponding Lozinskii logarithmic norm. Then from the proof of Lemma 2.2 we see that equilibrium $z = 0$ is a global attractor for the system

$$\dot{z} = \varepsilon\,\bar{f}(z). \tag{4.18}$$

Now we are ready to prove the following results [28].

Theorem 4.5. *Suppose that* (4.17) *holds for a norm* $|\cdot|_0$ *on* $\mathbb{R}^n$ *and a constant* $\alpha > 0$ *with the corresponding Lozinskii logarithmic norm* μ. *Take* $R_0 > R_1 > 0$ *and put*

$$K_0 := \max_{|x|_0 \le R_0, |\varepsilon| \le 1, t \in \mathbb{R}} |g(x,t,\varepsilon)|_0,$$

and

$$K_1 := \max_{|x|_0 \le R_0, |\varepsilon| \le 1, t \in \mathbb{R}} \|D_x g(x,t,\varepsilon)\|_0.$$

If

$$0 < \varepsilon < \min\left\{1, \frac{R_0 - R_1}{K_0}\alpha, \frac{R_0\alpha}{3K_0}, \frac{K_1}{\alpha}\right\}, \tag{4.19}$$

then any solution of (4.16) *starting from the ball* $B(0, R_1)$ *remains in* $B(0, R_0)$ *and tends asymptotically to a unique* pT-*periodic solution of* (4.16).

Moreover, if $x(\cdot)$ *and* $z(\cdot)$ *are solutions of* (4.16) *and* (4.18) *on* $\mathbb{R}_+$ *such that* $x(0) = z(0)$ *and* $|x(0)| \le R_1$, *then*

$$|x(t) - z(t)|_0 \le \frac{K_0}{\alpha}\varepsilon \tag{4.20}$$

for all $t \in \mathbb{R}_+$.

Proof. We take ε satisfying (4.19). Let $x(t)$ be a solution of (4.16) such that $|x(0)| \leq R_1$ and $|x(t)| \leq R_0$ for all $t \in [0, \delta)$. Then from the proof of Lemma 2.2, we obtain

$$|x(t)|_0 \leq \mathrm{e}^{-\alpha \varepsilon t} R_1 + \frac{\varepsilon K_0}{\alpha} \leq R_1 + \frac{\varepsilon K_0}{\alpha} \leq R_0$$

for any $t \in [0, \delta)$. Thus, in fact, $\delta = +\infty$. Since

$$\frac{\varepsilon K_0}{\alpha} \leq \frac{R_0}{3} < \frac{R_0}{2},$$

all solutions starting from $B(0, R_1)$ terminate in $B(0, R_0/2)$ as $t \to \infty$. Next, again from the proof of Lemma 2.2, we obtain

$$|x_1(t) - x_2(t)|_0 \leq \mathrm{e}^{-\varepsilon(\alpha - \varepsilon K_1)t} |x_1(t) - x_2(t)|_0 \leq 2\,\mathrm{e}^{-\varepsilon(\alpha - \varepsilon K_1)t} R_1 \quad \forall t \geq 0$$

for any two solutions $x_1(t)$ and $x_2(t)$ of (4.16) starting in $B(0, R_1)$. Consequently, (4.16) has a unique pT-periodic solution in $B(0, R_1)$ and it is attracting all solutions starting in this ball. Finally, like above we see that any solution of (4.18) starting in ball $B(0, R_1)$ remains in $B(0, R_1)$ and it asymptotically tends to an equilibrium $z = 0$. Estimate (4.20) is derived similarly for any solutions $x(\cdot)$ and $z(\cdot)$ of (4.16) and (4.18) on $\mathbb{R}_+$ with $x(0) = z(0)$ and $|x(0)| \leq R_1$. The proof is complete. $\square$

We get immediately from Theorem 4.5 the following result.

Corollary 4.1. *Assume that* (4.17) *holds for a norm* $|\cdot|_0$ *on* $\mathbb{R}^n$ *and a constant* $\alpha > 0$ *with the corresponding Lozinskii logarithmic norm* μ. *Then for any bounded subset* $\mathcal{S} \subset \mathbb{R}^n$ *there are constants* $\widetilde{K} > 0$ *and* $\widetilde{\varepsilon} > 0$ *such that for any* $0 < \varepsilon < \widetilde{\varepsilon}$, *if* $x(t)$ *and* $z(t)$ *are solutions of* (1.1) *and* (4.18) *on* $\mathbb{R}_+$ *with* $x(0) = z(0) \in \mathcal{S}$, *then*

$$|x(t) - z(t)|_0 \leq \widetilde{K}_0 \varepsilon \tag{4.21}$$

for all $t \in \mathbb{R}_+$. *Moreover,* $x(t)$ *asymptotically tends to a unique periodic and symmetric solution of* (1.1).

Remark 4.1. Results similar to the estimates (4.20) and (4.21) above are obtained in the paper [27].

5. AVERAGING THEORY

In this section we survey classical results [5, 9, 18, 21, 22, 26, 27] on the local uniqueness and stability of pT-periodic solutions of equation (1.1). We already know that pT-periodic solutions are described by mapping (3.5). So, now we recall some known results for this mapping. Taking into account (3.9), we have

$$P_\varepsilon(x) = \varphi_0(x, pT) + \varepsilon \varphi_1(x, pT) + \cdots \tag{5.1}$$

From (3.6) we have

$$P_0(x) + \varepsilon P_1(x) + \cdots = x,$$

which, in view of the relation $P_0(x) = \varphi_0(x, pT) = x$, becomes $\varepsilon P_1(x) + \cdots = 0$. It is easy to see that $P_1(x) = \frac{d}{d\varepsilon} P_\varepsilon(x)|_{\varepsilon=0}$, and using (5.1) we get

$$P_1(x) = \frac{d}{d\varepsilon} \varphi_0(x, pT). \tag{5.2}$$

Similar as it is done in Section 4 we solve the problem (4.7), (4.8) and considering (5.2), one can get that

$$P_1(x) = \frac{d}{d\varepsilon} D\varphi_0(0, pT) = \int_0^{pT} f(x,s)ds \tag{5.3}$$

and

$$D_x P_1(x) = \int_0^{pT} D_x f(x,s)ds. \tag{5.4}$$

Now repeating the procedure of the previous sections, we are ready to state the following classical result.

Theorem 5.1. *If there exists $x_0 \in \mathbb{R}^n$ such that*

$$\int_0^{pT} f(x_0,s)ds = 0$$

and

$$\det \int_0^{pT} D_x f(x_0,s)ds \neq 0,$$

then, for any $\varepsilon \neq 0$ small, equation (1.1) has a unique pT-periodic solution $x(t) = x_0 + O(\varepsilon)$. Moreover, for $\varepsilon > 0$ small, the following statements are true:

(1) *If $\mathrm{Re}\left\{\sigma\left(\int_0^{pT} D_x f(x_0,s)ds\right)\right\} \subset (-\infty, 0)$ then $x(t)$ is asymptotically stable.*

(2) *If $\mathrm{Re}\left\{\sigma\left(\int_0^{pT} D_x f(x_0,s)ds\right)\right\} \cap (0, \infty) \neq \varnothing$ then $x(t)$ is unstable.*

(3) *If $\mathrm{Re}\left\{\sigma\left(\int_0^{pT} D_x f(x_0,s)ds\right)\right\} \subset (0, \infty)$ then $x(t)$ is a repeller.*

(4) *If $\mathrm{Re}\left\{\sigma\left(\int_0^{pT} D_x f(x_0,s)ds\right)\right\} \cap \{0\} = \varnothing$ then $x(t)$ is hyperbolic with the same hyperbolicity type as $\int_0^{pT} D_x f(x_0,s)ds$.*

Remark 5.1. By (1.2), (5.3), and (5.4), we have

$$A P_1(x) = P_1(Ax), \quad A D_x P_1(x) = D_x P_1(Ax)A \quad \forall x \in \mathbb{R}^n. \tag{5.5}$$

So if $P_1(x_0) = 0$ then $P_1(x_j) = 0$ and $\sigma(D_x P_1(x_j)) = \sigma(D_x P_1(x_0))$ for $x_j := A^j x_0$ and $j = 0, 1, \ldots, p-1$. Assuming $x_0 \neq 0$ from $1 \notin \sigma(A)$, there is the maximal $p_0 > 1$, $p_0|p$ such that x_j are different for $j = 0, 1, \ldots, p_0 - 1$. Then if

$$\det D_x P_1(x_0) \neq 0,$$

then, by Theorem 5.1, we have p_0 distinct pT-periodic solutions $x_j(t) = x_j + O(\varepsilon)$ for $j = 0, 1, \ldots, p_0 - 1$. By (1.2), the function $\widetilde{x}_j(t) := A^j x_0(t - jT) = A^j x_0 + O(\varepsilon) = x_j + O(\varepsilon)$ is also a solution of (1.1). The uniqueness from Theorem 5.1 yields $\widetilde{x}_j(t) = x_j(t)$. Consequently, $x_j(t)$, $j = 0, 1, \ldots, p_0 - 1$ are distinct solutions of (1.1), but geometrically they are A-symmetric to one other with same local asymptotic properties.

6. k-HYPERBOLICITY

To study hyperbolicity of periodic solutions of equation (1.1) we need the following results from [22].

Definition 6.1. A continuous matrix function $L_\varepsilon \colon \mathbb{R}^n \to \mathbb{R}^n$ of $\varepsilon \geq 0$ is k-hyperbolic if, for every matrix function N_ε defined for $\varepsilon \geq 0$ satisfying $N_\varepsilon = o(\varepsilon^k)$, there exists an interval $0 < \varepsilon < \varepsilon_1$ in which $L_\varepsilon + N_\varepsilon$ is hyperbolic of the same type (i. e., with the same number of eigenvalues on each side of the unit circle).

Definition 6.2. A continuous matrix function $L_\varepsilon \colon \mathbb{R}^n \to \mathbb{R}^n$ of $\varepsilon \geq 0$ is *strongly k-hyperbolic* if there exists a continuous real matrix C_ε defined in an interval $0 \leq \varepsilon < \varepsilon_0$ such that C_ε is regular (even for $\varepsilon = 0$) and such that

$$C^{-1} L_\varepsilon C_\varepsilon = \begin{pmatrix} A_\varepsilon & 0 \\ 0 & B_\varepsilon \end{pmatrix}$$

for $0 < \varepsilon < \varepsilon_0$, where A_ε and B_ε are $r \times r$ and $s \times s$ blocks, respectively, and $\|A_\varepsilon\| < 1 - c\varepsilon^k$, $\|B_\varepsilon^{-1}\| < 1 - c\varepsilon^k$ for some $c > 0$.

Now repeating proofs of Theorems 2.2 and 2.4 of [22], we arrive at the following generalizations.

Theorem 6.1. *Strong k-hyperbolicity implies k-hyperbolicity.*

Theorem 6.2. *If $L_\varepsilon = L_0 + \varepsilon L_1 + \cdots + \varepsilon^k L_k$, if the eigenvalues of L_0 are distinct numbers on the unit circle, and if the eigenvalues $\lambda_i(\varepsilon)$ of L_ε suitably numbered satisfy $|\lambda_i(\varepsilon)| < 1 - c\varepsilon^k$ for $i = 1, \ldots r$, $|\lambda_i(\varepsilon)| > 1 - c\varepsilon^k$ for $i = r+1, \ldots, n$, for some constant $c > 0$ and $\varepsilon > 0$ small, then L_ε is strongly k-hyperbolic.*

Remark 6.1. A difference between our Theorem 6.2 and [22, Theorem 2.2] is that it is supposed $L_\varepsilon = \mathbb{1} + \varepsilon L_1 + \cdots + \varepsilon^k L_k$ and the eigenvalues of L_1 are distinct in [22, Theorem 2.2].

Now we can improve Theorem 4.4 as follows.

Theorem 6.3. *Suppose $1 \notin \sigma(A)$ and all eigenvalues of A are distinct complex numbers on the unit circle. Let $x_\varepsilon = O(\varepsilon)$ be the unique solution of (3.10). If*

$$D_x g_\varepsilon(x_\varepsilon) = G_{\varepsilon,k} + o(\varepsilon^k) := A^{-1} + \varepsilon G_1 + \cdots + \varepsilon^k G_k + o(\varepsilon^k)$$

with a k-hyperbolic matrix function $G_{\varepsilon,k}$ then $D_x g_\varepsilon(x_\varepsilon)$ is hyperbolic of the same type as $G_{\varepsilon,k}$. In particular, $G_{\varepsilon,k}$ is k-hyperbolic if it fulfills assumptions of Theorem 6.2.

According to (4.2), from Theorem 6.3, we obtain the following consequence.

Theorem 6.4. *Suppose $1 \notin \sigma(A)$ and all eigenvalues of A are distinct complex numbers on the unit circle. If the eigenvalues $\lambda_i(\varepsilon)$ of $A^{-1} + \varepsilon G_1 + \cdots + \varepsilon^k G_k$ suitably numbered satisfy $|\lambda_i(\varepsilon)| < 1 - c\varepsilon^k$ for $i = 1, \ldots r$, $|\lambda_i(\varepsilon)| > 1 - c\varepsilon^k$ for $i = r+1, \ldots, n$, for some constant $c > 0$ and $\varepsilon > 0$ small.*

Then the unique symmetric and T-periodic solution of (1.1) is hyperbolic for any $\varepsilon > 0$ small.

7. Applications to weakly nonlinear problems

In this section, we present two concrete weakly nonlinear ordinary differential equations to illustrate our theory.

7.1. A scalar problem

Let us consider a scalar differential equation

$$x' = \varepsilon(x - x^3 + \sin t), \quad x \in \mathbb{R}, \tag{7.1}$$

with $T = \pi$, $p = 2$ and $f(x,t) = x - x^3 + \sin t$, $Ax = -x$. Indeed, we verify condition (1.2):

$$Af(x,t) = -f(x,t) = -(x - x^3 + \sin t) =$$
$$= (-x) - (-x^3) + \sin(t + \pi) = f(-x, t + \pi) = f(Ax, t + \pi).$$

Let us establish the stability of the pT-periodic solution of the symmetric problem (1.1), (2.2). Using (5.3) we get

$$P_1(x) = \int_0^{2\pi} (x - x^3 + \sin t)dt = 2\pi(x - x^3).$$

Solving the equation $P_1(x) = 0$ we find that

$$x = 0, \qquad x = -1, \qquad x = 1.$$

It is easy to see, using (5.4), that

$$P_1'(x) = 2\pi(1 - 3x^2),$$

$$P_1'(-1) < 0, \quad P_1'(0) > 0, \quad P_1'(1) < 0.$$

From Theorem 5.1 we know that $x_0(t) = O(\varepsilon)$, $x_\pm(t) = \pm 1 + O(\varepsilon)$ are the only 2π-periodic solutions of the problem (7.1) for $\varepsilon > 0$ small, while $x_0(t)$ is unstable (a repeller) and $x_\pm(t)$ are asymptotically stable. Note by Remark 5.1, curves $x_\pm(t)$

are symmetric with respect to the origin to each others. Furthermore, from Theorem 3.2 we know that $x_0(t)$ is the only π-antiperiodic solution of (7.1). Now using Theorem 4.4 together with a calculation

$$\int_0^\pi f'(0,s)ds = \pi > 0,$$

we reverify its instability.

7.2. *A planar problem*

Let us consider a planar differential system

$$\begin{aligned}
x_1' &= \varepsilon\left(f_1(x_1,x_2)+h_1(t)\right), \\
x_2' &= \varepsilon\left(f_2(x_1,x_2)+h_2(t)\right)
\end{aligned} \tag{7.2}$$

with smooth functions $f_{1,2}$, $h_{1,2}$, $T = \pi/2$, $p = 4$, and with

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{7.3}$$

Then symmetry condition (1.2) implies

$$f_1(x_1,x_2) = f_2(-x_2,x_1), \qquad f_2(x_1,x_2) = -f_1(-x_2,x_1), \tag{7.4}$$

and

$$h_1\left(t+\frac{\pi}{2}\right) = -h_2(t), \qquad h_2\left(t+\frac{\pi}{2}\right) = h_1(t). \tag{7.5}$$

Note that (7.5) gives $h_{1,2}(t+\pi) = -h_{1,2}(t)$ and, thus,

$$\int_0^{2\pi} h_{1,2}(t)dt = 0.$$

Consequently, (5.3) now has the form

$$P_1(x) = 2\pi\left(f_1(x_1,x_2), f_2(x_1,x_2)\right), \tag{7.6}$$

where $x = (x_1,x_2)$. Symmetry conditions (7.4) are satisfied, e. g., for the polynomials

$$f_1(x_1,x_2) = a_0x_1 + b_0x_2 + \sum_{\substack{j,k=0,\,j+k\geq 3 \\ 2\,|\,j+k+1}}^m \left(a_{jk}x_1^j x_2^k + b_{kj}x_1^k x_2^j\right),$$

$$f_2(x_1,x_2) = -b_0x_1 + a_0x_2 \tag{7.7}$$

$$- \sum_{\substack{j,k=0,\,j+k\geq 3 \\ 2\,|\,j+k+1}}^m \left((-1)^k b_{kj}x_1^j x_2^k + (-1)^j a_{jk}x_1^k x_2^j\right).$$

Hence $D_x P_1(0) = \begin{pmatrix} a_0 & b_0 \\ b_0 & a_0 \end{pmatrix}$. Then from Theorem 5.1 we know that $x_0(t) = O(\varepsilon)$ is the only symmetric and $\pi/2$-periodic solution of (7.2) which is in addition asymptotically stable (a repeller) if $a_0 < 0$ $(a_0 > 0)$, respectively, for $\varepsilon > 0$ small. We need to compute higher-order terms for $a_0 = 0$, but it is not carried out here. Since in general polynomials (7.7) are difficult to handle, we consider the following particular cases.

7.2.1. *Degenerate cases*

If $a_0 = b_0 = 0$ then $D_x P_1(0) = 0$ and so Theorem 5.1 is not applicable. But Theorem 3.2 still guarantees the existence and uniqueness of $x_0(t)$. This is one of advantages of our results. For instance, consider

$$
\begin{aligned}
x_1' &= \varepsilon \left(a_j x_1^j + b_j x_2^j + h_1(t) \right), \\
x_2' &= \varepsilon \left(-b_j x_1^j + a_j x_2^j + h_2(t) \right),
\end{aligned}
\tag{7.8}
$$

where $a_j, b_j \in \mathbb{R} \setminus \{0\}$, $j \in \mathbb{N}$, $j \geq 3$ is odd and $h_{1,2} \neq 0$ satisfy (7.5). The linearization of (7.8) along $x_0(t) = \left(x_{1,0}(t), x_{2,0}(t) \right)$ has the form

$$
\begin{aligned}
x_1' &= \varepsilon \left(j a_j x_{1,0}(t)^{j-1} x_1 + j b_j x_{2,0}(t)^{j-1} x_2 \right), \\
x_2' &= \varepsilon \left(-j b_j x_{1,0}(t)^{j-1} x_1 + j a_j x_{2,0}(t)^{j-1} x_2 \right).
\end{aligned}
\tag{7.9}
$$

Let $X_\varepsilon(t)$ be the fundamental matrix solution of (7.9). Then in notations of Sections 4 and 5, we have $D_x g_\varepsilon(x_\varepsilon) = A^{-1} X_\varepsilon(\pi/2)$ and $D_x P_\varepsilon(x_\varepsilon) = X_\varepsilon(2\pi)$ for $x_\varepsilon = x_0(0)$. Since

$$
\sigma \left(A^{-1} X_0(\pi/2) \right) = \left\{ e^{\pm \pi \imath/2} \right\},
$$

it follows that $\sigma \left(A^{-1} X_\varepsilon(\pi/2) \right) = \left\{ \lambda_\varepsilon, \overline{\lambda}_\varepsilon \right\}$ is close to $\left\{ e^{\pm \pi \imath/2} \right\}$. Let $\sigma \left(X_\varepsilon(2\pi) \right) = \{ \lambda_{\varepsilon,1}, \lambda_{\varepsilon,2} \}$. Then, due to (4.2), we have $\lambda_{\varepsilon,1} = \lambda_\varepsilon^4$ and $\lambda_{\varepsilon,2} = (\overline{\lambda}_\varepsilon)^4 = \overline{\lambda_\varepsilon^4} = \overline{\lambda}_{\varepsilon,1}$. On the other hand, by the Liouville theorem [11], we know that

$$
\lambda_{\varepsilon,1} \lambda_{\varepsilon,2} = \det X_\varepsilon(2\pi) = e^{\int_0^{2\pi} j a_j \left(x_{1,0}(t)^{j-1} + x_{2,0}(t)^{j-1} \right) dt}.
$$

Therefore, since $\int_0^{2\pi} \left(x_{1,0}(t)^{j-1} + x_{2,0}(t)^{j-1} \right) dt > 0$, we see that $|\lambda_{\varepsilon,1}| = |\lambda_{\varepsilon,2}| < 1$ for $a_j < 0$ and $|\lambda_{\varepsilon,1}| = |\lambda_{\varepsilon,2}| > 1$ for $a_j > 0$. Furthermore, the relation $P_1(x) = 0$ has the form

$$
a_j x_1^j + b_j x_2^j = 0, \quad -b_j x_1^j + a_j x_2^j = 0,
\tag{7.10}
$$

which gives $(x_1 x_2)^j (a_j^2 + b_j^2) = 0$. Thus, (7.10) has only zero solution $x_1 = x_2 = 0$. Summarizing, we have the following result.

Theorem 7.1. *Weakly nonlinear system* (7.8) *has the only 2π-periodic solution $x_0(t) = O(\varepsilon)$ for $\varepsilon > 0$ small, which is moreover $-\pi/2$-symmetric, i.e., has the property*

$$x_0(t + \pi/2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x_0(t) \quad \forall t \in \mathbb{R}. \tag{7.11}$$

If $a_j < 0$ then $x_0(t)$ is globally asymptotically stable, and if $a_j > 0$ then $x_0(t)$ is a global repeller.

Proof. We concentrate on the case $a_j < 0$, since the case $a_j > 0$ can be handled similarly. We already know that (7.8) possesses a solution $x_0(t)$ which is asymptotically stable. Now we apply Lemma 2.5. By (2.11), we have

$$J(x,t) = ja_j \varepsilon \begin{pmatrix} x_1^{j-1} & 0 \\ 0 & x_2^{j-1} \end{pmatrix},$$

which clearly satisfies (2.13). The proof is finished. $\qquad\square$

7.2.2. *Nondegenerate cases*

First we consider a linear perturbation of (7.8) of the form

$$\begin{aligned} x_1' &= \varepsilon \left(a_0 x_1 + b_0 x_2 + a_j x_1^j + h_1(t) \right) \\ x_2' &= \varepsilon \left(-b_0 x_1 + a_0 x_2 + a_j x_2^j + h_2(t) \right), \end{aligned} \tag{7.12}$$

where $a_0, b_0, a_j \in \mathbb{R}$, $j \in \mathbb{N}$, $j \geq 3$ is odd and $h_{1,2}$ satisfy (7.5). Then

$$J(x,t) = \varepsilon \begin{pmatrix} a_0 + ja_j x_1^{j-1} & 0 \\ 0 & a_0 + ja_j x_2^{j-1} \end{pmatrix}$$

which, clearly, satisfies relation (2.12) for $a_0 < 0$, $a_j \leq 0$ and any $\varepsilon > 0$. Thus, applying Lemma 2.4, we obtain the following result.

Theorem 7.2. *Assume $a_0 < 0$, $a_j \leq 0$. Then nonlinear system* (7.12) *has the only 2π-periodic solution $x_0(t) = O(\varepsilon)$ for any $\varepsilon > 0$, which is moreover $-\pi/2$-symmetric, i.e., it satisfies* (7.11) *and it is globally asymptotically stable.*

Remark 7.1. Except of Theorem 7.2 in this paper, we do not know anything about dynamics of (1.1) as ε is increasing.

Finally, to get further interesting results, we consider the following simple form of (7.7):

$$\begin{aligned} f_1(x_1, x_2) &= x_1 + x_2 + ax_1^2 x_2 + bx_1 x_2^2, \\ f_2(x_1, x_2) &= -x_1 + x_2 + bx_1^2 x_2 - ax_1 x_2^2, \end{aligned} \tag{7.13}$$

where $a, b \in \mathbb{R}$ are constants with $(a,b) \neq (0,0)$. Hence $x_0(t)$ is a repeller for $\varepsilon > 0$ small. We intend to find more 2π-periodic solutions of (7.2) with (7.13). For this

reason, we solve

$$x_1 + x_2 + ax_1^2 x_2 + bx_1 x_2^2 = 0,$$
$$-x_1 + x_2 + bx_1^2 x_2 - ax_1 x_2^2 = 0,$$
(7.14)

which gives

$$x_2 \left(2 + (a+b)x_1^2 - (a-b)x_1\right) = 0.$$
(7.15)

For $x_2 = 0$ from (7.14) we derive $x_1 = 0$ which gives $x_0(t)$. If $x_2 \neq 0$ and $a = b \geq 0$ then (7.15) has no more solutions. So we suppose $a \neq b$ and then (7.15) gives $x_1 \neq 0$ along with

$$x_2 = \frac{(a+b)x_2^2 + 2}{(a-b)x_1}.$$
(7.16)

Inserting (7.16) into the first equation of (7.14), after some computation, we arrive at

$$2(a+b) + 4(a^2+b^2)x_1^2 + (a+b)(a^2+b^2)x_1^4 = 0.$$
(7.17)

If $a + b = 0$ then (7.17) gives $x_1 = 0$ and so we again obtain $x_2 = 0$. Hence we suppose $a + b \neq 0$, and then (7.17) gives

$$x_{1,+}^2 = \frac{\sqrt{2}(a-b) - 2\sqrt{a^2+b^2}}{(a+b)\sqrt{a^2+b^2}},$$

$$x_{1,-}^2 = -\frac{\sqrt{2}(a-b) + 2\sqrt{a^2+b^2}}{(a+b)\sqrt{a^2+b^2}}.$$
(7.18)

It is elementary to observe

$$\sqrt{2}(a-b) - 2\sqrt{a^2+b^2} < 0, \quad \sqrt{2}(a-b) + 2\sqrt{a^2+b^2} > 0$$

whenever $a + b \neq 0$. So (7.18) has a real solution if and only if $a + b < 0$, which is from now supposed. Then by (7.16), we get four solutions of (7.14). But since according to (7.4), system of equations (7.14) is equivariant with respect to the symmetry (7.3), these solutions are rotationally symmetric (see also Remark 5.1). So we concentrate on the first solution given by the formulae

$$x_{1,+} = \sqrt{\frac{\sqrt{2}(a-b) - 2\sqrt{a^2+b^2}}{(a+b)\sqrt{a^2+b^2}}},$$

$$x_{2,+} = \sqrt{\frac{2(a+b)}{\sqrt{a^2+b^2}\left(\sqrt{2}(a-b) - 2\sqrt{a^2+b^2}\right)}}.$$
(7.19)

Then by (7.6) and (7.13) we get

$$D_x P_1(x_{1,+}, x_{2,+}) = \frac{2\pi}{(a+b)\sqrt{a^2+b^2}} \begin{pmatrix} u & v \\ w & w \end{pmatrix},$$

where

$$u = (a-b)\sqrt{a^2+b^2} + \sqrt{2}\left(2a^2+ab+b^2\right),$$
$$v = (b-a)\sqrt{a^2+b^2} + \sqrt{2}\left(a^2+ab+2b^2\right),$$

and

$$w = (a-b)\sqrt{a^2+b^2} + \sqrt{2}\left(a^2+ab+2b^2\right).$$

It is interesting to note $\det D_x P_1(x_{1,+}, x_{2,+}) = -16\pi$. The eigenvalues of the matrix $D_x P_1(x_{1,+}, x_{2,+})$ are given by the formulae

$$\lambda_{\pm} = 2\pi \frac{a-b \mp \sqrt{9a^2+14ab+9b^2}}{a+b}.$$

Since $a+b < 0$, we see that $\lambda_+ > 0$ and $\lambda_- < 0$. Summarizing, we obtain the following result.

Theorem 7.3. *Consider the system*

$$\begin{aligned}
x_1' &= \varepsilon\left(x_1 + x_2 + ax_1^2x_2 + bx_1x_2^2 + h_1(t)\right), \\
x_2' &= \varepsilon\left(-x_1 + x_2 + bx_1^2x_2 - ax_1x_2^2 + h_2(t)\right),
\end{aligned} \tag{7.20}$$

where $a, b \in \mathbb{R}$ are constants with $(a,b) \neq (0,0)$, $h_{1,2}$ are continuous functions satisfying (7.5) and $\varepsilon > 0$ is a small parameter. Then:

(1) *If $a+b \geq 0$ then (7.20) has the only 2π-periodic solution $x_0(t) = O(\varepsilon)$ which is in addition a repeller and $-\pi/2$-symmetric, i. e., it satisfies (7.11).*
(2) *If $a+b < 0$ then (7.20) has in addition precisely four more 2π-periodic solutions $x_j(t)$, $j = 1, 2, 3, 4$ which are hyperbolic with the same hyperbolicity type and orbitally $-\pi/2$-symmetric to each others.*

Proof. The proof follows immediately from the above computations together with application of Theorem 5.1 and Remark 5.1. $\qquad\square$

8. APPLICATIONS TO PLANAR WEAKLY LINEAR PROBLEMS

We consider

$$x' = \varepsilon B(t) x, \tag{8.1}$$

where $B(t) = \left(B_{ik}(t)\right)_{i,k=1}^2$ is a smooth 2×2-matrix function, 2π-periodic in $t \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$ is a small real parameter.

We first study (8.1) without symmetries using standard approach [8, 9, 13, 19, 21, 22, 29]. If $X_{\varepsilon}(t)x = \varphi_{\varepsilon}(x,t)$ is the Cauchy solution of (8.1), where $X_{\varepsilon}(t)$ is its fundamental matrix solution, then we expand $X_{\varepsilon}(t)$ in the form

$$X_{\varepsilon}(t) = \sum_{i \geq 0} Y_i(t)\varepsilon^i, \quad Y_0(0) = \mathbb{1}, \quad Y_i(0) = 0 \quad \forall i \geq 1 \tag{8.2}$$

to derive from (8.1) the recurrence formulas

$$Y_{i+1}(t) = \int_0^t B(s)Y_i(s)ds, \quad Y_0(t) = \mathbb{1}.$$

Thus, we have

$$X_\varepsilon(2\pi) = \mathbb{1} + \varepsilon \int_0^{2\pi} B(t)dt + \varepsilon^2 \int_0^{2\pi} \int_0^t B(t)B(s)dsdt + O(\varepsilon^3). \tag{8.3}$$

If $\int_0^{2\pi} B(t)dt \neq 0$ then we can apply results of Section 6 (see Remark 6.1). Here we concentrate on the degenerate cases by supposing (see also [9])

$$\int_0^{2\pi} B(t)dt = 0. \tag{8.4}$$

Then (8.3) becomes

$$X_\varepsilon(2\pi) = \mathbb{1} + \varepsilon^2 \int_0^{2\pi} \int_0^t B(t)B(s)dsdt + O(\varepsilon^3). \tag{8.5}$$

The results of [19, 21, 22] are difficult to apply to (8.5) (see Remark 6.1). We show that if we consider some symmetries to (8.1), then our theory helps to study such degenerate cases. Now $f(x,t) = B(t)x$, so symmetry condition (1.2) gives

$$AB(t) = B(t+T)A, \tag{8.6}$$

where $A^p = \mathbb{1}$ for some $p \in \mathbb{N}$, $p \geq 2$, and $T = 2\pi/p$. Then we have $X_\varepsilon(iT) = A^i\left(A^{-1}X_\varepsilon(T)\right)^i$ for all $i \geq 1$, which implies

$$\left(A^{-1}X_\varepsilon\left(\frac{2\pi}{p}\right)\right)^p = X_\varepsilon(2\pi). \tag{8.7}$$

We already know (8.7) from Lemma 3.2. In spite of the fact that $Y_1(2\pi) = 0$, in many cases we could have $Y_1(T) \neq 0$. Thus, using (8.7), we could still study the existence and stability of the zero solution $x = 0$ of (8.1) also in the degenerate case (8.4).

Finally, it is better to pass in computations to the complex variable $z = x_1 + \iota x_2 \in \mathbb{C}$ for $x = (x_1, x_2) \in \mathbb{R}^2$. Then (8.1) has the form

$$\dot{z} = \varepsilon(a(t)z + b(t)\overline{z}) \tag{8.8}$$

in which

$$a(t) = \frac{B_{11}(t) + B_{22}(t)}{2} + \iota\frac{B_{21}(t) - B_{12}(t)}{2},$$

$$b(t) = \frac{B_{11}(t) - B_{22}(t)}{2} + \iota\frac{B_{12}(t) + B_{21}(t)}{2}, \tag{8.9}$$

and $\overline{z} = x_1 - \iota x_2$. Next we again expand

$$z_\varepsilon(t) = \sum_{i \geq 0} w_i(t)\varepsilon^i, \quad w_0(0) = w, \quad w_i(0) = 0 \quad \forall i \geq 1 \tag{8.10}$$

to derive from (8.8) the recurrence formulas

$$w_{i+1}(t) = \int_0^t (a(s)w_i(s) + b(s)\bar{w}_i(s))\,ds, \quad w_0(t) = w. \tag{8.11}$$

In the next sections, we demonstrate the above approach by considering two different classes of symmetries A of (8.1).

8.1. *Rotation symmetries*

We consider the rotation $Az = e^{\frac{2\pi}{n}\iota}z$, so $p = n$ and $T = \frac{2\pi}{n}$. Note $n \in \mathbb{N}$ and $n \geq 3$. Now $f(z,t) = a(t)z + b(t)\bar{z}$, so the symmetry condition (1.2) (see also (8.6)) implies

$$a(t+T) = a(t), \quad b(t+T) = e^{\frac{4\pi}{n}\iota}b(t). \tag{8.12}$$

Note that (8.4) gives

$$\int_0^{2\pi} a(t)dt = 0, \quad \int_0^{2\pi} b(t)dt = 0. \tag{8.13}$$

So both (8.12) and (8.13) imply $\int_0^T a(t)dt = 0$. But we need either $\int_0^T a(t)dt \neq 0$ or $\int_0^T b(t)dt \neq 0$. For this reason, we further consider $f(z,t) = b(t)\bar{z}$. Next, expanding $b(t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} b_k e^{k\iota t}$ and using (8.12) we derive

$$b(t) = \sum_{m\in\mathbb{Z}} b_m e^{(2+mn)\iota t}. \tag{8.14}$$

Then

$$\tilde{b} = \int_0^T b(t)dt = \int_0^{2\pi/n} b(t)dt = \left(e^{\frac{4\pi}{n}\iota} - 1\right) \sum_{m\in\mathbb{Z}} \frac{b_m}{(2+mn)\iota}.$$

In general $\tilde{b} \neq 0$, so by (4.12) we consider the linear map $z \mapsto \tilde{b}\bar{z}$, which for $\tilde{b} = \tilde{b}_1 + \iota\tilde{b}_2$ and $z = x_1 + \iota x_2$ has the form

$$(x_1, x_2) \mapsto \begin{pmatrix} \tilde{b}_1 & \tilde{b}_2 \\ \tilde{b}_2 & -\tilde{b}_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

However, the matrix $\begin{pmatrix} \tilde{b}_1 & \tilde{b}_2 \\ \tilde{b}_2 & -\tilde{b}_1 \end{pmatrix} \neq 0$ is always hyperbolic and not asymptotically stable. So we can not apply Theorem 4.1. Consequently, we get now only that the equation

$$\dot{z} = \varepsilon b(t)\bar{z} \tag{8.15}$$

has zero solution the only symmetric one for $\varepsilon \neq 0$ small, when $b(t)$ is given by (8.14). Furthermore, by (8.2) we obtain

$$
\begin{aligned}
A^{-1} X_\varepsilon \left(\frac{2\pi}{n} \right) &= A^{-1} \left(\mathbb{1} + \varepsilon Y_1 \left(\frac{2\pi}{n} \right) \right) + O(\varepsilon^2) \\
&= \begin{pmatrix} \cos \frac{2\pi}{n} & \sin \frac{2\pi}{n} \\ -\sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{pmatrix} \begin{pmatrix} 1 + \varepsilon \widetilde{b}_1 & \varepsilon \widetilde{b}_2 \\ \varepsilon \widetilde{b}_2 & 1 - \varepsilon \widetilde{b}_1 \end{pmatrix} + O(\varepsilon^2).
\end{aligned}
$$

The eigenvalues of $A^{-1} \left(\mathbb{1} + \varepsilon Y_1 \left(\frac{2\pi}{n} \right) \right)$ are given by

$$
\lambda_\pm = \cos \frac{2\pi}{n} \pm \frac{\iota}{\sqrt{2}} \sqrt{1 - \cos \frac{4\pi}{n} - 2|\widetilde{b}|^2 \varepsilon^2}
$$

for ε small with

$$
|\lambda_\pm| = \sqrt{1 - |\widetilde{b}|^2 \varepsilon^2} = 1 - \frac{|\widetilde{b}|^2}{2} \varepsilon^2 + O(\varepsilon^3).
$$

So the matrix $A^{-1} \left(\mathbb{1} + \varepsilon Y_1 \left(\frac{2\pi}{n} \right) \right)$ is by Theorem 6.2 strongly 2-hyperbolic, but we should have its strongly 1-hyperbolicity in order to apply results of Section 6 concerning the hyperbolicity of the zero solution. We should need to compute higher order terms in (8.2).

Next, by (8.11) and (8.13), we get

$$
z_\varepsilon(2\pi) = \left(1 + \varepsilon^2 \int_0^{2\pi} \int_0^t b(t) \overline{b}(s) \, ds \, dt \right) w + O\left(\varepsilon^3 \right),
$$

and using (8.14), we derive

$$
z_\varepsilon(2\pi) = \left(1 + \varepsilon^2 \vartheta \iota \right) w + O\left(\varepsilon^3 \right)
$$

for

$$
\vartheta = 2\pi \sum_{m \in \mathbb{Z}} \frac{|b_m|^2}{2 + mn},
$$

which is a nonzero constant for $b(t) \neq 0$. In the variable x, we get

$$
X_\varepsilon(2\pi) = B_2(\varepsilon) + O(\varepsilon^3) \tag{8.16}
$$

for

$$
B_2(\varepsilon) = \begin{pmatrix} 1 & -\varepsilon^2 \vartheta \\ \varepsilon^2 \vartheta & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 0 & -\vartheta \\ \vartheta & 0 \end{pmatrix}.
$$

Since

$$
1 \notin \sigma \left(\begin{pmatrix} 0 & -\vartheta \\ \vartheta & 0 \end{pmatrix} \right),
$$

equation (8.15) has the only zero 2π-periodic solution. Note the eigenvalues of $B_2(\varepsilon)$ are $1 \pm \varepsilon^2 \vartheta \iota$. Furthermore, we see that

$$\left\| B_2(\varepsilon)^{-1} \right\| = \frac{1}{\sqrt{1 + \varepsilon^4 \vartheta^2}} \leq 1 - \frac{\vartheta^2}{3} \varepsilon^4$$

for $\varepsilon \neq 0$ small, so by Definition 6.2, $B_2(\varepsilon)$ is strongly 4-hyperbolic, but we should have its strongly 2-hyperbolicity in order to apply results of Section 6 concerning the hyperbolicity of the zero solution. Consequently, the expansion approach (8.2) seems to be difficult for application to (8.15), and hence our approach is as follows. Since $\sigma \left(A^{-1} X_0 \left(\frac{2\pi}{n} \right) \right) = \left\{ e^{\pm 2\pi \iota / n} \right\}$, so $\sigma \left(A^{-1} X_\varepsilon \left(\frac{2\pi}{n} \right) \right) = \left\{ \lambda_\varepsilon, \overline{\lambda}_\varepsilon \right\}$ is close to $\left\{ e^{\pm 2\pi \iota / n} \right\}$ for ε small. Next, (8.15) has the form

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \varepsilon \begin{pmatrix} b_1(t) & b_2(t) \\ b_2(t) & -b_1(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Hence $\sigma \left(X_\varepsilon(2\pi) \right) = \{\lambda_{\varepsilon,1}, \lambda_{\varepsilon,2}\}$ with $\lambda_{\varepsilon,1}\lambda_{\varepsilon,2} = 1$. But (8.7) gives $\lambda_{\varepsilon,1} = \lambda_\varepsilon^p$ and $\lambda_{\varepsilon,2} = \overline{\lambda}_\varepsilon^p = \overline{\lambda}_{\varepsilon,1}$. This implies that $|\lambda_{\varepsilon,1}| = |\lambda_{\varepsilon,2}| = 1$. Summarizing, we obtain the following result.

Theorem 8.1. *Assume* (8.13) *and* $b(t) \neq 0$ *is given by* (8.14)*. Then the zero solution is the only symmetric one of equation* (8.15) *for* $\varepsilon \neq 0$ *small. It is stable but not asymptotically stable. Moreover, it is the only* 2π-*periodic solution of* (8.15)*.*

We see that simple arguments using symmetries of (8.15) provide the stability of this equation, while expansion method seems to be rather awkward.

8.2. *Reflection symmetries*

Now we consider $Az = e^{\frac{\pi}{n} \iota} \overline{z}$ with $p = 2n$ and $T = \frac{\pi}{n}$. Note that A has the form

$$A = \begin{pmatrix} \cos \frac{\pi}{n} & -\sin \frac{\pi}{n} \\ -\sin \frac{\pi}{n} & -\cos \frac{\pi}{n} \end{pmatrix},$$

satisfies the relation $A^2 = \mathbb{1}$, and has the eigenvalues 1 and -1 and the corresponding orthogonal eigenvectors $\left(\sin \frac{\pi}{n}, \cos \frac{\pi}{n} - 1 \right)$ and $\left(\sin \frac{\pi}{n}, 1 + \cos \frac{\pi}{n} \right)$, respectively. Thus, A is the reflection with respect to the line $x_2 = -\tan \left(\frac{\pi}{2n} \right) x_1$.

Next, the symmetry condition (1.2) (see also (8.6)) gives

$$a(t + T) = \overline{a}(t), \quad b(t + T) = e^{\frac{2\pi}{n} \iota} \overline{b}(t). \tag{8.17}$$

Using the above approach, we derive

$$a(t) = \sum_{k \in \mathbb{N}} \left(a_k + \overline{a}_k e^{k \frac{\pi}{n} \iota} \right) e^{kt\iota} \tag{8.18}$$

and

$$b(t) = \sum_{k \in \mathbb{N}} \left(b_k + \overline{b}_k e^{(2+k) \frac{\pi}{n} \iota} \right) e^{kt\iota}. \tag{8.19}$$

Then

$$\widehat{a} = \int_0^T a(t)\,dt = \int_0^{\pi/n} a(t)\,dt = \sum_{k \in \mathbb{N}} \left(a_k + \overline{a}_k\, \mathrm{e}^{k\frac{\pi}{n}\iota} \right) \frac{\mathrm{e}^{k\frac{\pi}{n}\iota} - 1}{k\iota} \qquad (8.20)$$

and

$$\widehat{b} = \int_0^T b(t)\,dt = \int_0^{\pi/n} b(t)\,dt = \sum_{k \in \mathbb{N}} \left(b_k + \overline{b}_k\, \mathrm{e}^{(2+k)\frac{\pi}{n}\iota} \right) \frac{\mathrm{e}^{k\frac{\pi}{n}\iota} - 1}{k\iota}. \qquad (8.21)$$

In general $\widehat{a} \neq 0$ and $\widehat{b} \neq 0$, so by (4.12) we consider the linear map $z \mapsto \widehat{a}z + \widehat{b}\overline{z}$.

Now we want to show that for any given $\widehat{a}, \widehat{b} \in \mathbb{C}$, we can find $a(t)$, $b(t)$ satisfying (8.20) and (8.21). For this reason, we first derive

$$\left(a_k + \overline{a}_k\, \mathrm{e}^{k\frac{\pi}{n}\iota} \right) \frac{\mathrm{e}^{k\frac{\pi}{n}\iota} - 1}{k\iota} = \left(a_k\, \mathrm{e}^{-k\frac{\pi}{2n}\iota} + \overline{a}_k\, \mathrm{e}^{k\frac{\pi}{2n}\iota} \right) \mathrm{e}^{k\frac{\pi}{n}\iota}\, \frac{\mathrm{e}^{k\frac{\pi}{2n}\iota} - \mathrm{e}^{-k\frac{\pi}{2n}\iota}}{k\iota}$$

$$= \left(4\,\mathrm{Re}\left\{ a_k\, \mathrm{e}^{-k\frac{\pi}{2n}\iota} \right\} \frac{\sin k\frac{\pi}{2n}}{k} \right) \mathrm{e}^{k\frac{\pi}{n}\iota}.$$

So the range of the mapping

$$\mathbb{C} \ni a_k \mapsto \left(a_k + \overline{a}_k\, \mathrm{e}^{k\frac{\pi}{n}\iota} \right) \frac{\mathrm{e}^{k\frac{\pi}{n}\iota} - 1}{k\iota} \in \mathbb{C} \sim \mathbb{R}^2$$

in the plain $\mathbb{R}^2$ is the line $x_2 = \tan k\frac{\pi}{n} x_1$. Consequently, the mapping

$$\mathbb{C} \times \mathbb{C} \ni (a_1, a_2) \mapsto \left(a_1 + \overline{a}_1\, \mathrm{e}^{\frac{\pi}{n}\iota} \right) \frac{\mathrm{e}^{\frac{\pi}{n}\iota} - 1}{\iota} + \left(a_2 + \overline{a}_2\, \mathrm{e}^{2\frac{\pi}{n}\iota} \right) \frac{\mathrm{e}^{2\frac{\pi}{n}\iota} - 1}{2\iota} \in \mathbb{C}$$

is onto (surjective). Hence, for any $\widehat{a} \in \mathbb{C}$, there are $a_1, a_2 \in \mathbb{C}$ such that (8.20) holds with

$$a(t) = \left(a_1 + \overline{a}_1\, \mathrm{e}^{\frac{\pi}{n}\iota} \right) \mathrm{e}^{t\iota} + \left(a_2 + \overline{a}_2\, \mathrm{e}^{2\frac{\pi}{n}\iota} \right) \mathrm{e}^{2t\iota}. \qquad (8.22)$$

Similarly we have

$$\left(b_k + \overline{b}_k\, \mathrm{e}^{(2+k)\frac{\pi}{n}\iota} \right) \frac{\mathrm{e}^{k\frac{\pi}{n}\iota} - 1}{k\iota} = \left(4\,\mathrm{Re}\left\{ b_k\, \mathrm{e}^{-(2+k)\frac{\pi}{2n}\iota} \right\} \frac{\sin k\frac{\pi}{2n}}{k} \right) \mathrm{e}^{(1+k)\frac{\pi}{n}\iota}.$$

So the range of the mapping

$$\mathbb{C} \ni b_k \mapsto \left(b_k + \overline{b}_k\, \mathrm{e}^{(2+k)\frac{\pi}{n}\iota} \right) \frac{\mathrm{e}^{k\frac{\pi}{n}\iota} - 1}{k\iota} \in \mathbb{C} \sim \mathbb{R}^2$$

in the plain $\mathbb{R}^2$ is the line $x_2 = \tan(1+k)\frac{\pi}{n} x_1$. Consequently, the mapping

$$\mathbb{C} \times \mathbb{C} \ni (b_1, b_2) \mapsto \left(b_1 + \overline{b}_1\, \mathrm{e}^{3\frac{\pi}{n}\iota} \right) \frac{\mathrm{e}^{\frac{\pi}{n}\iota} - 1}{\iota} + \left(b_2 + \overline{b}_2\, \mathrm{e}^{4\frac{\pi}{n}\iota} \right) \frac{\mathrm{e}^{2\frac{\pi}{n}\iota} - 1}{2\iota} \in \mathbb{C}$$

is onto (surjective). Hence, for any $\hat{b} \in \mathbb{C}$, there are $b_1, b_2 \in \mathbb{C}$ such that (8.21) holds with

$$b(t) = \left(b_1 + \bar{b}_1 e^{3\frac{\pi}{n}\iota}\right) e^{t\iota} + \left(b_2 + \bar{b}_2 e^{4\frac{\pi}{n}\iota}\right) e^{2t\iota}. \tag{8.23}$$

Summarizing we see that, for any given $a, b, c, d \in \mathbb{R}$, by adjusting suitable $a_1, a_2, b_1, b_2 \in \mathbb{C}$, we have $Y_1\left(\frac{\pi}{n}\right) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Next we derive

$$A^{-1}X_\varepsilon\left(\frac{\pi}{n}\right) = A^{-1}\left(\mathbb{1} + \varepsilon Y_1\left(\frac{\pi}{n}\right)\right) + O(\varepsilon^2)$$

$$= \begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} \\ -\sin\frac{\pi}{n} & -\cos\frac{\pi}{n} \end{pmatrix} \begin{pmatrix} 1+\varepsilon a & \varepsilon b \\ \varepsilon c & 1+\varepsilon d \end{pmatrix} + O(\varepsilon^2).$$

The eigenvalues $\widetilde{\lambda}_\pm$ of $A^{-1}\left(\mathbb{1} + \varepsilon Y_1\left(\frac{\pi}{n}\right)\right)$ are given by the equalities

$$\widetilde{\lambda}_\pm = \frac{1}{2}\left(\varepsilon(a-d)\cos\frac{\pi}{n} - \varepsilon(b+c)\sin\frac{\pi}{n}\right.$$

$$\left. \pm \sqrt{4 + 4\varepsilon(a+d) + \varepsilon^2\left(ad - bc + \left((d-a)\cos\frac{\pi}{n} + (b+c)\sin\frac{\pi}{n}\right)^2\right)}\right)$$

for ε small, which yield

$$\left. \begin{array}{l} \widetilde{\lambda}_+ = 1 + \varepsilon\dfrac{\sin\frac{\pi}{n}}{2}\left(a\cot\dfrac{\pi}{2n} + d\tan\dfrac{\pi}{2n} - (b+c)\right) + O(\varepsilon^2), \\[3mm] \widetilde{\lambda}_- = -1 - \varepsilon\dfrac{\sin\frac{\pi}{n}}{2}\left(a\tan\dfrac{\pi}{2n} + d\cot\dfrac{\pi}{2n} + (b+c)\right) + O(\varepsilon^2). \end{array} \right\} \tag{8.24}$$

The two planes

$$x_1\cot\frac{\pi}{2n} + x_2\tan\frac{\pi}{2n} - x_3 = 0,$$

$$x_1\tan\frac{\pi}{2n} + x_2\cot\frac{\pi}{2n} + x_3 = 0$$

decompose the space $\mathbb{R}^3$ into four subspaces where, by (8.24), we get four different types of a strong 1-hyperbolicity of $A^{-1}\left(\mathbb{1} + \varepsilon Y_1\left(\frac{\pi}{n}\right)\right)$ with the corresponding hyperbolicities of $A^{-1}X_\varepsilon\left(\frac{\pi}{n}\right)$. For instance, $A^{-1}X_\varepsilon\left(\frac{\pi}{n}\right)$ is asymptotically stable for $\varepsilon > 0$ small if

$$a\cot\frac{\pi}{2n} + d\tan\frac{\pi}{2n} < b + c < -a\tan\frac{\pi}{2n} - d\cot\frac{\pi}{2n}. \tag{8.25}$$

Note that (8.25) yields $a + d < 0$. On the other hand, the real parts of the eigenvalues of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ lie in $\subset (-\infty, 0)$ if and only if

$$a + d < 0, \quad ad - bc > 0, \tag{8.26}$$

which condition follows from the Routh–Hurwitz criterion [6]. The relationship between constants $a, b, c, d \in \mathbb{R}$ and $a_1, a_2, b_1, b_2 \in \mathbb{C}$ are given by (8.20), (8.21) and

(see (8.9)) also by the equalities

$$a = \operatorname{Re}\widehat{a} + \operatorname{Re}\widehat{b}, \quad b = \operatorname{Im}\widehat{b} - \operatorname{Im}\widehat{a}, \quad c = \operatorname{Im}\widehat{a} + \operatorname{Im}\widehat{b}, \quad d = \operatorname{Re}\widehat{a} - \operatorname{Re}\widehat{b}, \quad (8.27)$$

which are obtained by solving the equality

$$\widehat{a}z + \widehat{b}\overline{z} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where $z = x_1 + \iota x_2$, for any x_1 and x_2 from $\mathbb{R}$. Inserting (8.27) into (8.25) and (8.26), summarizing, we obtain the following result.

Theorem 8.2. *Assume* (8.13)*, and* $a(t)$ *and* $b(t)$ *are given by* (8.22) *and* (8.23)*, respectively. Then the equation*

$$\dot{z} = \varepsilon(a(t)z + b(t)\overline{z}) \tag{8.28}$$

has zero solution the only symmetric one for $\varepsilon \neq 0$ *small which can be either hyperbolic, asymptotically stable or unstable, respectively.*[†] *For instance, it is asymptotically stable for* $\varepsilon > 0$ *small if* $\operatorname{Re}\widehat{a} < 0$ *and either* $|\widehat{a}| > |\widehat{b}|$ *or the inequalities*

$$\operatorname{Re}\widehat{a} \sec\frac{\pi}{n} + \operatorname{Re}\widehat{b} \cot\frac{\pi}{n} < \operatorname{Im}\widehat{b} < -\operatorname{Re}\widehat{a} \sec\frac{\pi}{n} + \operatorname{Re}\widehat{b} \cot\frac{\pi}{n}$$

are satisfied. Constants $\widehat{a}$ *and* $\widehat{b}$ *are given by* (8.20) *and* (8.21)*, respectively.*

We do not compute $Y_2(2\pi)$ in (8.2) for (8.28) since it is an awkward formula, but we again see that in general, for instance in the hyperbolic cases, the zero solution is the only 2π-periodic solution of (8.28).

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[†]Of course, depending on the choice of $a_1, a_2, b_1, b_2 \in \mathbb{C}$.

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