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ON PERIODIC-TYPE BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH A POSITIVELY HOMOGENEOUS OPERATOR

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Abstract. Consider the problem

$$u'(t) = H(u)(t) + Q(u)(t), \qquad u(a) - \lambda u(b) = h(u),$$

where $H, Q : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ are continuous operators satisfying the Caratheodory condition, the operator H is positively homogeneous, $\lambda \in \mathbb{R}_+$, and $h : C([a,b];\mathbb{R}) \to \mathbb{R}$ is a continuous functional. In this paper, efficient sufficient conditions guaranteeing the solvability and unique solvability of the problem considered are established.

Mathematics Subject Classification: 34K06, 34K10

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1. Introduction

In this paper, we establish efficient sufficient conditions guaranteeing the existence and uniqueness of a solution of a periodic-type boundary value problem for a scalar functional differential equation involving a positively homogeneous operator.

The following notation is used throughout the paper.

$\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$, $[x]_+ = \frac{1}{2}(|x| + x)$, $[x]_- = \frac{1}{2}(|x| - x)$.

$C([a,b];\mathbb{R})$ is the Banach space of continuous functions $u : [a,b] \to R$ with the norm $\|u\|_C = \max\{|u(t)| : t \in [a,b]\}$.

$\tilde{C}([a,b];\mathbb{R})$ is the set of absolutely continuous functions $u : [a,b] \to \mathbb{R}$.

$L([a,b];\mathbb{R})$ is the Banach space constituted by the Lebesgue integrable functions $p : [a,b] \to \mathbb{R}$ with the norm $\|p\|_L = \int_a^b |p(s)|ds$.

$L([a,b];\mathbb{R}_+) = \{p \in L([a,b];\mathbb{R}) : p(t) \geq 0 \text{ for } t \in [a,b]\}$.

$\mathcal{M}_{ab}$ is the set of measurable functions $\tau : [a,b] \to [a,b]$.

$\mathcal{K}_{ab}$ is the set of continuous operators $F : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ satisfying the Caratheodory condition, i. e., for every $r > 0$ there exists $q_r \in L([a,b];\mathbb{R}_+)$ such that

$$|F(v)(t)| \le q_r(t) \qquad \text{for } t \in [a,b], \quad v \in C([a,b];\mathbb{R}), \quad \|v\|_C \le r.$$

$\mathcal{H}_{ab}$ is the set of operators $H \in \mathcal{K}_{ab}$ which are positively homogeneous, i. e., such that, for an arbitrary number $\alpha > 0$, we have

$$H(\alpha v)(t) = \alpha H(v)(t) \qquad \text{for } t \in [a,b], \quad v \in C([a,b];\mathbb{R}).$$

$\mathcal{P}_{ab}$ is the set of operators $H \in \mathcal{H}_{ab}$ which are nondecreasing, i. e., for all $H \in \mathcal{P}_{ab}$ and $u, v \in C([a,b];\mathbb{R})$ such that $u(t) \le v(t)$ for $t \in [a,b]$, we have

$$H(u)(t) \le H(v)(t) \qquad \text{for } t \in [a,b].$$

$K([a,b] \times \mathbb{R}_+;\mathbb{R}_+)$ is the set of functions $\eta : [a,b] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the Caratheòdory conditions, i. e., $\eta(\cdot, x) : [a,b] \to \mathbb{R}_+$ is a measurable function for all $x \in \mathbb{R}_+$, $\eta(t,\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function for almost all $t \in [a,b]$, and for every $r > 0$ there exists $q_r \in L([a,b];\mathbb{R}_+)$ such that

$$|\eta(t,x)| \le q_r(t) \qquad \text{for } t \in [a,b], \quad 0 \le x \le r.$$

By a solution of the equation

$$u'(t) = H(u)(t) + Q(u)(t), \tag{1.1}$$

where $H \in \mathcal{H}_{ab}$ and $Q \in \mathcal{K}_{ab}$, we understand a function $u \in \tilde{C}([a,b];\mathbb{R})$ satisfying equality (1.1) almost everywhere in $[a,b]$.

Consider the problem on the existence and uniqueness of a solution of (1.1) satisfying the condition

$$u(a) - \lambda u(b) = h(u), \tag{1.2}$$

where $h : C([a,b];\mathbb{R}) \to \mathbb{R}$ is a continuous operator such that for every $r > 0$ there exists $M_r \in \mathbb{R}_+$ such that

$$|h(v)| \le M_r \qquad \text{for } v \in C([a,b];\mathbb{R}), \quad \|v\|_C \le r.$$

The differential equations with "maxima" are one of the special cases of equation (1.1). Various types of boundary value problems for this type of equations were studied, e. g., in [3, 4, 33, 35, 36, 38, 41].

In the case where H in (1.1) is a linear operator, the results presented here coincide with those obtained earlier in [6, 17] (see also [22]). Another type of conditions guaranteeing the solvability of (1.1), (1.2) with a linear operator H can be found, e. g., in [5, 8, 10–15, 18–21, 28, 34]. Conditions for the solvability and unique solvability for other types of boundary value problems for equation (1.1) with a linear operator H are established, e. g., in [9, 16, 22, 24, 27, 39, 42].

There are many interesting results concerning the solvability of general boundary value problems for functional differential equations (see, e. g., [1, 2, 7, 9, 22–26, 29–32, 37, 40] and references therein). In spite of this, the general theory of boundary value problems for functional differential equations is not still complete. Here, we try to fill this gap in a certain way. More precisely, in Section 2, we establish unimprovable, in a certain sense, efficient conditions sufficient for the solvability of problem (1.1), (1.2). In Section 3, we prove some auxiliary propositions. Section 4 is devoted to the proofs of the main results.

The results are concretized for the boundary value problem of the form

$$u'(t) = p(t)\max\{u(s) : \tau_1(t) \le s \le \tau_2(t)\} + q(t), \tag{1.3}$$

$$u(a) - \lambda u(b) = c, \tag{1.4}$$

where $p, q \in L([a,b];\mathbb{R})$, $\tau_1, \tau_2 \in \mathcal{M}_{ab}$, $\tau_1(t) \le \tau_2(t)$ for $t \in [a,b]$, and $c \in \mathbb{R}$.

Definition 1.1. We will say that an operator $H \in \mathcal{K}_{ab}$ is subadditive if for any $u, v \in C([a,b];\mathbb{R})$ we have

$$H(u+v)(t) \le H(u)(t) + H(v)(t) \qquad \text{for } \ t \in [a,b].$$

Definition 1.2. We will say that an operator $H \in \mathcal{K}_{ab}$ is superadditive if for any $u, v \in C([a,b];\mathbb{R})$ we have

$$H(u+v)(t) \ge H(u)(t) + H(v)(t) \qquad \text{for } \ t \in [a,b].$$

2. Main results

Throughout the paper, $\eta \in K([a,b]\times\mathbb{R}_+;\mathbb{R}_+)$ and $\eta_0 : \mathbb{R}_+ \to \mathbb{R}_+$ are such that

$$\lim_{x\to+\infty}\left(\frac{\eta_0(x)}{x} + \frac{1}{x}\int_a^b \eta(s,x)ds\right) = 0.$$

Theorem 2.1. *Let $\lambda \in\,]0,1]$ and let H admit the representation $H = H_0 - H_1$, where $H_0, H_1 \in \mathcal{P}_{ab}$. Let, furthermore,*

$$|Q(v)(t)| \le \eta(t, \|v\|_C) \qquad \textit{for } \ t \in [a,b], \quad v \in C([a,b];\mathbb{R}), \tag{2.1}$$

$$|h(v)| \le \eta_0(\|v\|_C) \qquad \textit{for } \ v \in C([a,b];\mathbb{R}). \tag{2.2}$$

If, moreover,

$$\int_a^b H_0(1)(s)ds < 1, \qquad \int_a^b |H_0(-1)(s)|ds < 1, \tag{2.3}$$

$$\frac{\int_a^b H_0(1)(s)ds}{1 - \int_a^b H_0(1)(s)ds} - \frac{1-\lambda}{\lambda} < \int_a^b H_1(1)(s)ds, \tag{2.4}$$

$$\frac{\int_a^b |H_0(-1)(s)|ds}{1 - \int_a^b |H_0(-1)(s)|ds} - \frac{1-\lambda}{\lambda} < \int_a^b |H_1(-1)(s)|ds, \tag{2.5}$$

$$\left[\int_s^t H_1(1)(\xi)d\xi - 1\right]_+ \left[\int_I |H_1(-1)(\xi)|d\xi - \lambda\right]_+ <$$
$$< \left(1 - \int_s^t |H_0(-1)(\xi)|d\xi\right)\left(1 - \int_I H_0(1)(\xi)d\xi\right)$$
$$\text{for } \ s,t \in [a,b], \quad s \le t, \tag{2.6}$$

and

$$\left[\int_s^t |H_1(-1)(\xi)|d\xi - 1\right]_+ \left[\int_I H_1(1)(\xi)d\xi - \lambda\right]_+ <$$
$$< \left(1 - \int_s^t H_0(1)(\xi)d\xi\right)\left(1 - \int_I |H_0(-1)(\xi)|d\xi\right)$$
$$\text{for } \ s,t \in [a,b], \quad s \le t, \tag{2.7}$$

where $I = [a,b]\setminus\,]s,t[$ *, then problem* (1.1), (1.2) *has at least one solution.*

The following assertion can be regarded as a supplement to the previous theorem in the case when $\lambda = 0$.

Theorem 2.1′. *Let* H *admit the representation* $H = H_0 - H_1$*, where* $H_0, H_1 \in \mathcal{P}_{ab}$*. Let, furthermore,* (2.1) *and* (2.2) *be fulfilled. If, moreover, the inequalities* (2.3),

$$\int_I |H_1(-1)(\xi)|d\xi \left[\int_s^t H_1(1)(\xi)d\xi - 1\right]_+ <$$
$$< \left(1 - \int_s^t |H_0(-1)(\xi)|d\xi\right)\left(1 - \int_I H_0(1)(\xi)d\xi\right)$$
$$\text{for } \ s,t \in [a,b], \quad s \le t, \tag{2.8}$$

and

$$\int_I H_1(1)(\xi)d\xi \left[\int_s^t |H_1(-1)(\xi)|d\xi - 1\right]_+ <$$
$$< \left(1 - \int_s^t H_0(1)(\xi)d\xi\right)\left(1 - \int_I |H_0(-1)(\xi)|d\xi\right)$$
$$\text{for } \ s,t \in [a,b], \quad s \le t \tag{2.9}$$

are fulfilled, where $I = [a,b]\setminus\,]s,t[$ *, then problem* (1.1), (1.2) *with* $\lambda = 0$ *has at least one solution.*

Theorem 2.2. *Let $\lambda \in \,]0,1]$ and let H admit the representation $H = H_0 - H_1$, where $H_0, H_1 \in \mathcal{P}_{ab}$. Let, furthermore,* (2.1) *and* (2.2) *be fulfilled. If, moreover,*

$$\int_a^b H_1(1)(s)ds < \lambda, \qquad \int_a^b |H_1(-1)(s)|ds < \lambda, \tag{2.10}$$

$$\frac{1}{\lambda - \int_a^b H_1(1)(s)ds} - 1 < \int_a^b H_0(1)(s)ds, \tag{2.11}$$

$$\frac{1}{\lambda - \int_a^b |H_1(-1)(s)|ds} - 1 < \int_a^b |H_0(-1)(s)|ds, \tag{2.12}$$

$$\begin{gathered}\left[\int_I H_0(1)(\xi)d\xi - 1\right]_+ \left[\int_s^t |H_0(-1)(\xi)|d\xi - 1\right]_+ < \\ < \left(\lambda - \int_I |H_1(-1)(\xi)|d\xi\right)\left(1 - \int_s^t H_1(1)(\xi)d\xi\right) \\ \text{for } \; s,t \in [a,b], \quad s \le t,\end{gathered} \tag{2.13}$$

and

$$\begin{gathered}\left[\int_I |H_0(-1)(\xi)|d\xi - 1\right]_+ \left[\int_s^t H_0(1)(\xi)d\xi - 1\right]_+ < \\ < \left(\lambda - \int_I H_1(1)(\xi)d\xi\right)\left(1 - \int_s^t |H_1(-1)(\xi)|d\xi\right) \\ \text{for } \; s,t \in [a,b], \quad s \le t,\end{gathered} \tag{2.14}$$

where $I = [a,b] \setminus \,]s,t[$, then problem (1.1), (1.2) *has at least one solution.*

Theorem 2.3. *Let $\lambda \in \,]0,1]$ and let H admit the representation $H = H_0 - H_1$, where $H_0, H_1 \in \mathcal{P}_{ab}$. Let, furthermore, each of the operators H_0 and H_1 be either a subadditive or a superadditive operator, and let there exist $q \in L([a,b];\mathbb{R})$ and $c \in \mathbb{R}$ such that*

$$Q(v)(t) = q(t) \qquad \text{for } \; t \in [a,b], \quad v \in C([a,b];\mathbb{R}), \tag{2.15}$$

$$h(v) = c \qquad \text{for } \; v \in C([a,b];\mathbb{R}). \tag{2.16}$$

If, moreover,

$$\int_a^b \overline{H}_0(1)(s)ds < 1, \tag{2.17}$$

$$\frac{\int_a^b \overline{H}_0(1)(s)ds}{1 - \int_a^b \overline{H}_0(1)(s)ds} - \frac{1-\lambda}{\lambda} < \int_a^b |\overline{H}_1(-1)(s)|ds, \tag{2.18}$$

$$\left[\int_s^t \overline{H}_1(1)(\xi)d\xi - 1\right]_+ \left[\int_I \overline{H}_1(1)(\xi)d\xi - \lambda\right]_+ <$$
$$< \left(1 - \int_s^t \overline{H}_0(1)(\xi)d\xi\right)\left(1 - \int_I \overline{H}_0(1)(\xi)d\xi\right)$$
$$\text{for} \quad s, t \in [a, b], \quad s \le t, \tag{2.19}$$

where $I = [a, b] \setminus]s, t[$ *and*

$$\overline{H}_i(v)(t) \stackrel{\text{def}}{=} \begin{cases} H_i(v)(t) & \text{if } H_i \text{ is subadditive} \\ -H_i(-v)(t) & \text{if } H_i \text{ is superadditive} \end{cases} \tag{2.20}$$

for $t \in [a, b]$ *and* $i = 0, 1$*, then problem* (1.1), (1.2) *has a unique solution.*

The following assertion can be regarded as a supplement of the previous theorem in the case when $\lambda = 0$.

Theorem 2.3′. *Let* H *admit the representation* $H = H_0 - H_1$*, where* $H_0, H_1 \in \mathcal{P}_{ab}$*. Let, furthermore, each of the operators* H_0 *and* H_1 *be either a subadditive or a superadditive operator, and let there exist* $q \in L([a, b]; \mathbb{R})$ *and* $c \in \mathbb{R}$ *such that* (2.15) *and* (2.16) *are fulfilled. If, moreover, inequalities* (2.17) *and*

$$\int_I \overline{H}_1(1)(\xi)d\xi \left[\int_s^t \overline{H}_1(1)(\xi)d\xi - 1\right]_+ <$$
$$< \left(1 - \int_s^t \overline{H}_0(1)(\xi)d\xi\right)\left(1 - \int_I \overline{H}_0(1)(\xi)d\xi\right)$$
$$\text{for} \quad s, t \in [a, b], \quad s \le t \tag{2.21}$$

hold, where $I = [a, b] \setminus]s, t[$*, and* $\overline{H}_i$ $(i = 0, 1)$ *are given by* (2.20)*, then problem* (1.1), (1.2) *with* $\lambda = 0$ *has a unique solution.*

Theorem 2.4. *Let* $\lambda \in]0, 1]$ *and let* H *admit the representation* $H = H_0 - H_1$*, where* $H_0, H_1 \in \mathcal{P}_{ab}$*. Let, furthermore, each of the operators* H_0 *and* H_1 *be either a subadditive or a superadditive operator, and let there exist* $q \in L([a, b]; \mathbb{R})$ *and* $c \in \mathbb{R}$ *such that* (2.15) *and* (2.16) *are fulfilled. If, moreover,*

$$\int_a^b \overline{H}_1(1)(s)ds < \lambda, \tag{2.22}$$

$$\frac{1}{\lambda - \int_a^b \overline{H}_1(1)(s)ds} - 1 < \int_a^b |\overline{H}_0(-1)(s)|ds, \tag{2.23}$$

$$\left[\int_I \overline{H}_0(1)(\xi)d\xi - 1\right]_+ \left[\int_s^t \overline{H}_0(1)(\xi)d\xi - 1\right]_+ <$$
$$< \left(\lambda - \int_I \overline{H}_1(1)(\xi)d\xi\right)\left(1 - \int_s^t \overline{H}_1(1)(\xi)d\xi\right)$$
$$\textit{for} \quad s,t \in [a,b], \quad s \le t, \tag{2.24}$$

where $I = [a,b] \setminus]s,t[$ *, and* $\overline{H}_i$ $(i = 0,1)$ *are given by* (2.20)*, then problem* (1.1), (1.2) *has a unique solution.*

Corollary 2.1. *Let* $\lambda \in]0,1]$ *and let the inequalities*

$$\int_a^b [p(s)]_+ ds < 1, \tag{2.25}$$

$$\frac{\int_a^b [p(s)]_+ ds}{1 - \int_a^b [p(s)]_+ ds} - \frac{1-\lambda}{\lambda} < \int_a^b [p(s)]_- ds < 1 + \lambda + 2\sqrt{1 - \int_a^b [p(s)]_+ ds} \tag{2.26}$$

be fulfilled. Then problem (1.3), (1.4) *has a unique solution.*

Corollary 2.1′. *Let inequalities* (2.25) *and*

$$\int_a^b [p(s)]_- ds < 1 + 2\sqrt{1 - \int_a^b [p(s)]_+ ds}$$

be fulfilled. Then problem (1.3), (1.4) *with* $\lambda = 0$ *has a unique solution.*

Corollary 2.2. *Let* $\lambda \in]0,1]$ *and let the inequalities*

$$\int_a^b [p(s)]_- ds < \lambda, \tag{2.27}$$

$$\frac{1}{\lambda - \int_a^b [p(s)]_- ds} - 1 < \int_a^b [p(s)]_+ ds < 2 + 2\sqrt{\lambda - \int_a^b [p(s)]_- ds} \tag{2.28}$$

be fulfilled. Then problem (1.3), (1.4) *has a unique solution.*

Remark 2.1. Let $\lambda \in [1,+\infty[$. Define operator $\psi : L([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ by the equality

$$\psi(w)(t) \overset{\text{def}}{=} w(a+b-t) \qquad \text{for } t \in [a,b].$$

Let φ be the restriction of ψ to the space $C([a,b];\mathbb{R})$. Put

$$\hat{H}_0(w)(t) \overset{\text{def}}{=} \psi(H_0(\varphi(w)))(t), \quad \hat{H}_1(w)(t) \overset{\text{def}}{=} \psi(H_1(\varphi(w)))(t) \quad \text{for } t \in [a,b],$$
$$\hat{Q}(w)(t) \overset{\text{def}}{=} -\psi(Q(\varphi(w)))(t) \quad \text{for } t \in [a,b],$$

and

$$\hat{h}(w) \overset{\text{def}}{=} -\frac{1}{\lambda} h(\varphi(w)).$$

It is clear that if u is a solution of problem (1.1), (1.2), then the function $v \stackrel{\text{def}}{=} \varphi(u)$ is a solution of the problem

$$v'(t) = \hat{H}_1(v)(t) - \hat{H}_0(v)(t) + \hat{Q}(v)(t), \qquad v(a) - \frac{1}{\lambda}v(b) = \hat{h}(v), \tag{2.29}$$

and *vice versa*, if v is a solution of problem (2.29), then the function $u \stackrel{\text{def}}{=} \varphi(v)$ is a solution of problem (1.1), (1.2).

Furthermore, if H_i $(i \in \{0,1\})$ is a subadditive (resp., superadditive) operator, then $\hat{H}_i$ $(i \in \{0,1\})$ is a subadditive (resp., superadditive) operator, as well.

Theorems 2.5–2.8 and Corollaries 2.3 and 2.4 formulated below can be derived easily from Theorems 2.1–2.4 and Corollaries 2.1 and 2.2, respectively, by using the change of variables described in Remark 2.1.

Theorem 2.5. *Let $\lambda \in [1,+\infty[$ and let H admit the representation $H = H_0 - H_1$, where $H_0, H_1 \in \mathcal{P}_{ab}$. Let, furthermore,* (2.1) *and* (2.2) *be fulfilled. If, moreover,*

$$\int_a^b H_1(1)(s)ds < 1, \qquad \int_a^b |H_1(-1)(s)|ds < 1,$$

$$\frac{\int_a^b H_1(1)(s)ds}{1 - \int_a^b H_1(1)(s)ds} - (\lambda - 1) < \int_a^b H_0(1)(s)ds,$$

$$\frac{\int_a^b |H_1(-1)(s)|ds}{1 - \int_a^b |H_1(-1)(s)|ds} - (\lambda - 1) < \int_a^b |H_0(-1)(s)|ds,$$

and the inequalities

$$\left[\int_s^t H_0(1)(\xi)d\xi - 1\right]_+ \left[\int_I |H_0(-1)(\xi)|d\xi - \frac{1}{\lambda}\right]_+ <$$
$$< \left(1 - \int_s^t |H_1(-1)(\xi)|d\xi\right)\left(1 - \int_I H_1(1)(\xi)d\xi\right) \quad \text{for } s,t \in [a,b], \quad s \le t,$$

and

$$\left[\int_s^t |H_0(-1)(\xi)|d\xi - 1\right]_+ \left[\int_I H_0(1)(\xi)d\xi - \frac{1}{\lambda}\right]_+ <$$
$$< \left(1 - \int_s^t H_1(1)(\xi)d\xi\right)\left(1 - \int_I |H_1(-1)(\xi)|d\xi\right) \quad \text{for } s,t \in [a,b], \quad s \le t,$$

are satisfied, where $I = [a,b] \setminus]s,t[$, then problem (1.1), (1.2) *has at least one solution.*

Theorem 2.6. *Let $\lambda \in [1, +\infty[$ and let H admit the representation $H = H_0 - H_1$, where $H_0, H_1 \in \mathcal{P}_{ab}$. Let, furthermore,* (2.1) *and* (2.2) *be fulfilled. If, moreover,*

$$\int_a^b H_0(1)(s)ds < \frac{1}{\lambda}, \qquad \int_a^b |H_0(-1)(s)|ds < \frac{1}{\lambda},$$

$$\frac{\lambda}{1 - \lambda \int_a^b H_0(1)(s)ds} - 1 < \int_a^b H_1(1)(s)ds,$$

$$\frac{\lambda}{1 - \lambda \int_a^b |H_0(-1)(s)|ds} - 1 < \int_a^b |H_1(-1)(s)|ds,$$

and the inequalities

$$\left[\int_I H_1(1)(\xi)d\xi - 1\right]_+ \left[\int_s^t |H_1(-1)(\xi)|d\xi - 1\right]_+ < \\ < \left(\frac{1}{\lambda} - \int_I |H_0(-1)(\xi)|d\xi\right)\left(1 - \int_s^t H_0(1)(\xi)d\xi\right) \\ \text{for } s, t \in [a, b], \quad s \le t,$$

and

$$\left[\int_I |H_1(-1)(\xi)|d\xi - 1\right]_+ \left[\int_s^t H_1(1)(\xi)d\xi - 1\right]_+ < \\ < \left(\frac{1}{\lambda} - \int_I H_0(1)(\xi)d\xi\right)\left(1 - \int_s^t |H_0(-1)(\xi)|d\xi\right) \\ \text{for } s, t \in [a, b], \quad s \le t,$$

are satisfied, where $I = [a, b] \setminus]s, t[$, then problem (1.1), (1.2) *has at least one solution.*

Theorem 2.7. *Let $\lambda \in [1, +\infty[$ and let H admit the representation $H = H_0 - H_1$, where $H_0, H_1 \in \mathcal{P}_{ab}$. Let, furthermore, each of the operators H_0 and H_1 be either a subadditive or a superadditive operator, and let there exist $q \in L([a, b]; \mathbb{R})$ and $c \in \mathbb{R}$ such that* (2.15) *and* (2.16) *are fulfilled. If, moreover,*

$$\int_a^b \bar{H}_1(1)(s)ds < 1,$$

$$\frac{\int_a^b \bar{H}_1(1)(s)ds}{1 - \int_a^b \bar{H}_1(1)(s)ds} - (\lambda - 1) < \int_a^b |\bar{H}_0(-1)(s)|ds,$$

$$\left[\int_s^t \overline{H}_0(1)(\xi)d\xi - 1\right]_+ \left[\int_I \overline{H}_0(1)(\xi)d\xi - \frac{1}{\lambda}\right]_+ <$$
$$< \left(1 - \int_s^t \overline{H}_1(1)(\xi)d\xi\right)\left(1 - \int_I \overline{H}_1(1)(\xi)d\xi\right)$$
$$\text{for} \quad s,t \in [a,b], \quad s \le t,$$

where $I = [a,b] \setminus]s,t[$, *and* $\overline{H}_i$ $(i = 0,1)$ *are given by* (2.20)*, then problem* (1.1)*,* (1.2) *has a unique solution.*

Theorem 2.8. *Let* $\lambda \in [1,+\infty[$ *and let* H *admit the representation* $H = H_0 - H_1$*, where* $H_0, H_1 \in \mathcal{P}_{ab}$*. Let, furthermore, each of the operators* H_0 *and* H_1 *be either a subadditive or a superadditive operator, and let there exist* $q \in L([a,b];\mathbb{R})$ *and* $c \in \mathbb{R}$ *such that* (2.15) *and* (2.16) *are fulfilled. If, moreover,*

$$\int_a^b \overline{H}_0(1)(s)ds < \frac{1}{\lambda},$$
$$\frac{\lambda}{1 - \lambda\int_a^b \overline{H}_0(1)(s)ds} - 1 < \int_a^b |\overline{H}_1(-1)(s)|ds$$

and

$$\left[\int_I \overline{H}_1(1)(\xi)d\xi - 1\right]_+ \left[\int_s^t \overline{H}_1(1)(\xi)d\xi - 1\right]_+ <$$
$$< \left(\frac{1}{\lambda} - \int_I \overline{H}_0(1)(\xi)d\xi\right)\left(1 - \int_s^t \overline{H}_0(1)(\xi)d\xi\right)$$
$$\text{for} \quad s,t \in [a,b], \quad s \le t,$$

where $I = [a,b] \setminus]s,t[$*, and* $\overline{H}_i$ $(i = 0,1)$ *are given by* (2.20)*, then problem* (1.1)*,* (1.2) *has a unique solution.*

Corollary 2.3. *Let* $\lambda \in [1,+\infty[$ *and let the inequalities*

$$\int_a^b [p(s)]_- ds < 1,$$
$$\frac{\int_a^b [p(s)]_- ds}{1 - \int_a^b [p(s)]_- ds} - (\lambda - 1) < \int_a^b [p(s)]_+ ds < 1 + \frac{1}{\lambda} + 2\sqrt{1 - \int_a^b [p(s)]_- ds}$$

be fulfilled. Then problem (1.3)*,* (1.4) *has a unique solution.*

Corollary 2.4. *Let $\lambda \in [1, +\infty[$ and let the inequalities*

$$\int_a^b [p(s)]_+ ds < \frac{1}{\lambda},$$

$$\frac{\lambda}{1 - \lambda \int_a^b [p(s)]_+ ds} - 1 < \int_a^b [p(s)]_- ds < 2 + 2\sqrt{\frac{1}{\lambda} - \int_a^b [p(s)]_+ ds}$$

be fulfilled. Then problem (1.3), (1.4) *has a unique solution.*

3. Auxiliary propositions

The following two lemmas are consequences of the result obtained in [31, Corollary 1.4].

Lemma 3.1. *Let $\lambda \in [0, 1]$ and let the problem*

$$u'(t) = \delta H(u)(t), \qquad u(a) = \delta\lambda u(b)$$

have only the trivial solution for every $\delta \in [0, 1]$. Let, moreover, (2.1) *and* (2.2) *be fulfilled. Then problem* (1.1), (1.2) *has at least one solution.*

Lemma 3.2. *Let $\lambda \in]0, 1]$ and let the problem*

$$u'(t) = \delta H(u)(t), \qquad \delta u(a) = \lambda u(b)$$

have only the trivial solution for every $\delta \in [0, 1]$. Let, moreover, (2.1) *and* (2.2) *be fulfilled. Then problem* (1.1), (1.2) *has at least one solution.*

Lemma 3.3. *Let $\lambda \in]0, 1]$, $F_i, G_i \in \mathcal{P}_{ab}$ $(i = 0, 1)$, and let $w \in \tilde{C}([a, b]; \mathbb{R})$ be a non-trivial function satisfying*

$$w'(t) \le \delta F_0(w)(t) - \delta F_1(w)(t) \qquad \text{for } t \in [a, b], \tag{3.1}$$

$$w'(t) \ge \delta G_0(w)(t) - \delta G_1(w)(t) \qquad \text{for } t \in [a, b], \tag{3.2}$$

$$w(a) = \delta\lambda w(b) \tag{3.3}$$

for some $\delta \in]0, 1]$. If, moreover,

$$\int_a^b F_0(1)(s) ds < 1, \qquad \int_a^b |G_0(-1)(s)| ds < 1, \tag{3.4}$$

$$\frac{\int_a^b F_0(1)(s) ds}{1 - \int_a^b F_0(1)(s) ds} - \frac{1 - \lambda}{\lambda} < \int_a^b F_1(1)(s) ds, \tag{3.5}$$

$$\frac{\int_a^b |G_0(-1)(s)| ds}{1 - \int_a^b |G_0(-1)(s)| ds} - \frac{1 - \lambda}{\lambda} < \int_a^b |G_1(-1)(s)| ds, \tag{3.6}$$

then there exists $t_0 \in [a, b]$ such that

$$w(t_0) = 0. \tag{3.7}$$

Proof. Assume that, on the contrary, the function w has no zero. First suppose that

$$w(t) > 0 \qquad \text{for} \;\; t \in [a,b]. \tag{3.8}$$

Put

$$M = \max\{w(t) : t \in [a,b]\}, \qquad m = \min\{w(t) : t \in [a,b]\} \tag{3.9}$$

and choose $t_M, t_m \in [a,b]$ such that $t_M \neq t_m$ and

$$w(t_M) = M, \qquad w(t_m) = m. \tag{3.10}$$

According to (3.8) and (3.9), we have

$$M > 0, \qquad m > 0. \tag{3.11}$$

It is obvious that either

$$t_M < t_m \tag{3.12}$$

or

$$t_m < t_M. \tag{3.13}$$

If (3.12) is fulfilled, then the integration of (3.1) from a to t_M and from t_m to b, respectively, in view of (3.10), yields

$$M - w(a) \leq \delta \int_a^{t_M} F_0(w)(s)ds - \delta \int_a^{t_M} F_1(w)(s)ds,$$
$$w(b) - m \leq \delta \int_{t_m}^{b} F_0(w)(s)ds - \delta \int_{t_m}^{b} F_1(w)(s)ds.$$

Hence, in view of (3.8), (3.9), and the assumptions $\delta \in\,]0,1]$, $\lambda \in\,]0,1]$, $F_0, F_1 \in \mathcal{P}_{ab}$, we get

$$M - w(a) \leq M \int_a^{t_M} F_0(1)(s)ds,$$
$$\delta\lambda w(b) - m \leq M \int_{t_m}^{b} F_0(1)(s)ds.$$

Summing the last two inequalities, according to (3.3), (3.4), and (3.11), we obtain

$$0 < M\left(1 - \int_a^b F_0(1)(s)ds\right) \leq m. \tag{3.14}$$

If (3.13) is fulfilled, then the integration of (3.1) from t_m to t_M, in view of (3.10), yields

$$M - m \leq \delta \int_{t_m}^{t_M} F_0(w)(s)ds - \delta \int_{t_m}^{t_M} F_1(w)(s)ds.$$

Hence, on account of (3.8), (3.9), (3.11), and the assumptions $\delta \in\,]0,1]$ and $F_0, F_1 \in \mathcal{P}_{ab}$, we get (3.14).

Thus, in both cases (3.12) and (3.13), inequality (3.14) holds.

On the other hand, the integration of (3.1) from a to b yields

$$w(b) - w(a) \le \delta \int_a^b F_0(w)(s)ds - \delta \int_a^b F_1(w)(s)ds,$$

whence, in view of (3.3), (3.8), (3.9), (3.11), and the assumptions $\delta \in\,]0,1]$, $\lambda \in\,]0,1]$, and $F_0, F_1 \in \mathcal{P}_{ab}$, we obtain

$$0 \le m\delta \frac{1-\lambda}{\lambda} \le w(b) - w(a) \le M\delta \int_a^b F_0(1)(s)ds - m\delta \int_a^b F_1(1)(s)ds.$$

Consequently,

$$0 \le m\left(\int_a^b F_1(1)(s)ds + \frac{1-\lambda}{\lambda}\right) \le M \int_a^b F_0(1)(s)ds. \tag{3.15}$$

Multiplying the terms from (3.14) by the respective terms from (3.15), on account of (3.11), we obtain

$$\int_a^b F_1(1)(s)ds \le \frac{\int_a^b F_0(1)(s)ds}{1 - \int_a^b F_0(1)(s)ds} - \frac{1-\lambda}{\lambda},$$

which contradicts (3.5).

Now suppose that $w(t) < 0$ for $t \in [a,b]$. Then, according to (3.2) and (3.3), $v \overset{\text{def}}{=} -w$ is a non-trivial function satisfying

$$v'(t) \le \delta \tilde{F}_0(v)(t) - \delta \tilde{F}_1(v)(t) \qquad \text{for } t \in [a,b], \qquad v(a) = \delta\lambda v(b),$$

where

$$\tilde{F}_i(v)(t) = -G_i(-v)(t) \qquad \text{for } t \in [a,b] \quad (i = 0,1).$$

Obviously, $\tilde{F}_0, \tilde{F}_1 \in \mathcal{P}_{ab}$, and $v(t) > 0$ for $t \in [a,b]$. Therefore, following the steps taken above we get

$$\int_a^b |G_1(-1)(s)|ds \le \frac{\int_a^b |G_0(-1)(s)|ds}{1 - \int_a^b |G_0(-1)(s)|ds} - \frac{1-\lambda}{\lambda},$$

which contradicts (3.6). □

Lemma 3.4. *Let* $\lambda \in [0,1]$, $F_i, G_i \in \mathcal{P}_{ab}$ $(i = 0,1)$, *and let* $w \in \tilde{C}([a,b];\mathbb{R})$ *be a non-trivial function satisfying* (3.1)–(3.3) *for some* $\delta \in\,]0,1]$. *If, moreover, inequalities* (3.4),

$$\left[\int_s^t G_1(1)(\xi)d\xi - 1\right]_+ \left[\int_I |F_1(-1)(\xi)|d\xi - \lambda\right]_+ <$$
$$< \left(1 - \int_s^t |G_0(-1)(\xi)|d\xi\right)\left(1 - \int_I F_0(1)(\xi)d\xi\right)$$
$$\textit{for } s,t \in [a,b], \quad s \le t, \tag{3.16}$$

and

$$\left[\int_s^t |F_1(-1)(\xi)|d\xi - 1\right]_+ \left[\int_I G_1(1)(\xi)d\xi - \lambda\right]_+ <$$
$$< \left(1 - \int_s^t F_0(1)(\xi)d\xi\right)\left(1 - \int_I |G_0(-1)(\xi)|d\xi\right)$$
$$\text{for} \quad s,t \in [a,b], \quad s \leq t, \tag{3.17}$$

are fulfilled, where $I = [a,b] \setminus\,]s,t[$ *, then*

$$w(t) \neq 0 \qquad \text{for} \quad t \in [a,b]. \tag{3.18}$$

Proof. Assume that, on the contrary, there exists a point $t_0 \in [a,b]$ such that (3.7) is fulfilled. Put

$$M = \max\{w(t) : t \in [a,b]\}, \qquad m = -\min\{w(t) : t \in [a,b]\} \tag{3.19}$$

and choose $t_M, t_m \in [a,b]$ such that $t_M \neq t_m$ and

$$w(t_M) = M, \qquad w(t_m) = -m. \tag{3.20}$$

According to (3.7) and (3.19) we have

$$M \geq 0, \qquad m \geq 0, \qquad M + m > 0. \tag{3.21}$$

Obviously, either (3.12) or (3.13) is fulfilled.

Suppose that (3.12) is satisfied. Then the integration of (3.1) from a to t_M and from t_m to b, and the integration of (3.2) from t_M to t_m, respectively, by virtue of (3.20), yields

$$M - w(a) \leq \delta \int_a^{t_M} F_0(w)(s)ds - \delta \int_a^{t_M} F_1(w)(s)ds,$$
$$-m - M \geq \delta \int_{t_M}^{t_m} G_0(w)(s)ds - \delta \int_{t_M}^{t_m} G_1(w)(s)ds,$$
$$w(b) + m \leq \delta \int_{t_m}^{b} F_0(w)(s)ds - \delta \int_{t_m}^{b} F_1(w)(s)ds.$$

Hence, in view of (3.19), (3.21), and the assumptions $\delta \in\,]0,1]$, $\lambda \in [0,1]$, $F_i, G_i \in \mathcal{P}_{ab}$ $(i = 0,1)$, we get

$$\delta M - w(a) \leq \delta M \int_a^{t_M} F_0(1)(s)ds + \delta m \int_a^{t_M} |F_1(-1)(s)|ds, \tag{3.22}$$

$$M + m \leq m \int_{t_M}^{t_m} |G_0(-1)(s)|ds + M \int_{t_M}^{t_m} G_1(1)(s)ds, \tag{3.23}$$

$$\delta\lambda(w(b) + m) \leq \delta M \int_{t_m}^{b} F_0(1)(s)ds + \delta m \int_{t_m}^{b} |F_1(-1)(s)|ds. \tag{3.24}$$

Summing (3.22) and (3.24), in view of (3.3), (3.4), (3.21), and the assumption $\delta > 0$, we obtain

$$0 \le M\left(1 - \int_I F_0(1)(s)ds\right) \le m\left[\int_I |F_1(-1)(s)|ds - \lambda\right]_+, \tag{3.25}$$

where $I = [a,b] \setminus]t_M, t_m[$. Further on, (3.23) according to (3.4) and (3.21), results in

$$0 \le m\left(1 - \int_{t_M}^{t_m} |G_0(-1)(s)|ds\right) \le M\left[\int_{t_M}^{t_m} G_1(1)(s)ds - 1\right]_+. \tag{3.26}$$

Now from (3.25) and (3.26) it follows that $M > 0$, $m > 0$. Therefore, multiplying (3.25) and (3.26) we obtain

$$\left(1 - \int_I F_0(1)(s)ds\right)\left(1 - \int_{t_M}^{t_m} |G_0(-1)(s)|ds\right) \le$$
$$\le \left[\int_I |F_1(-1)(s)|ds - \lambda\right]_+ \left[\int_{t_M}^{t_m} G_1(1)(s)ds - 1\right]_+,$$

which contradicts (3.16).

Now suppose that (3.13) holds. Then, according to (3.1)–(3.3), $v \overset{\text{def}}{=} -w$ is a non-trivial function satisfying the relations

$$v'(t) \le \delta\tilde{F}_0(v)(t) - \delta\tilde{F}_1(v)(t) \qquad \text{for } t \in [a,b],$$
$$v'(t) \ge \delta\tilde{G}_0(v)(t) - \delta\tilde{G}_1(v)(t) \qquad \text{for } t \in [a,b],$$
$$v(a) = \delta\lambda v(b),$$

where

$$\tilde{F}_i(v)(t) = -G_i(-v)(t), \quad \tilde{G}_i(v)(t) = -F_i(-v)(t) \quad \text{for } t \in [a,b] \quad (i = 0,1).$$

Obviously, $\tilde{F}_i, \tilde{G}_i \in \mathcal{P}_{ab}$ $(i = 0,1)$, and

$$v(t_m) = \max\{v(t) : t \in [a,b]\}, \qquad v(t_M) = \min\{v(t) : t \in [a,b]\}.$$

Therefore, the substitution $v \overset{\text{def}}{=} -w$ transforms the case considered into the previous one and, following the steps above, we get

$$\left(1 - \int_I |G_0(-1)(s)|ds\right)\left(1 - \int_{t_m}^{t_M} F_0(1)(s)ds\right) \le$$
$$\le \left[\int_I G_1(1)(s)ds - \lambda\right]_+ \left[\int_{t_m}^{t_M} |F_1(-1)(s)|ds - 1\right]_+,$$

where $I = [a,b] \setminus]t_m, t_M[$, which contradicts (3.17). □

Lemma 3.5. *Let* $\lambda \in]0,1]$, $F_i, G_i \in \mathcal{P}_{ab}$ $(i = 0,1)$, *and let* $w \in \tilde{C}([a,b];\mathbb{R})$ *be a non-trivial function satisfying inequalities* (3.1), (3.2), *and*

$$\delta w(a) = \lambda w(b) \tag{3.27}$$

for some $\delta \in]0,1]$. *If, moreover,*

$$\int_a^b G_1(1)(s)ds < \lambda, \qquad \int_a^b |F_1(-1)(s)|ds < \lambda, \tag{3.28}$$

$$\frac{1}{\lambda - \int_a^b G_1(1)(s)ds} - 1 < \int_a^b G_0(1)(s)ds, \tag{3.29}$$

$$\frac{1}{\lambda - \int_a^b |F_1(-1)(s)|ds} - 1 < \int_a^b |F_0(-1)(s)|ds, \tag{3.30}$$

then there exists a point $t_0 \in [a,b]$ *such that* (3.7) *holds.*

Proof. Assume that, on the contrary, the function w has no zero. First suppose that (3.8) is fulfilled. Define numbers M and m by (3.9) and choose $t_M, t_m \in [a,b]$ such that $t_M \neq t_m$ and (3.10) is satisfied. According to (3.8) and (3.9) we have (3.11). Obviously, either (3.12) or (3.13) holds.

If (3.13) is fulfilled, then the integration of (3.2) from a to t_m and from t_M to b, respectively, in view of (3.10), yields

$$w(a) - m \le -\delta \int_a^{t_m} G_0(w)(s)ds + \delta \int_a^{t_m} G_1(w)(s)ds,$$

$$M - w(b) \le -\delta \int_{t_M}^b G_0(w)(s)ds + \delta \int_{t_M}^b G_1(w)(s)ds.$$

Hence, in view of (3.8), (3.9), and the assumptions $\delta \in]0,1]$, $\lambda \in]0,1]$, $G_0, G_1 \in \mathcal{P}_{ab}$, we get

$$\delta(w(a) - m) \le M \int_a^{t_m} G_1(1)(s)ds,$$

$$\lambda(M - w(b)) \le M \int_{t_M}^b G_1(1)(s)ds.$$

Summing the last two inequalities, and taking (3.11), (3.27), and (3.28) into account, we obtain

$$0 < M\left(\lambda - \int_a^b G_1(1)(s)ds\right) \le \delta m. \tag{3.31}$$

If (3.12) is fulfilled, then the integration of (3.2) from t_M to t_m, in view of (3.10), yields

$$M - m \le -\delta \int_{t_M}^{t_m} G_0(w)(s)ds + \delta \int_{t_M}^{t_m} G_1(w)(s)ds. \tag{3.32}$$

Suppose that $\lambda \le \delta$. Then from (3.32), in view of (3.8), (3.9), and the assumptions $\delta \in]0,1]$, $\lambda \in]0,1]$, and $G_0, G_1 \in \mathcal{P}_{ab}$, it follows that

$$\lambda M - \delta m \le \lambda(M - m) \le M \int_a^b G_1(1)(s)ds,$$

whence we get (3.31). If $\delta < \lambda$, then from (3.32), in view of (3.8), (3.9), and the assumptions $\delta \in]0,1]$, $\lambda \in]0,1]$, and $G_0, G_1 \in \mathcal{P}_{ab}$, we obtain

$$0 < M\left(\lambda - \int_a^b G_1(1)(s)ds\right) \leq m.$$

Thus in both cases (3.12) and (3.13), we have

$$0 < M\left(\lambda - \int_a^b G_1(1)(s)ds\right) \leq \delta m \qquad \text{if } \lambda \leq \delta, \tag{3.33}$$

$$0 < M\left(\lambda - \int_a^b G_1(1)(s)ds\right) \leq m \qquad \text{if } \lambda > \delta. \tag{3.34}$$

On the other hand, the integration of (3.2) from a to b yields

$$w(a) - w(b) \leq -\delta \int_a^b G_0(w)(s)ds + \delta \int_a^b G_1(w)(s)ds. \tag{3.35}$$

If now $\lambda \leq \delta$, then from (3.35), on account of (3.8), (3.9), (3.11), (3.27), and the assumptions $\delta \in]0,1]$, $\lambda \in]0,1]$, and $G_0, G_1 \in \mathcal{P}_{ab}$, we obtain

$$-M(1-\lambda) \leq w(b)(\lambda - 1) \leq w(a) - w(b) \leq -m\delta \int_a^b G_0(1)(s)ds + M\int_a^b G_1(1)(s)ds.$$

Therefore,

$$0 \leq m\delta \int_a^b G_0(1)(s)ds \leq M\left(\int_a^b G_1(1)(s)ds + 1 - \lambda\right) \qquad \text{if } \lambda \leq \delta. \tag{3.36}$$

If $\lambda > \delta$, from (3.35), on account of (3.8), (3.11), (3.27), and the assumption $\lambda \in]0,1]$, we have

$$-M\delta(1-\lambda) \leq 0 \leq w(a) - w(b) \leq -m\delta \int_a^b G_0(1)(s)ds + M\delta \int_a^b G_1(1)(s)ds.$$

Hence, in view of the assumption $\delta > 0$, we get

$$0 \leq m \int_a^b G_0(1)(s)ds \leq M\left(\int_a^b G_1(1)(s)ds + 1 - \lambda\right) \qquad \text{if } \lambda > \delta. \tag{3.37}$$

Multiplying the terms in (3.33) (resp., (3.34)) by the corresponding terms in (3.36) (resp., (3.37)), on account of (3.11) and the assumption $\delta > 0$, we obtain

$$\int_a^b G_0(1)(s)ds \leq \frac{1}{\lambda - \int_a^b G_1(1)(s)ds} - 1,$$

which contradicts (3.29).

Now suppose that $w(t) < 0$ for $t \in [a,b]$. Then, according to (3.1) and (3.27), $v \overset{\text{def}}{=} -w$ is a non-trivial function satisfying

$$v'(t) \geq \delta\tilde{G}_0(v)(t) - \delta\tilde{G}_1(v)(t) \qquad \text{for } t \in [a,b], \qquad \delta v(a) = \lambda v(b),$$

where

$$\tilde{G}_i(v)(t) = -F_i(-v)(t) \qquad \text{for } t \in [a,b] \quad (i = 0,1).$$

Obviously, $\tilde{G}_0, \tilde{G}_1 \in \mathcal{P}_{ab}$, and $v(t) > 0$ for $t \in [a,b]$. Therefore, following the steps taken above we get

$$\int_a^b |F_0(-1)(s)|ds \le \frac{1}{\lambda - \int_a^b |F_1(-1)(s)|ds} - 1,$$

which contradicts (3.30). $\square$

Lemma 3.6. *Let $\lambda \in]0,1]$, $F_i, G_i \in \mathcal{P}_{ab}$ $(i = 0,1)$, and let $w \in \tilde{C}([a,b];\mathbb{R})$ be a nontrivial function satisfying* (3.1), (3.2), *and* (3.27) *for some $\delta \in]0,1]$. If, moreover, inequalities* (3.28),

$$\left[\int_I F_0(1)(\xi)d\xi - 1\right]_+ \left[\int_s^t |G_0(-1)(\xi)|d\xi - 1\right]_+ < \\ < \left(\lambda - \int_I |F_1(-1)(\xi)|d\xi\right)\left(1 - \int_s^t G_1(1)(\xi)d\xi\right) \\ \textit{for } s,t \in [a,b], \quad s \le t, \quad (3.38)$$

and

$$\left[\int_I |G_0(-1)(\xi)|d\xi - 1\right]_+ \left[\int_s^t F_0(1)(\xi)d\xi - 1\right]_+ < \\ < \left(\lambda - \int_I G_1(1)(\xi)d\xi\right)\left(1 - \int_s^t |F_1(-1)(\xi)|d\xi\right) \\ \textit{for } s,t \in [a,b], \quad s \le t, \quad (3.39)$$

are fulfilled, where $I = [a,b] \setminus]s,t[$, then (3.18) *holds.*

Proof. Assume that, on the contrary, there exists $t_0 \in [a,b]$ such that (3.7) holds. Define numbers M and m by (3.19) and choose $t_M, t_m \in [a,b]$ such that $t_M \ne t_m$ and (3.20) is fulfilled. According to (3.7) and (3.19) we have (3.21). Obviously, either (3.12) or (3.13) is fulfilled.

Suppose that (3.12) is satisfied. Then the integration of (3.1) from a to t_M and from t_m to b, and the integration of (3.2) from t_M to t_m, respectively, by virtue of (3.20), yields

$$M - w(a) \le \delta \int_a^{t_M} F_0(w)(s)ds - \delta \int_a^{t_M} F_1(w)(s)ds,$$

$$-m - M \ge \delta \int_{t_M}^{t_m} G_0(w)(s)ds - \delta \int_{t_M}^{t_m} G_1(w)(s)ds,$$

$$w(b) + m \le \delta \int_{t_m}^{b} F_0(w)(s)ds - \delta \int_{t_m}^{b} F_1(w)(s)ds.$$

Hence, in view of (3.19), (3.21), and the assumptions $\delta \in\,]0,1]$, $\lambda \in\,]0,1]$, $F_i, G_i \in \mathcal{P}_{ab}$ $(i = 0,1)$, we get

$$\delta(M - w(a)) \leq \delta M \int_a^{t_M} F_0(1)(s)ds + \delta m \int_a^{t_M} |F_1(-1)(s)|ds, \tag{3.40}$$

$$M + m \leq m \int_{t_M}^{t_m} |G_0(-1)(s)|ds + M \int_{t_M}^{t_m} G_1(1)(s)ds, \tag{3.41}$$

$$\lambda(w(b) + \delta m) \leq \delta M \int_{t_m}^{b} F_0(1)(s)ds + \delta m \int_{t_m}^{b} |F_1(-1)(s)|ds. \tag{3.42}$$

Summing (3.40) and (3.42), according to (3.21), (3.27), (3.28), and the assumption $\delta > 0$, we obtain

$$0 \leq m\left(\lambda - \int_I |F_1(-1)(s)|ds\right) \leq M\left[\int_I F_0(1)(s)ds - 1\right]_+, \tag{3.43}$$

where $I = [a,b] \setminus\,]t_M, t_m[$. Furthermore, (3.41) in view of (3.28) and (3.21), results in

$$0 \leq M\left(1 - \int_{t_M}^{t_m} G_1(1)(s)ds\right) \leq m\left[\int_{t_M}^{t_m} |G_0(-1)(s)|ds - 1\right]_+. \tag{3.44}$$

Now from (3.43) and (3.44) it follows that $M > 0$, $m > 0$. Therefore, multiplying the terms in (3.43) by the corresponding terms in (3.44), we obtain

$$\left(\lambda - \int_I |F_1(-1)(s)|ds\right)\left(1 - \int_{t_M}^{t_m} G_1(1)(s)ds\right) \leq$$
$$\leq \left[\int_I F_0(1)(s)ds - 1\right]_+ \left[\int_{t_M}^{t_m} |G_0(-1)(s)|ds - 1\right]_+,$$

which contradicts (3.38).

Now suppose that (3.13) holds. Then, according to (3.1), (3.2), and (3.27), $v \overset{\text{def}}{=} -w$ is a non-trivial function satisfying

$$v'(t) \leq \delta\tilde{F}_0(v)(t) - \delta\tilde{F}_1(v)(t) \qquad \text{for } t \in [a,b],$$
$$v'(t) \geq \delta\tilde{G}_0(v)(t) - \delta\tilde{G}_1(v)(t) \qquad \text{for } t \in [a,b],$$
$$\delta v(a) = \lambda v(b),$$

where

$$\tilde{F}_i(v)(t) = -G_i(-v)(t), \quad \tilde{G}_i(v)(t) = -F_i(-v)(t) \quad \text{for } t \in [a,b] \quad (i = 0,1).$$

Obviously, $\tilde{F}_i, \tilde{G}_i \in \mathcal{P}_{ab}$ $(i = 0,1)$, and

$$v(t_m) = \max\{v(t) : t \in [a,b]\}, \qquad v(t_M) = \min\{v(t) : t \in [a,b]\}.$$

Therefore, the substitution $v \overset{\text{def}}{=} -w$ transforms the case considered into the previous one and, arguing similarly to the considerations above, we get

$$\left(\lambda - \int_I G_1(1)(s)ds\right)\left(1 - \int_{t_m}^{t_M} |F_1(-1)(s)|ds\right) \leq$$
$$\leq \left[\int_I |G_0(-1)(s)|ds - 1\right]_+ \left[\int_{t_m}^{t_M} F_0(1)(s)ds - 1\right]_+ ,$$

where $I = [a,b] \setminus]t_m, t_M[$, which contradicts (3.39). □

4. Proofs

Theorem 2.1 follows from Lemmas 3.1, 3.3, and 3.4, Theorem 2.1′ follows from Lemmas 3.1 and 3.4, and Theorem 2.2 follows from Lemmas 3.2, 3.5, and 3.6.

Proof of Theorem 2.3. Let us first note that, for every operator $T \in \mathcal{H}_{ab}$ and arbitrary $t \in [a,b]$ and $v \in C([a,b];\mathbb{R})$, we have

$$-T(-v)(t) \leq T(v)(t)$$

if T is subadditive, and

$$T(v)(t) \leq -T(-v)(t)$$

if T is superadditive. Therefore,

$$-\overline{H}_i(-1)(t) \leq \overline{H}_i(1)(t) \qquad \text{for } t \in [a,b] \quad (i = 0,1),$$

and, consequently, inequalities (2.17)–(2.19) yield inequalities (2.3)–(2.7). Thus, the assumptions of Theorem 2.1 are fulfilled, and problem (1.1), (1.2) has at least one solution.

We will show that (1.1), (1.2) has no more than one solution. Let u and v be solutions of (1.1), (1.2). Then, according to (2.15) and (2.16), the function $w(t) = u(t) - v(t)$ for $t \in [a,b]$ satisfies (3.1)–(3.3) with $\delta = 1$ and

$$F_0(w)(t) = \overline{H}_0(w)(t), \qquad F_1(w)(t) = -\overline{H}_1(-w)(t) \qquad \text{for } t \in [a,b],$$
$$G_0(w)(t) = -\overline{H}_0(-w)(t), \qquad G_1(w)(t) = \overline{H}_1(w)(t) \qquad \text{for } t \in [a,b].$$

According to Lemmas 3.3 and 3.4 we have $w \equiv 0$, i. e., $u \equiv v$. □

Theorems 2.3′ and 2.4 can be proved analogously.

Proof of Corollary 2.1. Let us put

$$H_0(v)(t) \overset{\text{def}}{=} [p(t)]_+ \max\{v(s) : \tau_1(t) \leq s \leq \tau_2(t)\} \qquad \text{for } t \in [a,b],$$
$$H_1(v)(t) \overset{\text{def}}{=} [p(t)]_- \max\{v(s) : \tau_1(t) \leq s \leq \tau_2(t)\} \qquad \text{for } t \in [a,b],$$
$$Q(v)(t) \overset{\text{def}}{=} q(t) \qquad \text{for } t \in [a,b],$$

and $h(v) \overset{\text{def}}{=} c$. Then the assumptions of Theorem 2.3 are fulfilled because

$$\left[\int_s^t \bar{H}_1(1)(\xi)d\xi - 1\right]_+ \left[\int_I \bar{H}_1(1)(\xi)d\xi - \lambda\right]_+ \leq$$
$$\leq \frac{1}{4}\left[\int_a^b [p(s)]_- ds - 1 - \lambda\right]_+^2 \qquad \text{for } \ s,t \in [a,b], \quad s \leq t$$

and

$$\left(1 - \int_s^t \bar{H}_0(1)(\xi)d\xi\right)\left(1 - \int_I \bar{H}_0(1)(\xi)d\xi\right) \geq$$
$$\geq 1 - \int_a^b [p(s)]_+ ds \qquad \text{for } \ s,t \in [a,b], \quad s \leq t,$$

where $I = [a,b] \setminus]s,t[$. □

In a similar way, it can be proved that Corollary 2.1′ follows from Theorem 2.3′ and Corollary 2.2 follows from Theorem 2.4.

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