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ON γ-R_0 AND γ-R_1 SPACES

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Abstract. In this paper, we investigate further properties of γ-R_0 and γ-R_1 spaces due to Ekici.

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1. INTRODUCTION AND PRELIMINARIES

The separation axioms R_0 and R_1 were introduced and studied by N. A. Shan in [22] and C. T. Yang [23]. In 1963, they were rediscovered by A. S. Davis [4]. Then, their some properties are obtained by some authors [6, 7, 15, 17]. The notion of *b*-open (or *sp*-open) sets was introduced by Andrijević [1] and Dontchev and Przemski [5] independently of each other in 1996. Since then it has been investigated in the literature (see [9, 13, 14, 18, 19]). Ekici [9, 10] introduced γ-R_0 and γ-R_1 spaces and obtained some properties of them. In this paper, we investigte further properties of γ-R_0 and γ-R_1 spaces due to Ekici [9, 10].

In this paper, by (X,τ) and (Y,φ) (or X and Y) we always mean topological spaces on which no separation axioms are assumed, unless otherwise mentioned. For a subset A of (X,τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ represent the closure of A and the interior of A with respect to τ, respectively.

Definition 1.1. Let (X,τ) be a topological space. A subset A is called

(a) b-open [1] or sp-open [5] or γ-open [14] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$,
(b) α-open [20] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$.

The family of all b-open (resp., α-open) sets in a topological space (X,τ) is denoted by $BO(X)$ (resp., $\alpha(X)$).

Definition 1.2. The complement of a b-open set is called b-closed [1].

Similarly, the family of all b-closed sets in a topological space (X,τ) is denoted by $BC(X)$.

Definition 1.3. The intersection of all b-closed sets containing A is called the b-closure [1] of A and is denoted by $b\mathrm{Cl}(A)$.

The union of all b-open sets contained in A is called the b-interior of A and is denoted by $\mathrm{b\,Int}(A)$. The following lemma is useful in the sequel:

Lemma 1.1 ([1]). *Let A be a subset of a space (X,τ). Then the following properties hold:*

(a) $\mathrm{b\,Cl}(A) = \mathrm{s\,Cl}(A) \cap \mathrm{p\,Cl}(A)$,
(b) $\mathrm{b\,Int}(A) = \mathrm{s\,Int}(A) \cup \mathrm{p\,Int}(A)$,
(c) $x \in \mathrm{b\,Cl}(A)$ *if and only if* $A \cap U \neq \varnothing$ *for every* $BO(X,x)$,
(d) $\mathrm{b\,Cl}(A) = A \cup \big[\mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Cl}(\mathrm{Int}(A))\big]$,
(e) $\mathrm{b\,Int}(A) = A \cap \big[\mathrm{Cl}(\mathrm{Int}(A)) \cup \mathrm{Int}(\mathrm{Cl}(A))\big]$.

The family of all b-open sets containing x of X will be denoted by $BO(X,x)$. The following basic properties of the b-closure are useful in the sequel:

Lemma 1.2 ([11]). *Let A and B be subsets of a space (X,τ). The following properties hold:*

(a) *If $A \subset B$, then* $\mathrm{b\,Cl}(A) \subset \mathrm{b\,Cl}(B)$,
(b) $\mathrm{b\,Cl}(X \setminus A) = X \setminus \mathrm{b\,Int}(A)$,
(c) $A \in BC(X,\tau)$ *if and only if* $A = \mathrm{b\,Cl}(A)$.

Lemma 1.3 ([1]). *The following assertions hold:*

(a) *The union of any family of b-open sets is a b-open set.*
(b) *The intersection of an open set and a b-open set is a b-open set.*

Definition 1.4. [10] Let X be a topological space and $S \subset X$. The γ-kernel of S, denoted by $\gamma\text{-}\ker(S)$, is defined to be the set $\gamma\text{-}\ker(S) = \cap\{U \in BO(X,\tau) \mid S \subset U\}$.

Remark 1.1. In [3], the set $\gamma\text{-}\ker(S)$ has been denoted by S^{Λ_b}.

Lemma 1.4 ([10]). *Let (X,τ) be a topological space and A be a subset of X. Then* $A^{\Lambda_b} = \{x \in X \mid \mathrm{b\,Cl}(\{x\}) \cap A \neq \varnothing\}$.

2. Preservation theorems

Definition 2.1. A topological space (X,τ) is called γ-R_0 [10] if its every b-open set contains the b-closure of each of its singletons.

Theorem 2.1 ([10]). *A topological space $(X,\tau$ is a γ-R_0 space if and only if for any x and y in X,* $\mathrm{b\,Cl}(\{x\}) \neq \mathrm{b\,Cl}(\{y\})$ *implies* $\mathrm{b\,Cl}(\{x\}) \cap \mathrm{b\,Cl}(\{y\}) = \varnothing$.

Definition 2.2. A function $f\colon (X,\tau) \to (Y,\varphi)$ is called

(a) b-continuous [14] (resp., α-continuous [16]) if $f^{-1}(V)$ is b-open (resp., α-open) in (X,τ) for every open set V of (Y,φ);
(b) α-open [16] if $f(U)$ is α-open in (Y,φ) for every open set U of (X,τ).

Lemma 2.1 ([14]). *Let $f\colon (X,\tau) \to (Y,\varphi)$ be an α-continuous and α-open function. Then the inverse image of each b-open set in Y is b-open in X.*

Definition 2.3. A function $f:(X,\tau)\to(Y,\varphi)$ is called strongly γ-closed [12] if $f(F)$ is γ-closed in (Y,φ) for every γ-closed set F of (X,τ).

Theorem 2.2. *If (X,τ) is a γ-R_0 space and $f:(X,\tau)\to(Y,\varphi)$ is an α-continuous, α-open and b-closed surjection, then (Y,φ) is a γ-R_0 space.*

Proof. Let V be a b-open set of Y and y any point of of V. Since f is α-continuous and α-open, $f^{-1}(V)$ is a b-open set in (X,τ), by Lemma 2.1. Since (X,τ) is γ-R_0, for a point $x\in f^{-1}(\{y\})$ by Definition 2.1, $b\mathrm{Cl}(\{x\})\subset f^{-1}(V)$. The b-closeness of f implies that $b\mathrm{Cl}(\{y\})=b\mathrm{Cl}(\{f(x)\})\subset f(b\mathrm{Cl}(\{x\}))\subset V$. Therefore, (Y,φ) is γ-R_0. □

Definition 2.4. A function $f:(X,\tau)\to(Y,\varphi)$ is called quasi γ-closed [12] if $f(F)$ is closed in (Y,φ) for every γ-closed set F of (X,τ).

Theorem 2.3. *If (X,τ) is a γ-R_0 space and $f:(X,\tau)\to(Y,\varphi)$ is a strongly b-closed and b-continuous surjection, then (Y,φ) is an R_0 space.*

Proof. The proof is similar to that of Theorem 2.2 and is thus omitted. □

3. γ-R_1 SPACES

Definition 3.1. A topological space (X,τ) is called γ-R_1 [9] if for x and y in X with $b\mathrm{Cl}(\{x\})\neq b\mathrm{Cl}(\{y\})$, there exist disjoint b-open sets U and V such that $b\mathrm{Cl}(\{x\})$ is a subset of U and $b\mathrm{Cl}(\{y\})$ is a subset of V.

Proposition 3.1. *Every γ-R_1 space is γ-R_0.*

Proof. Let U be a b-open set such that $x\in U$. If $y\notin U$, since $x\notin b\mathrm{Cl}(\{y\})$, we have $b\mathrm{Cl}(\{x\})\neq b\mathrm{Cl}(\{y\})$. So, there exists a b-open set V such that $b\mathrm{Cl}(\{y\})\subset V$ and $x\notin V$, which implies $y\notin b\mathrm{Cl}(\{x\})$. Hence $b\mathrm{Cl}(\{x\})\subset U$. Therefore, (X,τ) is γ-R_0. □

Remark 3.1. The converse of Proposition 3.1 need not be true as shown in the following example.

Example 3.1. Let (X,τ) be a topological space as in [2, Example 4.16], that is, p be a fixed point of (X,τ) with τ as the cofinite topology on X. If it meaned clearly, $\tau=\{\varnothing,X,U\}$ with $U\subset(X\setminus\{p\})$ and $X\setminus U$ finite. One can obtain $BO(X)=\{\varnothing,X,U,A\}$ such that for each $A\subset U$. Since $b\mathrm{Cl}(\{x\})=\{x\}$ for every $x\in X$, we obtain that (X,τ) is γ-R_0 by using Theorem 2.1. On the other hand, one can easily show that (X,τ) is not γ-R_1 by using Definition 3.1.

Theorem 3.1 ([9])**.** *A topological space (X,τ) is γ-R_1 if and only if for $x,y\in X$, $\{x\}^{\Lambda_b}\neq\{y\}^{\Lambda_b}$, there exist disjoint b-open sets U and V such that $b\mathrm{Cl}(\{x\})\subset U$ and $b\mathrm{Cl}(\{y\})\subset V$.*

Definition 3.2. A set U in a topological space (X,τ) is called b-neighborhood [11] of x if U contains a b-open set V such that $x\in V$.

Theorem 3.2. *A topological space* (X,τ) *is* γ-R_1 *if and only if the inclusion* $x \in X \setminus \mathrm{bCl}(\{y\})$ *implies that* x *and* y *have disjoint b-open neighborhoods.*

Proof. Necessity: Let $x \in X \setminus \mathrm{bCl}(\{y\})$. Then $\mathrm{bCl}(\{x\}) \neq \mathrm{bCl}(\{y\})$ and x, y have disjoint b-open neighborhoods.

Sufficiency: First, we show that (X,τ) is γ-R_0. Let U be a b-open set and $x \in U$. Suppose that $y \notin U$. Then, $\mathrm{bCl}(\{y\}) \cap U = \varnothing$ and $x \notin \mathrm{bCl}(\{y\})$. There exist b-open sets U_x and U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \varnothing$. Hence, $\mathrm{bCl}(\{x\}) \subset \mathrm{bCl}(U_x)$ and $\mathrm{bCl}(\{x\}) \cap U_y \subset \mathrm{bCl}(U_x) \cap U_y = \varnothing$. Therefore, $y \notin \mathrm{bCl}(\{x\})$. Consequently, $\mathrm{bCl}(\{x\}) \subset U$ and (X,τ) is γ-R_0. Next, we show that (X,τ) is γ-R_1. Suppose that $\mathrm{bCl}(\{x\}) \neq \mathrm{bCl}(\{y\})$. Then, we can assume that there exists $z \in \mathrm{bCl}(\{x\})$ such that $z \notin \mathrm{bCl}(\{y\})$. There exist b-open sets V_z and V_y such that $z \in V_z$, $y \in V_y$ and $V_z \cap V_y = \varnothing$. Since $z \in \mathrm{bCl}(\{x\})$, $x \in V_z$. Since (X,τ) is γ-R_0, we obtain $\mathrm{bCl}(\{x\}) \subset V_z$, $\mathrm{bCl}(\{y\}) \subset V_y$ and $V_z \cap V_y = \varnothing$. This shows that (X,τ) is γ-R_1. □

Corollary 3.1. *For a topological space* (X,τ)*, the following properties are equivalent:*

(1) (X,τ) *is a* γ-R_1 *space,*
(2) $\mathrm{bCl}(\{x\}) \neq \mathrm{bCl}(\{y\})$ *implies that* x *and* y *have disjoint b-open neighborhoods.*

Proof. The implication (1)⇒(2) is obvious.

(2)⟹(1): Suppose that $x \in X \setminus \mathrm{bCl}(\{y\})$. Then, $\mathrm{bCl}(\{x\}) \neq \mathrm{bCl}(\{y\})$ and by (2) x and y have disjoint b-open neighborhoods. By Theorem 3.2, (X,τ) is γ-R_1. □

The following notions are due to Park [21]. A point x of X is called a b-θ-cluster point of A if $\mathrm{bCl}(U) \cap A \neq \varnothing$ for every $U \in BO(X,x)$. The set of all b-θ-cluster points of A is called the b-θ-closure of A and is denoted by $\mathrm{bCl}_\theta(A)$. A subset A is said to be b-θ-closed if $A = \mathrm{bCl}_\theta(A)$. The complement of a b-$\theta$-closed set is said to be b-θ-open.

Lemma 3.1. *For any subset* A *of a topological space* (X,τ)*,* $\mathrm{bCl}(A) \subseteq \mathrm{bCl}_\theta(A)$.

Theorem 3.3. *A topological space* (X,τ) *is a* γ-R_1 *space if and only if for each* $x \in X$*,* $\mathrm{bCl}_\theta(\{x\}) = \mathrm{bCl}(\{x\})$.

Proof. Necessity: Let (X,τ) be γ-R_1 and x any point of X. Assume that $y \in X \setminus \mathrm{bCl}(\{x\}))$. By Theorem 3.2, there exist disjoint b-open sets U and V such that $x \in U$ and $y \in V$. Thus, we have $\{x\} \cap \mathrm{bCl}(V) \subset U \cap \mathrm{bCl}(V) = \varnothing$ and hence $y \notin \mathrm{bCl}_\theta(\{x\})$. Therefore, we obtain $\mathrm{bCl}_\theta(\{x\}) \subseteq \mathrm{bCl}(\{x\})$. So, by using Lemma 3.1, we have $\mathrm{bCl}_\theta(\{x\}) = \mathrm{bCl}(\{x\})$.

Sufficiency: Let x, y be distinct points of X and $y \in X \setminus \mathrm{bCl}(\{x\})$. Then $y \in X \setminus \mathrm{bCl}_\theta(\{x\}))$ and hence there exists a b-open neighborhood V of y such that $\mathrm{bCl}(V) \cap \{x\} = \varnothing$. Then V and $X \setminus \mathrm{bCl}(V))$ are disjoint b-open neighborhoods of y and x, respectively. By Theorem 3.2, (X,τ) is γ-R_1. □

The following notion is due to Ekici and Caldas [13].

Definition 3.3. A topological space (X,τ) is said to be b-regular [13] if for each b-closed set F and each point $x \in X \setminus F$, there exist disjoint b-open sets U and V such that $x \in U$ and $F \subset V$.

Now, we can give a characterization of b-regular spaces.

Theorem 3.4. *For a topological space (X,τ), the following properties are equivalent:*

(1) *(X,τ) is b-regular,*
(2) *$\mathrm{bCl}_\theta(A) = \mathrm{bCl}(A)$ for every subset A of X,*
(3) *$\mathrm{bCl}_\theta(A) = A$ for every b-closed subset A of X.*

Proof. (1)$\Rightarrow$(2): Since $\mathrm{bCl}(A) \subseteq \mathrm{bCl}_\theta(A)$ by Lemma 3.1, we will show that $bCl_\theta(A) \subseteq \mathrm{bCl}(A)$. Assume that $x \notin \mathrm{bCl}(A)$. Since (X,τ) is b-regular, there exist disjoint b-open sets U and V such that $x \in U$ and $\mathrm{bCl}(A) \subset V$. Hence $A \cap \mathrm{bCl}(U) \subset \mathrm{bCl}(A) \cap \mathrm{bCl}(U) = \varnothing$. This implies $x \notin \mathrm{bCl}_\theta(A)$.
(2)$\Rightarrow$(3): Let A be a b-closed subset of X, then we have $A = \mathrm{bCl}(A)$ and $\mathrm{bCl}_\theta(A) = A$.
(3)$\Rightarrow$(1): Let V be a b-open set containing x in X. Hence $X \setminus V$ is b-closed. By (3), $X \setminus V = \mathrm{bCl}_\theta(X \setminus V)$. Then there exists an $U \in BO(X,x)$ such that $\mathrm{bCl}(U) \cap (X \setminus V) = \varnothing$. This implies $\mathrm{bCl}(U) \subset V$. Consequently, the proof is completed. □

Corollary 3.2. *Every b-regular space is a γ-R_1 space.*

Proof. This immediately follows from Theorems 3.3 and 3.4. □

Definition 3.4. Let ∇ be a filter base of a topological space (X,τ).

(1) The b-adherence (resp., b-θ-adherence) of a filter base ∇, denoted by $\mathrm{ad}_b\,\nabla$ (resp., θ-$\mathrm{ad}_b\,\nabla$), is defined by $\cap\{\mathrm{bCl}(V) \mid V \in \nabla\}$ (resp., $\cap\{\mathrm{bCl}_\theta(V) \mid V \in \nabla\}$),
(2) ∇ b-converges [8] (resp., ∇ b-θ-converges) to x if for each b-open set U containing x, there is $V \in \nabla$ such that $V \subset U$ (resp., $V \subset \mathrm{bCl}(U)$).

Theorem 3.5. *If a topological space (X,τ) is γ-R_1, then $\mathrm{ad}_b\,\nabla \subset \mathrm{bCl}(\{x\})$ for each filter base ∇ b-θ-converging to $x \in X$.*

Proof. Let (X,τ) be a γ-R_1 space, $x \in X$, and ∇ be a filter base in X b-θ-converging to x. If $\mathrm{ad}_b\,\nabla = \varnothing$, then the assertion of the theorem holds trivially. Assume that $y \in \mathrm{ad}_b\,\nabla$ and $y \in X \setminus \mathrm{bCl}_\theta(\{x\})$. Then there exists a b-open set U containing y such that $x \in X \setminus \mathrm{bCl}(U)$. Since ∇ b-θ-converging to x, there exists $V \in \nabla$ such that $V \subset \mathrm{bCl}(X \setminus \mathrm{bCl}(U))$. This implies that $V \cap U = \varnothing$ which contradicts the assumption that $y \in \mathrm{ad}_b\,\nabla$. Thus $y \in \mathrm{bCl}_\theta(\{x\})$ and $\mathrm{ad}_b\,\nabla \subset \mathrm{bCl}_\theta(\{x\})$. Since (X,τ) is γ-R_1, by Theorem 3.3 we obtain $\mathrm{ad}_b\,\nabla \subset \mathrm{bCl}(\{x\})$. □

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