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MULTIVALENT FUNCTIONS BASED ON A LINEAR INTEGRAL OPERATOR

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Abstract. By using a linear integral operator, a subclass of multivalent functions is introduced. Some important properties of this class such as coefficient bounds are found. Moreover, applications are also introduced.

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1. INTRODUCTION

Let $T(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p \in \mathbb{N},\ z \in U), \tag{1.1}$$

which are analytic and p-valent (multivalent) in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Given two functions $f, g \in T(p)$, $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ and $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$ their convolution or Hadamard product $f(z) * g(z)$ is defined by

$$f(z) * g(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n, \quad (z \in U).$$

A function $f \in T(p)$ is said to be p-valent starlike of order μ, $0 \le \mu < p$ if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \mu, \quad (z \in U),$$

this class denoted by $S_p^*(\mu)$. A function $f \in T(p)$ is said to be p-valent convex of order μ, $0 \le \mu < p$ if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \mu, \quad (z \in U),$$

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this class denoted by $C_p(\mu)$.

For a function $f \in T(p)$ given by (1.1), we define the linear integral operator $I_{p,\alpha,\beta} f(z)$ by

$$\begin{aligned} I_{p,\alpha,\beta} f(z) &= (\alpha + p\beta + 1) \int_0^1 u^{\alpha} f(u^{\beta} z) du \\ &= z^p + \sum_{n=p+1}^{\infty} \left[\frac{\alpha + p\beta + 1}{\alpha + n\beta + 1} \right] a_n z^n, \quad z \in U, \end{aligned} \tag{1.2}$$

$(\alpha, \beta \in \mathbb{R},\ p \in \mathbb{N})$. Clearly, (1.2) yields

$$f \in T(p) \Rightarrow I_{p,\alpha,\beta} \in T(p).$$

Thus, by applying the operator $I_{p,\alpha,\beta}$ successively, we can obtain

$$\begin{aligned} I^k_{p,\alpha,\beta} f(z) &= \begin{cases} I_{p,\alpha,\beta}\big(I^{k-1}_{p,\alpha,\beta} f(z)\big) & \text{for } k \in \mathbb{N}, \\ f(z) & \text{for } k = 0, \end{cases} \\ &= z^p + \sum_{n=p+1}^{\infty} \left[\frac{\alpha + p\beta + 1}{\alpha + n\beta + 1} \right]^k a_n z^n, \quad (z \in U). \end{aligned} \tag{1.3}$$

Note that when $p = 1$, $\alpha = -1$ and $\beta = 1$, operator (1.3) poses an integral operator defined by Sălăgean (see [9]). Further operators of type (1.2) involving different type of integral operators were introduced by many authors (cf. [1, 3–5, 8]).

Let $S^*_p(\alpha, \beta, \mu)$ denote the class of all functions $f \in T(p)$ satisfying the condition

$$\Re \left\{ \frac{z[I^k_{p,\alpha,\beta} f(z)]'}{I^k_{p,\alpha,\beta} f(z)} \right\} > \mu, \quad (0 \le \mu < p,\ z \in U)$$

and let $C_p(\alpha, \beta, \mu)$ be the class of all functions $f \in T(p)$ such that

$$\Re \left\{ \frac{z[I^k_{p,\alpha,\beta} f(z)]''}{[I^k_{p,\alpha,\beta} f(z)]'} \right\} > \mu, \quad (0 \le \mu < p,\ z \in U).$$

In the present paper, using Jack's lemma, the authors investigate the differential inequality

$$\left| \frac{I^k_{p,\alpha,\beta} f(z)}{z^p} + \frac{z[I^k_{p,\alpha,\beta} f(z)]'}{p I^k_{p,\alpha,\beta} f(z)} - 2 \right| < \mu, \quad (0 \le \mu < p,\ z \in U). \tag{1.4}$$

Let $S^{**}_p(\alpha, \beta, \mu)$ denote the class of analytic functions $f \in T(p)$ satisfying the inequality (1.4).

To establish our results, we need the following lemma given by Jack [2] (see also [7]).

Lemma 1. *Let $w(z)$ be analytic in U with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point z_0, then*

$$z_0 w'(z_0) = c w(z_0), \tag{1.5}$$

where c is a real number and $c \geq 1$.

2. Main results

In this section, we establish some theorems involving inequalities on p-valent functions. We first prove the following theorem.

Theorem 1. *If $f(z) \in T(p)$ satisfies the inequality*

$$\Re\left\{1+\frac{z[I^k_{p,\alpha,\beta}f(z)]''}{[I^k_{p,\alpha,\beta}f(z)]'}-p\right\} < \frac{2\mu}{\mu+2}, \quad (0 \leq \mu < p,\ z \in U), \tag{2.1}$$

then $f \in S^{**}_p(\alpha,\beta,\mu)$.

Proof. Let the function $f(z) \in T(p)$ and satisfy the inequality (2.1). Define the function $w(z)$, $z \in U$, such that

$$\frac{I^k_{p,\alpha,\beta}f(z)}{z^p} = 1+\frac{\mu}{2}w(z), \quad (z \in U,\ p \in \mathbb{N}), \tag{2.2}$$

we have that $w(z)$ is analytic in U and $w(0)=0$. Differentiating both sides of (2.2), we obtain

$$\frac{[I^k_{p,\alpha,\beta}f(z)]'}{z^{p-1}} = \frac{\mu}{2}zw'(z)+p\left(1+\frac{\mu}{2}w(z)\right), \quad (z \in U,\ p \in \mathbb{N},\ p>k,\ k \in \mathbb{N}_0). \tag{2.3}$$

Then, we have from (2.2) and (2.3) that

$$\begin{aligned} F(z) &:= \frac{z[I^k_{p,\alpha,\beta}f(z)]'}{I^k_{p,\alpha,\beta}f(z)} - p \\ &= \frac{\frac{\mu}{2}zw'(z)}{1+\frac{\mu}{2}w(z)}, \quad (z \in U,\ p \in \mathbb{N},\ p>k,\ k \in \mathbb{N}_0). \end{aligned} \tag{2.4}$$

Now, suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = 1.$$

Then, using Lemma 1 and letting $w(z_0) = e^{i\theta}$ such that $w(z_0) \neq -1$ in the equation (2.4), yields

$$\begin{aligned}\Re\{F(z_0)\} &= \Re\left\{\frac{\frac{\mu}{2} z_0 w'(z_0)}{1+\frac{\mu}{2} w(z_0)}\right\} = \Re\left\{\frac{\frac{\mu c}{2} w(z_0)}{1+\frac{\mu}{2} w(z_0)}\right\} \\ &= c\Re\left\{\frac{\frac{\mu}{2} e^{i\theta}}{1+\frac{\mu}{2} e^{i\theta}}\right\} = c\,\frac{\mu}{2+\mu} \geq \frac{\mu}{2+\mu}.\end{aligned} \tag{2.5}$$

On the other hand define the function $\overline{w}(z)$, $z \in U$, such that

$$\frac{z[I^k_{p,\alpha,\beta} f(z)]'}{p I^k_{p,\alpha,\beta} f(z)} = 1 + \frac{\mu}{2}\,\overline{w}(z), \quad (z \in U,\ p \in \mathbb{N}), \tag{2.6}$$

we find that $\overline{w}(z)$ is analytic in U and $\overline{w}(0) = 0$. Differentiating both sides of (2.6), yields

$$\begin{aligned}&\frac{z}{p}\left\{\frac{z[I^k_{p,\alpha,\beta} f(z)]''}{I^k_{p,\alpha,\beta} f(z)} - z\left(\frac{[I^k_{p,\alpha,\beta} f(z)]'}{I^k_{p,\alpha,\beta} f(z)}\right)^2\right\} \\ &= \frac{\mu}{2} z\overline{w}'(z) - \left(1 + \frac{\mu}{2}\overline{w}(z)\right), \quad (z \in U,\ p \in \mathbb{N},\ p > k,\ k \in \mathbb{N}_0).\end{aligned} \tag{2.7}$$

Then, we have from (2.6) and (2.7)

$$\begin{aligned}\bar{F}(z) &:= 1 + \frac{z[I^k_{p,\alpha,\beta} f(z)]''}{[I^k_{p,\alpha,\beta} f(z)]'} - \frac{z[I^k_{p,\alpha,\beta} f(z)]'}{I^k_{p,\alpha,\beta} f(z)} \\ &= \frac{\frac{\mu}{2} z\overline{w}'(z)}{1+\frac{\mu}{2}\overline{w}(z)}, \quad (z \in U,\ p \in \mathbb{N},\ p > k,\ k \in \mathbb{N}_0).\end{aligned} \tag{2.8}$$

Now, suppose that there exists a point $\zeta \in U$ such that

$$\max_{|z| \leq |\zeta|} |\overline{w}(z)| = |\overline{w}(\zeta)| = 1.$$

Again in view of Lemma 1 and letting $\overline{w}(\zeta) = e^{i\vartheta}$ such that $\overline{w}(\zeta) \neq -1$ in the equation (2.8), we receive

$$\begin{aligned}\Re\{\bar{F}(\zeta)\} &= \Re\left\{\frac{\frac{\mu}{2}\zeta\overline{w}'(\zeta)}{1+\frac{\mu}{2}\overline{w}(\zeta)}\right\} = \Re\left\{\frac{\frac{\mu c}{2}\overline{w}(\zeta)}{1+\frac{\mu}{2}\overline{w}(\zeta)}\right\} \\ &= c\Re\left\{\frac{\frac{\mu}{2} e^{i\vartheta}}{1+\frac{\mu}{2} e^{i\vartheta}}\right\} = c\,\frac{\mu}{2+\mu} \geq \frac{\mu}{2+\mu}.\end{aligned} \tag{2.9}$$

Combining (2.5) and (2.9), we obtain

$$\Re\left\{1+\frac{z[I^k_{p,\alpha,\beta}f(z)]''}{[I^k_{p,\alpha,\beta}f(z)]'}-p\right\}\geq\frac{2\mu}{2+\mu},\quad(0\leq\mu<p,\ z\in U),$$

which contradicts the hypothesis (2.1). Therefore, we conclude that $|w(z)|<1$ and $|\overline{w}(z)|<1$ for all $z\in U$, and it yields the inequality

$$\begin{aligned}\left|\frac{I^k_{p,\alpha,\beta}f(z)}{z^p}+\frac{z[I^k_{p,\alpha,\beta}f(z)]'}{pI^k_{p,\alpha,\beta}f(z)}-2\right|&\leq\left|\frac{\mu}{2}w(z)\right|+\left|\frac{\mu}{2}\overline{w}(z)\right|\\&=\frac{\mu}{2}\big(|w(z)|+|\overline{w}(z)|\big)\\&<\mu,\quad(0\leq\mu<p,\ z\in U),\end{aligned}$$

that is, $f(z)\in S_p^{**}(\alpha,\beta,\mu)$. Thus, the proof is completed. □

Corollary 1. *Let $f(z)\in T(p)$ and satisfy the inequality*

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}-p\right\}<\frac{2\mu}{\mu+2},\quad(0\leq\mu<p,\ z\in U).$$

Then

$$\left|\frac{f(z)}{z^p}+\frac{zf'(z)}{pf(z)}-2\right|<\mu,\quad(0\leq\mu<p,\ z\in U).$$

Proof. It is sufficient to put $k=0$. □

Corollary 2. *Let $f(z)\in T(p)$ and satisfy the inequality*

$$\Re\left\{\frac{zf''(z)}{f'(z)}\right\}<\frac{2\mu}{\mu+2},\quad(0\leq\mu<1,\ z\in U).$$

Then

$$\left|\frac{f(z)}{z}+\frac{zf'(z)}{f(z)}-2\right|<\mu,\quad(0\leq\mu<1,\ z\in U).$$

Proof. It proceeds by setting $p=1$ and $k=0$. □

As an application of Theorem 1, we have the following results which determine the bound estimates for functions $f\in T(p)$ to be in the class $S_p^{**}(\alpha,\beta,\mu)$.

Theorem 2. *Let $f(z)\in T(p)$ and satisfy*

$$\begin{aligned}\sum_{n=p+1}^{\infty}n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k|a_n|\\<2p|2(\mu+1)-p(\mu+2)|,\quad(0\leq\mu<p,\ z\in U).\end{aligned}\tag{2.10}$$

Then $f\in S_p^{**}(\alpha,\beta,\mu)$.

Proof. Let the function $f(z) \in T(p)$ and satisfy the inequality (2.10). Thus we obtain

$$\left|1 + \frac{z[I^k_{p,\alpha,\beta} f(z)]''}{[I^k_{p,\alpha,\beta} f(z)]'} - p\right| \le \frac{2p(p-1) + \sum_{n=p+1}^{\infty} n(n-p)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |a_n|}{p - \sum_{n=p+1}^{\infty} n\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |a_n|}$$

the last inequality is less than $\frac{\mu}{\mu+2}$ if the assertion (2.10) holds. Now by using the fact that $\Re\{z\} \le |z|$ then in view of Theorem 1 yields $f \in S^{**}_p(\alpha, \beta, \mu)$. □

Theorem 3. *Let*

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$$

satisfy the inequality (2.10). *Then the weighted mean of f and g defined by*

$$W_i(z) = \frac{1}{2}\big[(1-i)f(z) + (1+i)g(z)\big], \quad (0 < i < 1)$$

is in $S^{**}_p(\alpha, \beta, \mu)$.

Proof. By the hypotheses of the theorem then in view of Theorem 2, we have

$$\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2) + 2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |a_n| < 2p|2(\mu+1) - p(\mu+2)|$$

and

$$\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2) + 2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |b_n| < 2p|2(\mu+1) - p(\mu+2)|.$$

After a simple calculation we obtain

$$W_i(z) = z^p + \sum_{n=p+1}^{\infty}\left[\frac{(1-i)}{2}a_n + \frac{(1+i)}{2}b_n\right] z^n.$$

However,

$$\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2) + 2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k \left|\frac{(1-i)}{2}a_n + \frac{(1+i)}{2}b_n\right|$$

$$\le \sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2) + 2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k \frac{1-i}{2}|a_n|$$

$$+\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k \frac{1+i}{2}|b_n|$$
$$=\frac{1-i}{2}\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |a_n|$$
$$+\frac{1+i}{2}\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |b_n|$$
$$\leq 2p|2(\mu+1)-p(\mu+2)|\frac{1-i}{2}+2p|2(\mu+1)-p(\mu+2)|\frac{1+i}{2}$$
$$=2p|2(\mu+1)-p(\mu+2)|.$$

Consequently, by Theorem 2, $W_i(z)\in S_p^{**}(\alpha,\beta,\mu)$. □

Theorem 4. *Let*

$$f_j(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n, \quad j=1,2,3,\dots,m$$

satisfy the inequality (2.10)*. Then the arithmetic mean of* $f_j(z)$ *defined by*

$$F(z)=\frac{1}{m}\sum_{j=1}^{m} f_j(z), \quad (z\in U)$$

is in $S_p^{**}(\alpha,\beta,\mu)$.

Proof. Since f_j satisfies (2.10) then, in view of Theorem 2, we have

$$\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |a_{n,j}|$$
$$<2p|2(\mu+1)-p(\mu+2)|.$$

A computation gives

$$F(z)=\frac{1}{m}\sum_{j=1}^{m} f_j(z)$$
$$=\frac{1}{m}\sum_{j=1}^{m}\left[z^p+\sum_{n=p+1}^{\infty} a_{n,j}z^n\right]$$
$$=z^p+\sum_{n=p+1}^{\infty}\left[\frac{1}{m}\sum_{j=1}^{m} a_{n,j}\right]z^n.$$

Hence

$$\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k \left|\frac{1}{m}\sum_{j=1}^{m} a_{n,j}\right|$$

$$\leq \sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k \frac{1}{m}\sum_{j=1}^{m} |a_{n,j}|$$

$$= \frac{1}{m}\sum_{j=1}^{m}\left[\sum_{n=p+1}^{\infty} n = \big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |a_{n,j}|\right]$$

$$< \frac{1}{m}\sum_{j=1}^{m} 2p|2(\mu+1)-p(\mu+2)|$$

$$= 2p|2(\mu+1)-p(\mu+2)|.$$

This implies that $F(z) \in S_p^{**}(\alpha,\beta,\mu)$. □

Theorem 5. *Let*

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$$

be in $T(p)$ such that $f(U)$ is convex. Assume that f satisfies the inequality (2.10). *Then, for the Cesàro operator [6, 10–12] of f defined by the relation*

$$\sigma_n(z) = \sum_{n=p+1}^{\infty} \frac{1}{n+1}\left(\sum_{i=0}^{n} a_{n,i}\right) z^n, \quad (p = 1, 2, \ldots,\ z \in U)$$

with $\sigma_0(z) = 0$, $\sigma_1(z) = z^p$, we have $\sigma_n(z) \in S_p^{**}(\alpha,\beta,\mu)$.

Proof. Since f_j satisfies (2.10) then in view of Theorem 2, we have

$$\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |a_{n,j}| < 2p|2(\mu+1)-p(\mu+2)|.$$

For all $n \in \mathbb{N}_0$ we have

$$\sigma_n(z) = 0 + z^p + \sum_{n=p+1}^{\infty} \frac{1}{n+1}\left(\sum_{i=0}^{n} a_{n,i}\right) z^n, \quad (z \in U).$$

Hence

$$\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k \left|\frac{1}{n+1}\left(\sum_{i=0}^{n} a_{n,i}\right)\right|$$

$$\leq \sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k \frac{1}{n+1}\left(\sum_{i=0}^{n}|a_{n,i}|\right)$$
$$= \frac{1}{n+1}\sum_{i=0}^{n}\left(\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |a_{n,j}|\right)$$
$$< \frac{1}{n+1}\sum_{i=0}^{n} 2p|2(\mu+1)-p(\mu+2)|$$
$$= 2p|2(\mu+1)-p(\mu+2)|.$$

This implies that $F(z) \in S_p^{**}(\alpha,\beta,\mu)$. □

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