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ASYMPTOTIC REPRESENTATIONS FOR SOLUTIONS OF A CLASS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. The asymptotic representations for solutions of a class of nonlinear non-autonomous differential equations of the second order are established.

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1. STATEMENT OF THE PROBLEM AND MAIN THEOREMS

Consider the differential equation

$$y'' = \alpha_0 p(t) y |\ln|y||^{\sigma}, \tag{1.1}$$

where $\alpha_0 \in \{-1, 1\}$, $\sigma \in \mathbb{R}$, $p\colon [a,\omega[\to]0,+\infty[$ is a continuous function, $-\infty < a < \omega \le +\infty$.

This equations is a special case of the important class of equations

$$y'' = \alpha_0 p(t) y L(y),$$

where the function L is continuous in one-sided neighborhood Δ_{Y_0} of Y_0 (Y_0 is either zero, or $\pm\infty$). L is positive and slowly varying [5] as $y \to Y_0$ also.

A solution y of Eq. (1.1) defined on the interval $[t_y, \omega[\subseteq [a,\omega[$ is called a $P_\omega(\lambda_0)$-solution if it satisfies the following conditions

$$\lim_{t \uparrow \omega} y^{(k)}(t) \in \{0, \pm\infty\} \text{ for } k = 0, 1 \quad \text{and} \quad \lim_{t \uparrow \omega} \frac{(y'(t))^2}{y''(t) y(t)} = \lambda_0. \tag{1.2}$$

In [1, 3] it was shown that, the set of all $P_\omega(\lambda_0)$-solutions of Eq. (1.1) and its asymptotic properties are divided into four classes of solutions associated with the values $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$, $\lambda_0 = \pm\infty$, $\lambda_0 = 0$ and $\lambda_0 = 1$. The necessary and sufficient conditions for the existence of $P_\omega(\lambda_0)$-solutions for each class were given. Also the asymptotic representations as $t \uparrow \omega$ for $\ln|y(t)|$ are established. Moreover in [1] for $P_\omega(\lambda_0)$-solutions, associated with the values of $\lambda_0 = 0$ and $\lambda_0 = \pm\infty$, the conditions for which the exact asymptotic formulas for the solution of Eq. (1.1) was established. In this paper we obtain similar results for $P_\omega(1)$-solutions of Eq. (1.1).

We introduce the following auxiliary notation:

$$\pi_\omega(t) = \begin{cases} t & \text{if } \omega = +\infty, \\ t-\omega & \text{if } \omega < +\infty, \end{cases} \qquad I_B(t) = \int_B^t p^{\frac{1}{2}}(\tau)\,d\tau,$$

where

$$B = \begin{cases} a & \text{if } \int_a^\omega p^{\frac{1}{2}}(\tau)\,d\tau = +\infty, \\ \omega & \text{if } \int_a^\omega p^{\frac{1}{2}}(\tau)\,d\tau < +\infty. \end{cases}$$

The next theorem for the equation (1.1) was established in [1].

Theorem 1.1. *Let $\sigma \neq 2$. Then for the existence of $P_\omega(1)$-solutions of Eq. (1.1) it is necessary that*

$$\alpha_0 > 0, \qquad \lim_{t\uparrow\omega} \pi_\omega(t)\,p^{\frac{1}{2}}(t)\,|I_B(t)|^{\frac{\sigma}{2-\sigma}} = \infty. \tag{1.3}$$

If besides that function $p\colon [a,\omega[\to]0,+\infty[$ is continuously differentiable and there is the finite of equal $\pm\infty$ limit

$$\lim_{t\uparrow\omega} \frac{\left(p^{\frac{1}{2}}(t)\,|I_B(t)|^{\frac{\sigma}{2-\sigma}}\right)'}{p(t)\,|I_B(t)|^{\frac{2\sigma}{2-\sigma}}}, \tag{1.4}$$

it is considered to be sufficient. Moreover, each of these solutions admits the following asymptotic representations as $t \uparrow \omega$

$$\ln|y(t)| \sim \pm\left(\frac{2-\sigma}{2}\,I_B(t)\right)^{\frac{2}{2-\sigma}}, \qquad \frac{y'(t)}{y(t)} \sim \pm p^{\frac{1}{2}}(t)\left(\frac{2-\sigma}{2}\,I_B(t)\right)^{\frac{\sigma}{2-\sigma}}. \tag{1.5}$$

Remark 1.1. Note that by virtue of (1.1)

$$\lim_{t\uparrow\omega} |I_B(t)|^{\frac{2}{2-\sigma}} = +\infty. \tag{1.6}$$

Moreover, it was shown in [3], that if the second condition of (1.1) is satisfied, then

$$\lim_{t\uparrow\omega} \frac{\left(p^{\frac{1}{2}}(t)\,|I_B(t)|^{\frac{\sigma}{2-\sigma}}\right)'}{p(t)\,|I_B(t)|^{\frac{2\sigma}{2-\sigma}}} = 0 \tag{1.7}$$

if it exists.

From Theorem 1.1 when $\sigma = 0$, that is, for the linear differential equation

$$y'' = \alpha_0\,p(t)\,y, \tag{1.8}$$

where $\alpha_0 \in \{-1,1\}$, $p\colon [a,\omega[\to]0,+\infty[$ is a continuous function, and $-\infty < a < \omega \leq +\infty$, we obtain

Corollary 1.1. *For the existence of $P_\omega(1)$-solutions of Eq. (1.8), it is necessary that*

$$\alpha_0 > 0, \qquad \lim_{t\uparrow\omega} \pi_\omega^2(t)p(t) = +\infty. \tag{1.9}$$

If besides that function $p\colon [a,\omega[\to]0,+\infty[$ is continuously differentiable and there is the finite of equal $\pm\infty$ limit $\lim_{t\uparrow\omega} p'(t)p^{-\frac{3}{2}}(t)$ it is sufficient. Moreover, each of these solutions admits the following asymptotic representations

$$\ln|y(t)| \sim \pm I_B(t), \qquad \frac{y'(t)}{y(t)} \sim \pm p^{\frac{1}{2}}(t) \quad \text{as} \quad t\uparrow\omega.$$

Remark 1.2. In the case where the function $p\colon [a,\omega[\to]0,+\infty[$ is continuously differentiable and the limit $\lim_{t\uparrow\omega} p'(t)p^{-\frac{3}{2}}(t)$ is finite or equal to $\pm\infty$, it is easy to prove that every non-oscillatory solution y of the linear differential equation (1.8), different from solutions that admit one of the asymptotic representations $y(t)\sim c$ or $y(t)\sim c\pi_\omega(t)$ as $t\uparrow\omega$ ($c\neq 0$), is certainly a $P_\omega(\lambda_0)$-solution, where $-\infty\le\lambda_0\le +\infty$.

The aim of the present paper is to establish conditions for the existence of $P_\omega(1)$-solutions of Eq. (1.1) that admits as $t\uparrow\omega$ the asymptotic representations

$$\begin{gathered} y(t) = \big(C+o(1)\big)\exp\left(\pm\left(\frac{2-\sigma}{2}I_B(t)\right)^{\frac{2}{2-\sigma}}\right), \\ \frac{y'(t)}{y(t)} = \pm p^{\frac{1}{2}}(t)\left(\frac{2-\sigma}{2}I_B(t)\right)^{\frac{\sigma}{2-\sigma}}\big(1+o(1)\big) \end{gathered} \tag{1.10}$$

in the case $\sigma(2-\sigma)\neq 0$, and

$$y(t)\sim C\exp\big(\pm I_B(t)\big), \qquad y'(t)\sim \pm p^{\frac{1}{2}}(t)y(t)$$

in the case $\sigma=0$, where C is a nonzero real constant.

The following theorems are true for equation (1.1).

Theorem 1.2. *Let $\sigma(2-\sigma)\neq 0$, the function $p\colon [a,\omega[\to]0,+\infty[$ be continuously differentiable, the conditions (1.4) are satisfied, and*

$$\lim_{t\uparrow\omega} \frac{\left(p^{\frac{1}{2}}(t)\,|I_B(t)|^{\frac{\sigma}{2-\sigma}}\right)'}{p(t)\,|I_B(t)|^{\frac{2(\sigma-1)}{2-\sigma}}} = 0. \tag{1.11}$$

Then, for $C=\pm 1$, the Eq. (1.1) has a $P_\omega(1)$-solution that admits the asymptotic representations (1.10) as $t\uparrow\omega$.

Theorem 1.3. *Let the function $p\colon [a,\omega[\to]0,+\infty[$ be continuously differentiable, the conditions (1.1) are satisfied, and*

$$\int_a^\omega \left|\frac{p'(t)}{p(t)}\right| dt < +\infty. \tag{1.12}$$

Then Eq. (1.8) *has a fundamental set* $P_\omega(1)$*-solutions, that admits the following asymptotic representations as* $t \uparrow \omega$*:*

$$y_i(t) \sim \exp\Big((-1)^{i-1} I_B(t)\Big), \quad y_i'(t) \sim (-1)^{i-1} p^{\frac{1}{2}}(t) y_i(t) \quad (i = 1,2). \quad (1.13)$$

Remark 1.3. Theorem 1.3 and Corollary 1.1 complement the known results (see [4]) on the asymptotic properties of solutions of linear differential equations.

2. Proof of main theorems

To prove the Theorems 1.2 and 1.3 presented above, we need the auxiliary statements on the existence of solutions vanishing at infinity for the system of differential equations

$$\frac{dz_i}{d\tau} = f_i(\tau) + \sum_{k=1}^{2} p_{ik}(\tau) z_k + g_i(\tau) Z_i(\tau, z_1, z_2) \quad (i = 1,2), \quad (2.1)$$

where the functions $f_i, g_i : [\tau_0, +\infty[\to \mathbb{R}$ $(i = 1,2)$, $p_{ik} : [\tau_0, +\infty[\to \mathbb{R}$ $(i,k = 1,2)$, $Z_i : [\tau_0, +\infty[\times \mathbb{R}_b^2 \to \mathbb{R}$ $(i = 1,2)$ are continuous, and $\mathbb{R}_b^2 = \{(z_1, z_2) \in \mathbb{R}^2 : |z_i| \le b \ (i = 1,2)\}$ for $b > 0$. For this system of equations, using Theorem 1.3 and Remark 1.4 from [2], the following Lemmas take place:

Lemma 2.1. *Suppose that*

$$p_{ii}(\tau) \neq 0 \quad \textit{for} \quad \tau \ge \tau_0, \qquad \int_{\tau_0}^{+\infty} |p_{ii}(\tau)| d\tau = +\infty \quad (i = 1,2), \quad (2.2)$$

and also assume that the following conditions are satisfied:

$$\lim_{\tau \to +\infty} \frac{f_i(\tau)}{p_{ii}(\tau)} = 0, \qquad \lim_{\tau \to +\infty} \frac{g_i(\tau)}{p_{ii}(\tau)} = const. \quad (i = 1,2), \quad (2.3)$$

$$\lim_{\tau \to +\infty} \frac{p_{12}(\tau)}{p_{11}(\tau)} = 0, \qquad \lim_{\tau \to +\infty} \frac{p_{21}(\tau)}{p_{22}(\tau)} = const., \quad (2.4)$$

and

$$\lim_{|z_1|+|z_2| \to 0} \frac{\partial Z_i(\tau, z_1, z_2)}{\partial z_k} = 0 \quad \textit{uniformly in } [\tau_0, +\infty[\quad (i,k = 1,2). \quad (2.5)$$

Then the system (2.1) *has at least one solution* $(z_1, z_2) : [\tau_1, +\infty[\to \mathbb{R}_b^2$ $(\tau_1 \ge \tau_0)$*, that tends to zero as* $\tau \to +\infty$*. Moreover, there exists a one-parametric family of these solutions if* $p_{11}(\tau) p_{22}(\tau) < 0$ *for* $\tau \ge \tau_0$*, and a two-parametric family of these solutions if* $p_{ii}(\tau) < 0$ *for* $\tau \ge \tau_0$*,* $i = 1,2$*.*

Lemma 2.2. *Suppose that the conditions* (2.5) *are satisfied, and*

$$\int_{\tau_0}^{+\infty} |f_i(\tau)| d\tau < +\infty \quad (i = 1,2), \qquad \int_{\tau_0}^{+\infty} |g_2(\tau)| d\tau < +\infty,$$

$$\int_{\tau_0}^{+\infty} |p_{2k}(\tau)|d\tau < +\infty \quad (k = 1,2),$$

the functions p_{1k} $(k = 1,2)$ *and* g_1 *has the form*

$$p_{1k}(\tau) = p_{1k}^0 + \delta_{1k}(\tau) \quad (k = 1,2), \qquad g_1(\tau) = g_1^0 + \gamma_1(\tau),$$

where p_{11}^0, p_{12}^0, *and* g_1^0 *are constants,* $p_{11}^0 \neq 0$, *and*

$$\int_{\tau_0}^{+\infty} |\delta_{1k}(\tau)|d\tau < +\infty \quad (k = 1,2), \qquad \int_{\tau_0}^{+\infty} |\gamma_1(\tau)|d\tau < +\infty.$$

Then the system (2.1) *has at least one solution* $(z_1, z_2)\colon [\tau_1, +\infty[\to \mathbb{R}_b^2$ $(\tau_1 \geq \tau_0)$, *that tends to zero as* $\tau \to +\infty$. *Moreover, there exists a one-parametric family of these solutions if* $p_{11}^0 < 0$.

Proof of Theorem 1.2. We choose $C \in \{-1, 1\}$ and, using the transformation

$$\tau = \left(\frac{2-\sigma}{2} I_B(t)\right)^{\frac{2}{2-\sigma}},$$

$$y(t) = C\exp\left(\pm\left(\frac{2-\sigma}{2} I_B(t)\right)^{\frac{2}{2-\sigma}}\right)\big(1 + v_1(\tau)\big),$$

$$y'(t) = \pm C p^{\frac{1}{2}}(t)\left(\frac{2-\sigma}{2} I_B(t)\right)^{\frac{\sigma}{2-\sigma}} \exp\left(\pm\left(\frac{2-\sigma}{2} I_B(t)\right)^{\frac{2}{2-\sigma}}\right)\big(1 + v_2(\tau)\big), \tag{2.6}$$

we reduce Eq. (1.1) to the system of differential equations

$$\begin{aligned} v_1' &= \pm(v_2 - v_1), \\ v_2' &= \pm\big[(1 + v_1)R(\tau, v_1) - (1 + \delta_1(\tau))(1 + v_2)\big], \end{aligned} \tag{2.7}$$

where

$$R(\tau, v_1) = \big|1 \pm \delta_2(\tau)\ln|1 + v_1|\big|^{\sigma}, \qquad \delta_2(\tau) = \left(\frac{2-\sigma}{2} I_B(t)\right)^{-\frac{2}{2-\sigma}},$$

and

$$\delta_1(\tau) = \pm\frac{\left(p^{\frac{1}{2}}(t)\,|I_B(t)|^{\frac{\sigma}{2-\sigma}}\right)'}{\left|\frac{2-\sigma}{2}\right|^{\frac{\sigma}{2-\sigma}} p(t)\,|I_B(t)|^{\frac{2\sigma}{2-\sigma}}}.$$

Here

$$\tau(t) = \left(\frac{2-\sigma}{2} I_B(t)\right)^{\frac{2}{2-\sigma}} \to +\infty \quad \text{as} \quad t \uparrow \omega$$

and

$$\tau'(t) = p^{\frac{1}{2}}(t)\left(\frac{2-\sigma}{2} I_B(t)\right)^{\frac{\sigma}{2-\sigma}} > 0 \quad \text{for} \quad t \in]a, \omega[.$$

By the conditions (1.6) and (1.2), we get

$$\lim_{\tau\to+\infty}\delta_1(\tau)=0,\qquad \lim_{\tau\to+\infty}\delta_2(\tau)=0,\qquad \lim_{\tau\to+\infty}\frac{\delta_1(\tau)}{\delta_2(\tau)}=0. \tag{2.8}$$

Also we have

$$\delta_2'(\tau)=-\delta_1^2(\tau),$$

It follows from this, that

$$\delta_2(\tau)=\frac{1+o(1)}{\tau}\quad \text{as}\quad \tau\to+\infty. \tag{2.9}$$

Now we choose the number $\tau_0>1$ such that the inequality

$$|\delta_2(\tau)|\le\frac{1}{2}\quad \text{for}\quad \tau\ge\tau_0$$

holds and then we consider the system (2.7) on the set

$$\Omega=[\tau_0,+\infty[\times\mathbb{R}^2_{1/2},\quad \text{where}\quad \mathbb{R}^2_{1/2}=\Big\{(v_1,v_2)\in\mathbb{R}^2:|v_i|\le\frac{1}{2},\ i=1,2\Big\}.$$

On this set, the right-hand sides of system (2.7) are continuous. So, by separation of the linear part in the second equation, we rewrite the system to the form

$$v'=\pm\big[q(\tau)+(A+B(\tau))v+V(\tau,v)\big], \tag{2.10}$$

where

$$q(t)=\begin{pmatrix}0\\-\delta_1(\tau)\end{pmatrix},\qquad A=\begin{pmatrix}-1&1\\1&-1\end{pmatrix},$$

$$B(\tau)=\begin{pmatrix}0&0\\\pm\sigma\delta_2(\tau)&-\delta_1(\tau)\end{pmatrix},\qquad V(\tau,v)=\begin{pmatrix}0\\V_1(\tau,v_1)\end{pmatrix},$$

and

$$V_1(\tau,v_1)=(1+v_1)\big[(1\pm\delta_2(\tau)\ln(1+v_1))^{\sigma}\mp\sigma\delta_2(\tau)v_1\big]\pm\sigma\delta_2(\tau)v_1^2.$$

For the function V_1, that belongs to the nonlinear component of this system, we have

$$\frac{\partial V_1(\tau,v_1)}{\partial v_1}=\big[1\pm\delta_2(\tau)\ln(1+v_1)\big]^{\sigma-1}\big[1\pm\delta_2(\tau)\ln(1+v_1)\pm\sigma\delta_2(\tau)\big]-1\mp\sigma\delta_2(\tau).$$

From this it is clear that

$$\frac{1}{\delta_2(\tau)}\frac{\partial V_1(\tau,v_1)}{\partial v_1}\to0\quad \text{as}\quad v_1\to0\quad \text{uniformly in } [\tau_0,+\infty[. \tag{2.11}$$

Using an additional transformation to the system (2.10), namely

$$v=\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix}, \tag{2.12}$$

we get a system of differential equations (2.1), in which

$$f_1(\tau)=\frac{\nu}{2}\delta_1(\tau),\quad f_2(\tau)=-\frac{\nu}{2}\delta_1(\tau),\quad g_1(\tau)=-\frac{\nu}{2}\delta_2(\tau),\quad g_2(\tau)=\frac{\nu}{2}\delta_2(\tau),$$

$$p_{11}(\tau)=\nu\left[-2-\frac{1}{2}\delta_1(\tau)\mp\frac{\sigma}{2}\delta_2(\tau)\right],\qquad p_{12}(\tau)=\frac{\nu}{2}\big[\delta_1(\tau)\mp\sigma\delta_2(\tau)\big],$$

$$p_{21}(\tau)=\frac{\nu}{2}\big[\delta_1(\tau)\pm\sigma\delta_2(\tau)\big],\qquad p_{22}(\tau)=-\frac{\nu}{2}\big[\delta_1(\tau)\mp\sigma\delta_2(\tau)\big],$$

and

$$Z_i(\tau,z_1,z_2)=\frac{V_1(\tau,z_1+z_2)}{\delta_2(\tau)}\quad(i=1,2),\quad \nu=\pm1.$$

By virtue of conditions (2.8) and (2.9), we have

$$p_{11}(\tau)=\nu\big(-2+o(1)\big),\quad p_{22}(\tau)=\pm\frac{\nu\sigma}{2\tau}\big(1+o(1)\big)\quad\text{as}\quad\tau\to+\infty$$

Therefore the assumptions (2.2) of Lemma 2.1 hold for some sufficiently large value of τ_0. Without loss of generality we can assume that τ_0 equals to the chosen one before. Then, taking into account (2.8), we have

$$\lim_{\tau\to+\infty}\frac{f_i(\tau)}{p_{ii}(\tau)}=0\quad(i=1,2),\qquad\lim_{\tau\to+\infty}\frac{p_{12}(\tau)}{p_{11}(\tau)}=0,\qquad\lim_{\tau\to+\infty}\frac{p_{21}(\tau)}{p_{22}(\tau)}=1,$$

and

$$\lim_{\tau\to+\infty}\frac{g_1(\tau)}{p_{11}(\tau)}=0,\qquad\lim_{\tau\to+\infty}\frac{g_2(\tau)}{p_{22}(\tau)}=\pm\frac{1}{\sigma}.$$

Therefore, the assumptions (2.3) and (2.4) of Lemma 2.1 are satisfied. Moreover, by virtue of (2.11), conditions (2.5) hold. Hence, according to Lemma 2.1, the obtained system of differential equations (2.1) has at least one solution $(z_1,z_2)\colon[\tau_1,+\infty[\to\mathbb{R}^2_b$ $(\tau_1\ge\tau_0)$, that tends to zero as $\tau\to+\infty$. By virtue of the change of variables (2.12) and (2.6), this solution is associated with a solution y of the differential equation (1.1) that satisfies the asymptotic relations (1.10) as $t\uparrow\omega$. □

Remark 2.1. Using results from [4], we can show that, if the assumptions of Theorem 1.2 are satisfied, then Eq. (1.1), for $\sigma>0$, has a one-parametric family of $P_\omega(1)$-solutions that admit the asymptotic representations

$$y(t)\sim\pm\exp\left[\left(\frac{2-\sigma}{2}I_B(t)\right)^{\frac{2}{2-\sigma}}\right],\quad\frac{y'(t)}{y(t)}\sim p^{\frac{1}{2}}(t)\left(\frac{2-\sigma}{2}I_B(t)\right)^{\frac{\sigma}{2-\sigma}}\tag{2.13}$$

as $t\uparrow\omega$, and, for $\sigma<0$, has a two-parametric family of $P_\omega(1)$-solutions admitting the asymptotic representations (2.13), and a one-parametric family of $P_\omega(1)$-solutions that admit the following asymptotic representations as $t\uparrow\omega$:

$$y(t)\sim\pm\exp\left[-\left(\frac{2-\sigma}{2}I_B(t)\right)^{\frac{2}{2-\sigma}}\right],\quad\frac{y'(t)}{y(t)}\sim-p^{\frac{1}{2}}(t)\left(\frac{2-\sigma}{2}I_B(t)\right)^{\frac{\sigma}{2-\sigma}}.$$

Proof of Theorem 1.3. Let $i \in \{1,2\}$ be arbitrary. If we use the transformation

$$\begin{aligned} \tau = I_B(t), \qquad y(t) = \exp\Big[(-1)^{i-1} I_B(t)\Big]\big(1 + z_1(\tau) + z_2(\tau)\big), \\ y'(t) = (-1)^{i-1} p^{\frac{1}{2}}(t) \exp\Big[(-1)^{i-1} I_B(t)\Big]\big(1 - z_1(\tau) + z_2(\tau)\big) \end{aligned} \tag{2.14}$$

in the equation (1.8), we get the system of linear differential equations

$$z_i' = f_i(\tau) + \sum_{k=1}^{2} p_{ik}(\tau) z_k \quad (i = 1,2), \tag{2.15}$$

where

$$f_1(\tau) = \frac{(-1)^{i-1}}{2}\delta(\tau), \qquad f_2(\tau) = \frac{(-1)^{i}}{2}\delta(\tau),$$
$$p_{11}(\tau) = (-1)^i \left(2 + \frac{1}{2}\delta(\tau)\right), \qquad p_{12}(\tau) = \frac{(-1)^{i-1}}{2}\delta(\tau),$$

and

$$p_{21}(\tau) = \frac{(-1)^{i-1}}{2}\delta(\tau), \quad p_{22}(\tau) = \frac{(-1)^{i-1}}{2}\delta(\tau), \quad \delta(\tau) = \frac{1}{2} p'(t) p^{-\frac{3}{2}}(t).$$

By virtue of (1.12) and substitution $\tau = I_B(t)$, we have

$$\int_{\tau_0}^{+\infty} |\delta(\tau)| d\tau = \int_a^{\omega} \left|\frac{p'(t)}{p(t)}\right| dt < +\infty,$$

where $\tau_0 = I_B(a)$. For the system (2.15), all assumptions of Lemma 2.2 are satisfied. Hence, according to this lemma, the system of differential equations (2.15) has at least one solution $(z_1, z_2)\colon [\tau_1, +\infty[\to \mathbb{R}_b^2$ $(\tau_1 \geq \tau_0)$, that tends to zero as $\tau \to +\infty$, which, by virtue of (2.14), is associated with solutions y_i of the differential equation (1.8) that admits the asymptotic representations (1.13) as $t \uparrow \omega$. Taking these representations into account, it is not difficult to see that the solution indicated is a $P_\omega(1)$-solution.

Since $i \in \{1,2\}$ was chosen arbitrarily, we have proved the existence of two $P_\omega(1)$-solutions y_1 and y_2 admitting the asymptotic representations (1.13) as $t \uparrow \omega$. It is obvious that these two solutions form a fundamental set of solutions of Eq. (1.8). □

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