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HOW TO CHARACTERIZE SOME PROPERTIES OF MEASURABLE FUNCTIONS

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Dedicated to the memory of Prof. J. Mogyoródi

Abstract. Making use of the so-called optimal measures dealt with in [1-2], we characterize the boundedness of measurable functions, the uniform boundedness and some well-known asymptotic behaviours of sequences of measurable functions (such as discrete, equal as well as pointwise types of convergence). The so-called quasi-uniform convergence is also characterized in the fourth section.

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1. Introduction

Making use of the so-called 'optimal measures' (cf. [1-2]), our main goal here is to characterize some well-known notions in Analysis such as the boundedness of measurable functions, the uniform boundedness as well as some commonly used asymptotic behaviors of sequences of measurable functions. We should like to mention that our results in [1-2] and in this article might interest everyone who handles measurable functions. Before we tackle our paper, let us first recall the following results (as we need them later on).

All along $(\Omega, \mathcal{F})$ will stand for any measurable space (where the elements of $\mathcal{F}$ are referred to as measurable sets).

By an optimal measure we mean a set function $p : \mathcal{F} \to [0, 1]$ fulfilling the following axioms:

P1. $p(\emptyset) = 0$ and $p(\Omega) = 1$.

P2. $p(B \bigcup E) = p(B) \vee p(E)$ for all measurable sets B and E.

P3. $p\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} p(E_n)$, for every decreasing sequence of measurable sets (E_n).

(The symbols $\bigvee$ and $\bigwedge$ will stand for the maximum and minimum respectively.)

First we shall summarize the background of the Theory of Optimal Measures .

Let $s = \sum_{i=1}^{n} b_i \chi(B_i)$ be an arbitrary nonnegative measurable simple function, where $\{B_i : i = 1, \ldots, n\} \subset \mathcal{F}$ is a partition of Ω. Then the so-called optimal average of s is defined by $\wr_\Omega s dp = \bigvee_{i=1}^{n} b_i p(B_i)$, where $\chi(B)$ is the indicator function of the measurable set B. We note that this quantity does not depend on the decompositions of s (cf. [1], *Theorem 1.0.*, page 135).

The optimal average of a measurable function f is defined by $\wr_\Omega |f| dp = \sup \wr_\Omega s dp$, where the supremum is taken over all measurable simple functions $s \geq 0$ for which $s \leq |f|$. (From now on *m.f.*'s will stand for measurable functions.)

Let f be any m.f. We shall say that f belongs to:

1. $\mathcal{A}_\infty(p)$ if $p(|f| \leq b) = 1$ for some constant $b \in (0, \infty)$.
2. $\mathcal{A}_\alpha(p)$ if $\wr_\Omega |f|^\alpha dp < \infty$, $\alpha \in [1, \infty)$.

For any $\alpha \in [1, \infty]$, the space $\mathcal{A}_\alpha(p)$ endowed with the norm $\|\cdot\|_\alpha$, defined by

$$\|f\|_\alpha = \begin{cases} \inf\{b \in (0, \infty) : p(|f| \leq b) = 1\}, & \text{if } f \in \mathcal{A}_\infty(p),\ \alpha = \infty \\ \sqrt[\alpha]{\wr_\Omega |f|^\alpha dp}, & \text{if } f \in \mathcal{A}_\alpha(p),\ \alpha \in [1, \infty) \end{cases}$$

is a Banach space. (For more about this refer to [1].)

In [2] we have obtained the following results for all optimal measures p.

By (p-)atom we mean a measurable set H, $p(H) > 0$ such that whenever $B \in \mathcal{F}$, $B \subset H$, then $p(B) = p(H)$ or $p(B) = 0$.

A p-atom H is decomposable if there exists a subatom $B \subset H$ such that $p(B) = p(H) = p(H \backslash B)$. If no such subatom exists, we shall say that H is indecomposable.

Fundamental Optimal Measure Theorem. *Let $(\Omega, \mathcal{F})$ be a measurable space and p an optimal measure on it. Then there exists a collection $\mathcal{H}(p) = \{H_n : n \in J\}$ of disjoint indecomposable p-atoms, where J is some countable (i.e. finite or countably infinite) index-set such that for any measurable set B, with $p(B) > 0$, we have that*

$$p(B) = \max\left\{p\left(B \bigcap H_n\right) : n \in J\right\}.$$

Moreover the only limit point of the set $\{p(H_n) : n \in J\}$ is 0 provided that J is a countably infinite set. ($\mathcal{H}(p)$ is referred to as p-generating countable system.)

In proving the *Fundamental Optimal Measure Theorem* we used *Zorn's lemma* which, as we know, is equivalent to the *Axiom of Choice.* It is worth noting that in [6] the above structure theorem has been proven without *Zorn's lemma.*

By a quasi-optimal measure we mean a set function $q : \mathcal{F} \to [0, \infty)$ satisfying the axioms *P1.-P3.* with the hypothesis $q(\Omega) = 1$ in *P1.* replaced by $0 < q(\Omega) < \infty$.

We say that a quasi-optimal measure q is absolutely continuous relative to an optimal measure p (abbreviated $q << p$) if $q(B) = 0$ whenever $p(B) = 0$, $B \in \mathcal{F}$.

Remark A. Every m.f. is constant almost surely on each indecomposable atom (cf. [2], page 84, *Remark 2.1.*).

Theorem B. (cf. [1] page 139, *Theorem 3.1.*)

1. *If* (f_n) *is an increasing sequence of nonnegative m.f.'s, then* $\lim\limits_{n\to\infty} \wr_\Omega f_n dp = \underset{\Omega}{\wr} \left(\lim\limits_{n\to\infty} f_n\right) dp.$

2. *If* (g_n) *is a decreasing sequence of nonnegative m.f.'s with* $g_1 \leq b$ *for some* $b \in (0, \infty)$, *then* $\lim\limits_{n\to\infty} \wr_\Omega g_n dp = \wr_\Omega \left(\lim\limits_{n\to\infty} g_n\right) dp.$

Lemma C. *Let q be a quasi-optimal measure, absolutely continuous relative to an optimal measure p. Then $\mathcal{H}_*(p) = \{H \in \mathcal{H}(p) : q(H) > 0\}$ is a q-generating countable system (where $\mathcal{H}(p)$ denotes a p-generating countable system).*

Lemma D. (cf. [1] page 141, *Lemma 3.2.*) *If* (f_n) *and* (h_n) *are sequences of nonnegative m.f.'s, then for every optimal measure* p, *we have that*

1. $\wr_\Omega \left(\liminf\limits_{n\to\infty} f_n\right) dp \leq \liminf\limits_{n\to\infty} \wr_\Omega f_n dp;$

2. $\limsup\limits_{n\to\infty} \wr_\Omega h_n dp \leq \wr_\Omega \left(\limsup\limits_{n\to\infty} h_n\right) dp$, *provided that* (h_n) *is a uniformly bounded sequence.*

NOTATIONS.

1. $\mathcal{P}$ will denote the set of all optimal measures defined on $(\Omega, \mathcal{F})$.

2. $\mathcal{P}_{<\infty}$ is the collection of all optimal measures whose generating systems are finite.

3. $\mathcal{P}_\infty$ is the set of all optimal measures whose generating systems are countably infinite.

4. For every fixed $\omega \in \Omega$, the optimal measure p_ω (defined on $(\Omega, \mathcal{F})$ by $p_\omega(B) = 1$ if $\omega \in B$, and 0 if $\omega \notin B$) will be referred to as ω-concentrated optimal measure.

5. $|E|$ stands for the cardinality of the measurable set E.

6. $\mathbb{N}$ will stand for the set of positive integers.

2. Some preliminary results

We say that a nonempty measurable set E is closely related to some sequence $(\omega_n) \subset \Omega$ if

$$\left| E \bigcap \{\omega_n : n \in \mathbb{N}\} \right| = \begin{cases} \infty, & \text{if } |E| = \infty \\ |E|, & \text{if } |E| < \infty \end{cases}$$

(that is, if E is infinite, then infinitely many members of the sequence belong to E, otherwise all of its elements are members of the sequence).

Definition 2.1. Let E be closely related to a sequence $(\omega_n) \subset \Omega$, and let $(\alpha_n) \subset [0, 1]$ be any fixed sequence tending decreasingly to 0. The optimal measure $p_E : \mathcal{F} \to [0, 1]$, defined by $p_E(B) = \max\{\alpha_n : \omega_n \in B\}$, will be called 1st-type E-dependent optimal measure.

Proposition 2.1. *Let $p \in \mathcal{P}$ and f be any m.f. Then*

$$\wr_\Omega |f|\, dp = \sup\left\{\wr_{H_n} |f|\, dp : n \in J\right\},$$

where $\mathcal{H}(p) = \{H_n : n \in J\}$ is a p-generating countable system.

Moreover if $f \in \mathcal{A}_1(p)$, then $\wr_\Omega |f|\, dp = \sup\{c_n \cdot p(H_n) : n \in J\}$, where $c_n = f(\omega)$ for almost all $\omega \in H_n$, $n \in J$.

(The proof is straightforward.) The following remark is worth being noted.

Remark 2.1. Let $p, q \in \mathcal{P}$, $\mathcal{H}(p) = \{H_n : n \in J\}$ be a p-generating countable system and f any m.f. Suppose that $q << p$ and $q(H) \le p(H)$ for every $H \in \mathcal{H}(p)$. Then $\wr_\Omega |f|\, dq \le \wr_\Omega |f|\, dp$, provided that $\wr_\Omega |f|\, dp < \infty$.

(This is immediate from *Lemma C* and *Proposition 2.1.*)

Remark 2.2. If (x_n) is a sequence of real numbers such that $\limsup_{n\to\infty} |x_n| < \infty$, then for each of its subsequences (x_{n_k}) we have that $\limsup_{k\to\infty} |x_{n_k}| < \infty$.

NOTICE. For every fixed m.f. f, the mapping $z_f : \mathcal{P} \to [0, \infty]$, defined by $z_f(p) = \wr_\Omega |f|\, dp$, is a function.

Lemma 2.2. *Let $\omega \in \Omega$ be fixed. Then for every m.f. f, we have that $z_f(p_\omega) = |f(\omega)|$.*

Proof. Let $0 \le s = \sum_{i=1}^{k} b_i \chi(B_i)$ be a measurable simple function. Then it is obvious that $z_s(p_\omega) = s(\omega)$. Let (s_n) be a sequence of nonnegative measurable simple functions tending increasingly to $|f|$. Then by *Theorem D* it ensues that

$$z_f(p_\omega) = \lim_{n\to\infty} z_{s_n}(p_\omega) = \lim_{n\to\infty} s_n(\omega) = |f(\omega)|$$

which was to be proved. **q.e.d.**

Theorem 2.3. *Let f be any m.f. The following assertions are equivalent.*

1. *f is bounded.*
2. *$\lim_{x\to\infty} \wr_{(|f|\ge x)} |f|\, dp = 0$ for all $p \in \mathcal{P}_\infty$.*
3. *There exists a constant $b > 0$ such that $\wr_\Omega |f|\, dp \ne b$ for all $p \in \mathcal{P}_\infty$.*

(The proof will be carried out in two steps. In *Proposition 2.4* we shall show the equivalence 1. $\longleftrightarrow$ 2. and then the equivalence $\rceil$1. $\longleftrightarrow\rceil$3. in *Proposition 2.5.*)

Proposition 2.4. *A m.f. f is bounded if and only if $\lim\limits_{x\to\infty} \wr_{(|f|\geq x)} |f|\, dp = 0$ for all $p \in \mathcal{P}_\infty$.*

Proof. Suppose that f is bounded, and write $b > 0$ for its bound. Then for every $p \in \mathcal{P}_\infty$, we have that $\wr_{(|f|\geq x)} |f|\, dp \leq b \cdot p\,(|f| \geq x) \to 0$, as $x \to \infty$.

Conversely, assume that $\lim\limits_{k\to\infty} \wr_{(|f|\geq k)} |f|\, dp = 0$ *for all* $p \in \mathcal{P}_\infty$, but for every $n \in \mathbb{N}$ we have that $(|f| \geq n-1) \neq \emptyset$. It obviously ensues that

$$(|f| \geq n-1) \setminus (|f| \geq n) = H_n \neq \emptyset$$

for infinitely many $n \in \mathbb{N}$. (Suppose without loss of generality that $H_n \neq \emptyset$, $n \in \mathbb{N}$.) Further let $(\omega_n) \subset \Omega$ be such that $\omega_n \in H_n$ for all $n \in \mathbb{N}$. Define $p \in \mathcal{P}_\infty$ by $p\,(B) = \max\left\{\frac{1}{n} : \omega_n \in B\right\}$. Clearly (H_n) is a generating system for p. Then by assumption it follows that $\lim\limits_{k\to\infty} \wr_{(|f|\geq k)} |f|\, dp = 0$. Now note that $(|f| \geq k) = \bigcup\limits_{i=k+1}^{\infty} H_i$ for all $k \in \mathbb{N}$. Hence *Proposition 2.1.* entails that $\wr_{(|f|\geq k)} |f|\, dp = \sup\limits_{i\geq k+1} \wr_{H_i} |f|\, dp$. It is not difficult to check that $\wr_{H_i} |f|\, dp \geq 1 - \frac{1}{i}$, $i \geq k+1$. Consequently it results that $\wr_{(|f|\geq k)} |f|\, dp \geq 1 - \frac{1}{k+1}$ $(k \in \mathbb{N})$, leading to $0 = \lim\limits_{k\to\infty} \wr_{(|f|\geq k)} |f|\, dp \geq 1$, which is absurd. This contradiction concludes on the validity of the sufficiency, ending the proof. **q.e.d.**

Proposition 2.5. *Let f be a finite m.f. Then f is unbounded if and only if for every constant $c > 0$, there exists some $p_c \in \mathcal{P}_\infty$ such that*

$$z_f\,(p_c) = c. \tag{1.1}$$

Proof. *Necessity.* Assume that f is unbounded. For every $n \in \mathbb{N}$, write $E_n = (c \cdot (n-1) \leq |f| < c \cdot n)$ where $c > 0$ is an arbitrarily fixed constant. Clearly the members of the sequence (E_n) are pairwise disjoint and $\Omega = \bigcup\limits_{n=1}^{\infty} E_n$. Fix a sequence $(\omega_n) \subset \Omega$ in the following way: $\omega_n \in E_n$, $n \in \mathbb{N}$. Define $p_c \in \mathcal{P}_\infty$ by $p_c\,(B) = \max\left\{\frac{1}{n} : \omega_n \in B\right\}$. It is obvious that sequence (E_n) is a p_c-generating system such that $z_f\,(p_c) = \sup\limits_{n\geq 1} \wr_{E_n} |f|\, dp_c$, because of *Proposition 2.1.* But as $\left(1 - \frac{1}{n}\right) c \leq \wr_{E_n} |f|\, dp_c < c$ (for all $n \in \mathbb{N}$), it ensues that $c = \sup\limits_{n\geq 1} \wr_{E_n} |f|\, dp_c = z_f\,(p_c)$.

Sufficiency. Suppose that for every constant $c > 0$, identity (1.1) holds with a suitable $p \in \mathcal{P}_\infty$. Assume that f is bounded (and denote by b its bound). Now let $c > b$ be any fixed constant with a corresponding $p_c \in \mathcal{P}_\infty$ satisfy (1.1). Then we trivially obtain that $z_f\,(p_c) \leq b$. Hence we must have that $c \leq b$, which is in contradiction with the choice of c. This absurdity allows us to conclude on the validity of the proposition. **q.e.d.**

Lemma 2.6. *Let $p \in \mathcal{P}_\infty$ and (B_n) be a sequence of measurable sets tending increasingly to a measurable set $B \neq \emptyset$. Then there exists some $n_0 \in \mathbb{N}$ such that $p(B) = p(B_n)$ whenever $n \geq n_0$.*

(See the proof of *Lemma 0.1.*, [1] page 134.)

We shall next give a set of measurable functions including uniformly bounded ones.

Definition 2.2. We say that a sequence of measurable functions (f_n) is uniformly bounded starting from an index if there can be found a real number $b > 0$ and some positive integer n_0 such that $(f_n > b) = \emptyset$ for all integers $n > n_0$. (We shall simply say that (f_n) is *i-uniformly bounded.*)

The following two results are just the extensions of *Theorem B/2* and *Lemma D/2.* We shall omit their proofs as they can be similarly carried out.

Lemma 2.7. *Let (g_n) be a decreasing sequence of nonnegative m.f.'s and $\lim_{n\to\infty} g_n = g$ such that $(g_m \leq b) = \Omega$ for some $m \geq 1$ and some constant $b > 0$. Then $\lim_{n\to\infty} \wr_\Omega g_n dp = \wr_\Omega g dp$ for all $p \in \mathcal{P}$.*

Lemma 2.8. *Let (f_n) be an i-uniformly bounded sequence of nonnegative m.f.'s. Then $\limsup_{n\to\infty} \wr_\Omega f_n dp \leq \wr_\Omega \left(\limsup_{n\to\infty} f_n\right) dp$ for every $p \in \mathcal{P}$.*

Theorem 2.9. *Let (f_n) be an arbitrary sequence of m.f.'s. Then*

1. (f_n) is i-uniformly bounded,

if and only if the following two assertions hold simultaneously:

2. $z_f(p) \leq c$ for some constant $c > 0$ and all $p \in \mathcal{P}_\infty$;

3. $\limsup_{n\to\infty} z_n(p) \leq z_f(p)$, for all $p \in \mathcal{P}_\infty$ (where $f = \limsup_{n\to\infty} |f_n|$ and $z_n(p) = \wr_\Omega |f_n| dp$ with $n \in \mathbb{N}$, $p \in \mathcal{P}_\infty$).

Proof. *Necessity.* We just note that the implication 1. → 2. is obvious and on the other hand the implication 1. → 3. is no more than *Lemma 2.8.*

Sufficiency. Assume that 2. and 3. hold. Let us suppose further that 1. is false, i.e. for every real number $b > 0$ and any positive integer n_0 there is some integer $m > n_0$ such that $(|f_m| > b) \neq \emptyset$. Then we can choose by recurrence a sequence (n_k) of positive integers as follows. Write $n_1 = 1$ and $n_2 = \min\{m > n_1 : (|f_m| > n_1) \neq \emptyset\}$. If n_k has been defined, then write $n_{k+1} = \min\{m > n_k : (|f_m| > k \cdot n_k) \neq \emptyset\}$. Clearly the sequence (n_k) tends increasingly to infinity and for all positive integers $k \in \mathbb{N}$, $\left(\left|f_{n_{k+1}}\right| > k \cdot n_k\right) \neq \emptyset$. Now set $E = \bigcup_{k=1}^{\infty} B_{n_k}$, where $B_{n_k} = \left(\left|f_{n_{k+1}}\right| > k \cdot n_k\right)$, $k \in \mathbb{N}$. Write $H_1 = B_{n_1}$, and $H_k = \left(\bigcup_{j=1}^{k} B_{n_j}\right) \setminus \left(\bigcup_{j=1}^{k-1} B_{n_j}\right)$, $k > 2$. It is obvious that (H_k)

is a sequence of pairwise disjoint measurable sets with $E = \bigcup_{k=1}^{\infty} H_k$. Let $p \in \mathcal{P}_\infty$ be a 1st-type E-dependent optimal measure defined by $p(B) = \max\left\{\frac{1}{k} : \omega_k \in B\right\}$, where $(\omega_k) \subset \Omega$ is a fixed sequence so that $\omega_k \in H_k$ $(k \in \mathbb{N})$. It is clear that $\mathcal{H}(p) = \{H_k : k \in \mathbb{N}\}$ is a p-generating system. Then via 2. and 3. we have that $c \geq \wr_\Omega \left(\limsup_{n \to \infty} |f_n|\right) dp \geq \limsup_{n \to \infty} \wr_\Omega |f_n| \, dp$ and hence $b > \limsup_{k \to \infty} \wr_\Omega \left|f_{n_{k+1}}\right| dp$ for some $b > 0$ (this is true because of *Remark 2.2*). Consequently, as $p(H_k) = \frac{1}{k}$ for every $k \in \mathbb{N}$, we must have

$$
\begin{aligned}
b > \limsup_{k \to \infty} \wr_\Omega \left|f_{n_{k+1}}\right| dp &= \limsup_{k \to \infty} \wr_E \left|f_{n_{k+1}}\right| dp \\
&\geq \limsup_{k \to \infty} \wr_{H_k} \left|f_{n_{k+1}}\right| dp \\
&\geq \limsup_{k \to \infty} k \cdot n_k \cdot p(H_k) = \infty,
\end{aligned}
$$

which is absurd. This contradiction justifies the validity of the theorem. **q.e.d.**

3. The case of some well-known types of convergence

Definition 3.1. Let X be an arbitrary nonempty set. We say that a sequence of real-valued functions (h_n) converges to a real-valued function h:

(i) discretely if for every $x \in X$ there exists a positive integer $n_0(x)$ such that $h_n(x) = h(x)$, whenever $n > n_0(x)$;

(ii) equally if there is a sequence (b_n) of positive numbers tending to 0 and for every $x \in X$ there can be found an $n_0(x)$ such that $|h_n(x) - h(x)| < b_n$ whenever $n > n_0(x)$.

(For more about these notions, cf. [3 - 5].)

Theorem 3.0. *Let f and f_n $(n \in \mathbb{N})$ be any m.f.'s. Then (f_n) tends to f uniformly if and only if (z_n) tends to* 0 *uniformly on $\mathcal{P}_\infty$, where $z_n(p) = \wr_\Omega |f_n - f| \, dp$ with $n \in \mathbb{N}$, $p \in \mathcal{P}_\infty$.*

Proof. *Sufficiency.* Suppose that (z_n) tends to 0 uniformly. To prove the sufficiency it is enough to show that for every number $b > 0$, there can be found some $n_0(b) \in \mathbb{N}$ such that $(|f - f_n| \geq b) = \emptyset$ whenever $n \geq n_0(b) + 1$. In fact, let us assume that the contrary holds. Then for some $b_0 > 0$ and all $n_0 \in \mathbb{N}$, there is an integer $m > n_0$ such that $(|f - f_m| \geq b_0) \neq \emptyset$. Define

$$n_1 = \min\{m > n_1 : (|f - f_m| \geq b_0) \neq \emptyset\}$$

when $n_0 = 1$. If n_k has been selected, define

$$n_{k+1} = \min\{m > n_k : (|f - f_m| \geq b_0) \neq \emptyset\}$$

when $n_0 = n_k$. It is clear that sequence (n_k) tends increasingly to infinity alongside with k, so that $(|f - f_{n_k}| \geq b_0) \neq \emptyset$, $k \in \mathbb{N}$. Then by assumption some $n_m \in \{n_k : k \in \mathbb{N}\}$ exists such that $z_{n_k}(p) < b_0$, for all $k \geq m$ and $p \in \mathcal{P}_\infty$. Now let $E_m = \bigcup_{k=m}^{\infty} B_{n_k}$, (where $B_{n_k} = (|f - f_{n_k}| \geq b_0)$, $k \in \mathbb{N}$). Write $H_{n_m} = B_{n_m}$ and for $k \geq m+1$, set $H_{n_k} = \left(\bigcup_{j=m}^{k} B_{n_j}\right) \setminus \left(\bigcup_{j=m}^{k-1} B_{n_j}\right)$. Clearly $\mathcal{H} = \{H_{n_k} : k \geq m\}$ is a sequence of pairwise disjoint measurable sets with $E_m = \bigcup_{k=m}^{\infty} H_{n_k}$. Fix a sequence $(\omega_k) \in \Omega$ so that $\omega_k \in H_{n_k}$ whenever $k \geq m$. Next, let $p_0 \in \mathcal{P}_\infty$ be a 1st-type E_m-dependent optimal measure defined by $p_0(B) = n_m \cdot \max\left\{\frac{1}{n_k} : \omega_k \in B\right\}$. It is obvious that $\mathcal{H}$ is a p_0-generating system. Hence we have on the one hand that $z_{n_m}(p_0) < b_0$. Nevertheless on the other hand we also obtain that $z_{n_m}(p_0) \geq \wr_{H_{n_m}} |f_{n_m} - f|\, dp_0 \geq b_0$, since $p_0(H_{n_m}) = 1$. As these last two inequalities contradict each other, the sufficiency is thus proved.

Necessity. Assume that $f_n \to f$ uniformly, as $n \to \infty$. Then for every $b \in (0, \infty)$, there is some $n_0(b) \in \mathbb{N}$ such that $\left(|f_n - f| < \frac{b}{2}\right) = \Omega$ whenever $n > n_0(b)$. Consequently, for every $p \in \mathcal{P}_\infty$, it ensues that $z_n(p) \leq \frac{b}{2} < b$, $n > n_0(b)$. This completes the proof of the theorem. **q.e.d.**

Lemma 3.1. *Let f and f_n ($n \in \mathbb{N}$) be any m.f.'s. If (f_n) tends to f pointwise (equally or discretely), then* $\limsup_{n\to\infty} B_n = \emptyset$, *where* $B_n = (|f_n - f| = \infty)$, $n \in \mathbb{N}$.

Proof. It is enough to prove the lemma for the pointwise convergence, since proving the remaining cases is similarly done. Assume that $\limsup_{n\to\infty} B_n \neq \emptyset$. Let us pick an arbitrary $\omega \in \limsup_{n\to\infty} B_n$. Then it is clair that $\limsup_{n\to\infty} |f_n(\omega) - f(\omega)| = \infty$ and hence $\bigwedge_{n=k}^{\infty} \bigvee_{j=n}^{\infty} |f_j(\omega) - f(\omega)| = \infty$ for every $k \in \mathbb{N}$. But since (f_n) tends to f pointwise we must have that for every constant $b > 0$ there is a positive integer $m_0 = m_0(b, \omega)$ such that $|f_n(\omega) - f(\omega)| < b$ whenever $n > m_0$. Hence $b \geq \bigwedge_{n=m_0}^{\infty} \bigvee_{j=n}^{\infty} |f_j(\omega) - f(\omega)| = \infty$, which is absurd, completing the proof. **q.e.d.**

Theorem 3.2. *Let (f_n) be any sequence of m.f.'s. Then (f_n) tends to a m.f. f pointwise if and only if (z_n) tends to 0 pointwise on $\mathcal{P}_{<\infty}$, where for every $n \in \mathbb{N}$, z_n is defined on $\mathcal{P}_{<\infty}$ by $z_n(p) = \wr_\Omega |f_n - f|\, dp$.*

Proof. *Sufficiency.* Assume that for all $b > 0$ and $p \in \mathcal{P}_{<\infty}$ there is a positive integer $n_0 = n_0(b, p)$ such that $z_n(p) < b$ whenever $n > n_0$. Then since for every fixed $\omega \in \Omega$ the ω-concentrated measure p_ω depends solely upon $\omega \in \Omega$, index $n_0(b, p_\omega)$ also depends on ω. Hence via *Lemma 2.2* we have for all $n \geq n_0(b, \omega) = n_0(b, p_\omega)$ that $|f_n(\omega) - f(\omega)| = z_n(p_\omega) < b$.

Necessity. Suppose that for all $a > 0$ and $\omega \in \Omega$, there can be found some positive integer $m_0 = m_0(a, \omega)$ such that $|f_n(\omega) - f(\omega)| < a$, whenever $n \geq m_0$. Assume further that there is some $b > 0$ and some $p \in \mathcal{P}_{<\infty}$ such that for every $n \in \mathbb{N}$, there exists some $m \geq n$ with the property that $z_n(p) \geq b$. Let $H_1, \ldots, H_k$ be a p-generating system. Via *Lemma 3.1*, there is some $n_0 \in \mathbb{N}$, big enough so that $f_n - f$ is finite on Ω whenever $n \geq n_0$. Then for every $n \geq n_0$, a measurable set $A_n^{(i)}$ exists with $A_n^{(i)} \subset H_i$ and $p\left(A_n^{(i)}\right) = 0$ such that $f_n - f$ is constant on $H_i \backslash A_n^{(i)}$, $i = 1, \ldots, k$ (because of *Remark A*). Clearly $p\left(\bigcup_{j=n_0}^{\infty} A_j^{(i)}\right) = 0$, so that the identity $p\left(H_i \backslash \bigcup_{j=n_0}^{\infty} A_j^{(i)}\right) = p(H_i)$ holds. Hence $f_n - f$ is constant on $H_i \backslash \bigcup_{j=n_0}^{\infty} A_j^{(i)}$ whenever $i \in \{1, \ldots, k\}$ and $n \geq n_0$. Fix $\omega_i \in H_i \backslash \bigcup_{j=n_0}^{\infty} A_j^{(i)}$, $i \in \{1, \ldots, k\}$. Then by assumption there must be some positive integer $k_0^{(i)} = k_0(b, \omega_i)$ such that $|f_n(\omega_i) - f(\omega_i)| < b$, $n > k_0^{(i)}$. Thus for all $n \geq k_0$ (where $k_0 = \bigvee_{i=1}^{k} k_0^{(i)}$), we have that $\bigvee_{i=1}^{k} |f_n(\omega_i) - f(\omega_i)| < b$. Now write $k^* = \max(k_0, n_0)$. Then some integer $m > k^*$ exists such that $z_m(p) \geq b$. Therefore (via *Proposition 2.1* and *Remark A*) we obtain that

$$b \leq z_m(p) = \bigvee_{i=1}^{k} c_i \cdot p(H_i) \leq \bigvee_{i=1}^{k} c_i = \bigvee_{i=1}^{k} |f_m(\omega_i) - f(\omega_i)| < b$$

where for $i \in \{1, \ldots, k\}$, $c_i = |f_m(\omega) - f(\omega)|$ if $\omega \in H_i \backslash \bigcup_{j=n_0}^{\infty} A_j^{(i)}$. However, this is absurd, a contradiction which ends the proof of the theorem. **q.e.d.**

Theorem 3.3. *A sequence of m.f.'s (f_n) converges to some m.f. f equally if and only if (z_n) converges to 0 equally on $\mathcal{P}_{<\infty}$, where for every $n \in \mathbb{N}$, z_n is defined on $\mathcal{P}_{<\infty}$ by $z_n(p) = \wr_\Omega |f_n - f|\, dp$.*

Proof. *Necessity.* Suppose that there exists a sequence $(b_n) \subset (0, \infty)$ tending to 0 and for every $\omega \in \Omega$ there can be found a positive integer $n_0(\omega)$ such that $|f_n(\omega) - f(\omega)| < b_n$ for all $n \geq n_0(\omega)$. It is enough to show that the equal convergence of (z_n) holds true for this sequence (b_n). In fact, assume that for this sequence (b_n), there is some $p \in \mathcal{P}_{<\infty}$ such that for all $j \in \mathbb{N}$ an integer $m = m(p) > j$ can be found with the property that $z_m(p) \geq b_m$. Let $H_1, \ldots, H_k$ be a p-generating system. Via *Lemma 3.1*, there is some $n_0 \in \mathbb{N}$, big enough so that $f_n - f$ is finite on Ω whenever $n \geq n_0$. Then for every $n \geq n_0$, a measurable set $A_n^{(i)}$ exists with $A_n^{(i)} \subset H_i$ and $p\left(A_n^{(i)}\right) = 0$ such that $f_n - f$ is constant on $H_i \backslash A_n^{(i)}$, $i = 1, \ldots, k$. But as $p\left(\bigcup_{j=n_0}^{\infty} A_j^{(i)}\right) = 0$, we can easily observe that $p\left(H_i \backslash \bigcup_{j=n_0}^{\infty} A_j^{(i)}\right) = p(H_i)$, $i \in \{1, \ldots, k\}$. Hence $f_n - f$ is constant on $H_i \backslash \bigcup_{j=n_0}^{\infty} A_j^{(i)}$ for all $i \in \{1, \ldots, k\}$ and

$n \geq n_0$. Fix $\omega_i \in H_i \setminus \bigcup_{j=n_0}^{\infty} A_j^{(i)}$, $i \in \{1, \dots, k\}$. Then by assumption there must be some positive integer $k_0^{(i)} = k_0(\omega_i)$ such that $|f_n(\omega_i) - f(\omega_i)| < b_n$, $n > k_0^{(i)}$. Thus for all $n \geq k_0$ (where $k_0 = \bigvee_{i=1}^{k} k_0^{(i)}$), we have that $\bigvee_{i=1}^{k} |f_n(\omega_i) - f(\omega_i)| < b_n$. Consequently we have on the one hand that $z_m(p) \geq b_m$. But on the other hand, *Proposition 2.1* yields that

$$z_m(p) = \bigvee_{i=1}^{k} c_i \cdot p(H_i) \leq \bigvee_{i=1}^{k} c_i = \bigvee_{i=1}^{k} |f_m(\omega_i) - f(\omega_i)| < b_m$$

(where for $i \in \{1, \dots, k\}$, $c_i = |f_m(\omega) - f(\omega)|$ if $\omega \in H_i \setminus \bigcup_{j=n_0}^{\infty} A_j^{(i)}$), meaning that $b_m < b_m$, which is, however, absurd. This contradiction concludes the proof of the necessity.

Sufficiency. Assume that there is a sequence (b_n) of positive numbers tending to 0 and for every $p \in \mathcal{P}_{<\infty}$ there exists a positive integer $n_0(p)$ such that $z_n(p) < b_n$ whenever $n > n_0(p)$. Then for each fixed $\omega \in \Omega$, *Lemma 2.2.* entails that $|f_n(\omega) - f(\omega)| = z_n(p_\omega) < b_n$ whenever $n > n_0(p_\omega) = n_0(\omega)$. The sufficiency is thus proved, which completes the proof of the theorem. **q.e.d.**

Theorem 3.4. *A sequence of m.f.'s (f_n) converges to some m.f. f discretely if and only if (z_n) converges to 0 discretely on $\mathcal{P}_{<\infty}$, where for every $n \in \mathbb{N}$, z_n is defined on $\mathcal{P}_{<\infty}$ by $z_n(p) = \wr_\Omega |f_n - f| \, dp$.*

(The proof is omitted as it can be carried out "mutatis mutandis" as in *Theorems 3.3* and *3.4*)

4. Quasi-uniform convergence

Definition 4.1. A sequence of real-valued functions (g_n), defined on a nonempty set X, is said to converge quasi-uniformly to a real-valued function g if for every given number $\epsilon \in (0,\ 1)$ there exists some nonempty set $B_\epsilon \subset X$ and some positive integer $n_0 = n_0(\epsilon)$ such that $|g_n(x) - g(x)| < \epsilon$ whenever $n > n_0$ and $x \in B_\epsilon$.

Example 1. Every uniformly convergent sequence of m.f.'s also converges quasi-uniformly.

Example 2. Let us endow the real line $\mathbb{R}$ with the Borel σ-algebra $\mathcal{B}$ and let (f_n) be a sequence of Borel measurable functions defined by

$$f_n(x) = \frac{x}{n},\ n \in \mathbb{N},\ x \in \mathbb{R}.$$

It is not difficult to see that (f_n) converges to zero pointwise but not uniformly. We show that (f_n) converges to zero quasi-uniformly. In fact, pick an arbitrary number

$\varepsilon \in (0,\ 1)$ and fix any number $x \in \mathbb{R}$. Clearly with the choice $n_0 = n_0(\varepsilon,\ x) = \left[\frac{|x|}{\varepsilon}\right] + 1$, we have that $\frac{|x|}{n} < \varepsilon$ for all $n \geq n_0$. Now define the set $B_\varepsilon = \{t \in \mathbb{R} : \left[\frac{|t|}{\varepsilon}\right] < \left[\frac{|x|}{\varepsilon}\right]\}$. Obviously we have that $\frac{|t|}{n} < \varepsilon$ for all $t \in B_\varepsilon$. Therefore (f_n) converges to zero quasi-uniformly.

Lemma 4.1. *Let f be any m.f., $p \in \mathcal{P}_\infty$, H some indecomposable p-atom with $p(H) = 1$ and $\epsilon \in (0,\ 1)$ any number. Then $p(H \bigcap (|f| \geq \epsilon)) = 0$ if and only if $\wr_H |f|\, dp < \epsilon$.*

Proof. As the necessity is obvious we shall just show the sufficiency. Suppose that $\wr_H |f|\, dp < \epsilon$ but $p(H \bigcap (|f| \geq \epsilon)) > 0$. Then

$$\begin{array}{rcl} \epsilon & > & \wr_H |f|\, dp \\ & = & \left(\wr_{H\cap(|f|<\epsilon)} |f|\, dp\right) \bigvee \left(\wr_{H\cap(|f|\geq\epsilon)} |f|\, dp\right) \\ & \geq & \wr_{H\cap(|f|\geq\epsilon)} |f|\, dp \\ & \geq & \epsilon \cdot p(H \bigcap (|f| \geq \epsilon)). \end{array}$$

But since H is an indecomposable p-atom and $p(H \bigcap (|f| \geq \epsilon)) > 0$, it ensues that $p(H \bigcap (|f| \geq \epsilon)) = p(H) = 1$. Consequently we must have that $\epsilon > \epsilon$, which is absurd, indeed. This contradiction concludes the proof. **q.e.d.**

Theorem 4.2. *Let f and f_n $(n \in \mathbb{N})$ be any m.f.'s. Then (f_n) tends to f quasi-uniformly if and only if (z_n) tends to 0 quasi-uniformly on $\mathcal{P}_\infty$, where $z_n(p) = \wr_\Omega |f_n - f|\, dp$ with $n \in \mathbb{N}$, $p \in \mathcal{P}_\infty$.*

Proof. *Sufficiency.* Assume the quasi-uniform convergence of (f_n), i.e. for every $\epsilon \in (0,\ 1)$ we can find some nonempty measurable set $B_{\frac{\epsilon}{2}}$ and some positive integer $n_0 = n_0(\epsilon)$ such that

$$B_{\frac{\epsilon}{2}} \subset \left(|f_n - f| < \frac{\epsilon}{2}\right), n > n_0.$$

Write $\mathcal{P}_\infty(\epsilon) = \left\{p \in \mathcal{P}_\infty : p\left(\Omega \backslash B_{\frac{\epsilon}{2}}\right) = 0\right\}$. We note that $\mathcal{P}_\infty(\epsilon) \neq \emptyset$, since each 1st-type $B_{\frac{\epsilon}{2}}$-dependent optimal measure belongs to $\mathcal{P}_\infty(\epsilon)$. Clearly for all $n > n_0$ and $p \in \mathcal{P}_\infty(\epsilon)$

$$\wr_\Omega |f_n - f|\, dp = \wr_{B_{\frac{\epsilon}{2}}} |f_n - f|\, dp \leq \frac{\epsilon}{2} < \epsilon.$$

Necessity. Assume the quasi-uniform convergence of (z_n), but (f_n) fails to converge quasi-uniformly to f. Then the latter assumption means that for some $\epsilon_* \in (0,\ 1)$, all nonempty measurable sets B and every positive integer m_0 there exists an $M \geq m_0$ such that $(|f_M - f| \geq \epsilon_*) \bigcap B \neq \emptyset$. Nevertheless, because of the former assumption there can be found some $\mathcal{P}_\infty(\epsilon_*) \subset \mathcal{P}_\infty$ and some integer $m_* = m_*(\epsilon_*) \geq 1$ such that $z_m(p) < \epsilon_*$ for all $m \geq m_*$ and $p \in \mathcal{P}_\infty(\epsilon_*)$. Let us fix some $p \in \mathcal{P}_\infty(\epsilon_*)$ with (H_k) its generating system so that $z_m(p) < \epsilon_*$ whenever $m \geq m_*$. Then *Proposition 2.1* entails that $\wr_{H_k} |f_m - f|\, dp < \epsilon_*$, for all $k \geq 1$ and $m \geq m_*$. As the *Fundamental Theorem* guarantees that $\lim_{k\to\infty} p(H_k) = 0$, there must exist some integer j such that $p(H_j) = 1$. Next, noting that the conditions of *Lemma 4.1* are met, it results that $p((|f_m - f| \geq \epsilon_*) \bigcap H_j) = 0$, $m \geq m_*$. Write

$$S = H_j \backslash \left(\bigcup_{m=m_*}^{\infty} \left((|f_m - f| \geq \epsilon_*) \bigcap H_j \right) \right) = H_j \backslash \left(\bigcup_{m=m_*}^{\infty} (|f_m - f| \geq \epsilon_*) \right).$$

It is not difficult to see that $(|f_m - f| < \epsilon_*) \bigcap S = S$, $m \geq m_*$. Consequently, since we have rather assumed the negation of the conclusion, some integer $i > m_*$ must exist so that $(|f_i - f| \geq \epsilon_*) \bigcap S \neq \emptyset$. This, however, is in contradiction with $(|f_i - f| < \epsilon_*) \bigcap S = S$, which ends the proof of the theorem. **q.e.d.**

5. Concluding remarks

I would like to simply note that when preparing those two works (see [1-2]) I was not aware of the existence of the so-called 'maxitive measures' proposed by N. Shilkret. in [7]. Hereafter one can find a briefing of his work.

By *a-maxitive measures*, i.e. set functions $m : \mathcal{F} \to [0, \infty)$, satisfying the conditions $m(\emptyset) = 0$ and $m\left(\bigcup_{i\in I} E_i\right) = \sup_{i\in I} m(E_i)$ for every collection $\{E_i\}_{i\in I} \subset \mathcal{F}$, where $\mathcal{F}$ is a ring of subsets of an arbitrary nonempty set Ω; it is called a maxitive measure if I is finite or countably infinite. Shilkret realized that maxitive measures are not in general continuous from above and he proved that: *A maxitive measure m is continuous from above if and only if the following assertion is false "There exist some $k \in \mathbb{N}$ and some sequence of measurable sets $\{E_i\} \subset \mathcal{F}$ such that $\frac{1}{k} < m(E_i) < k$, $i \in \mathbb{N}$."*

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