Miskolc Mathematical Notes
Vol. 11 (2010), No 2, pp. 169-174

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2010.240

Some results on symmetric bi-(σ, τ)-derivations in near-rings

Mehmet Ali Öztürk and Hasret Yazarli

Miskolc Mathematical Notes
Vol. 11 (2010), No. 2, pp. 169–173

HU e-ISSN 1787-2413

SOME RESULTS ON SYMMETRIC BI-(σ,τ)-DERIVATIONS IN NEAR-RINGS

MEHMET ALI ÖZTÜRK AND HASRET YAZARLI

Received 1 December, 2009

Abstract. The aim of this paper is to investigate certain results on 3-prime near-rings and generalize these results on near-rings to semi-group ideals of 3-prime near-rings.

2000 *Mathematics Subject Classification:* 16A12, 16D30, 16S99, 16Y99, 16W25

Keywords: near-ring, prime near-ring, semigroup ideal, derivation, symmetric bi-derivation, symmetric bi-(σ,τ)-derivation

1. INTRODUCTION

For preliminary definitions and results related to near-rings, we refer for Pilz [5]. The concept of a symmetric bi-derivation has been introduced by Maksa [3]. In recent years, many mathematicians studied the commutativity of prime and semi-prime rings admitting suitably-constrained symmetric bi-derivations. In [4], Öztürk (together with Jun) introduced the notion of a symmetric bi-derivations in near-rings and proved some results. Moreover the concept of symmetric bi-(σ,τ)-derivation of near-ring has been introduced by Çeven (together with Öztürk) [2]. The aim of this paper is to investigate some results on 3-prime near-rings and generalize the results on near-rings to semi-group ideals of 3-prime near-rings.

2. PRELIMINARIES

Throughout this paper N will be a zero-symmetric left near-ring, and usually N will be 3-prime, that is, it will have the property that $xNy = \{0\}$ implies $x = 0$ or $y = 0$. The symbol Z will denote the multiplicative center of N. A nonempty subset U of N will be called semigroup right ideal (resp. semigroup left ideal) if $UN \subset U$ (resp. $NU \subset U$) and U is both a semigroup right ideal and a semigroup left ideal, it is called a semigroup ideal. For $x, y \in N$, the symbol $[x, y]$ will denote the commutator $xy - yx$, while the symbol (x, y) will denote the additive-group commutator $x + y - x - y$. A mapping $D : N \times N \to N$ is said to be symmetric if $D(x, y) = D(y, x)$ for all $x, y \in N$. A mapping $d : N \to N$ defined by $d(x) = D(x, x)$ is called the trace of D where $D : N \times N \to N$ is a symmetric mapping. It is obvious that, if $D : N \times N \to N$ is a symmetric mapping which is also bi-additive (i.e., additive in both arguments), then the trace of D satisfies the relation $d(x+y) = d(x) + 2D(x,y) + d(y)$ for all $x, y \in N$. A symmetric bi-additive mapping $D : N \times N \to N$ is called a symmetric bi-derivation if $D(xy,z) = D(x,z)y + xD(y,z)$ is fulfilled for all $x, y, z \in N$. Then, for any $y \in N$, a mapping $x \mapsto D(x,y)$ is a derivation. A symmetric bi-additive mapping $D : N \times N \to N$ is called symmetric bi-(σ,τ)-derivation if there exist automorphisms $\sigma, \tau : N \to N$ such that $D(xy,z) = D(x,z)\sigma(y) + \tau(x)D(y,z)$ for all $x, y, z \in N$. Note that if $\sigma = 1$ and $\tau = 1$ then D is a symmetric bi-derivation.

3. Results

The following lemmas and theorems are necessary for the paper.

Lemma 1 ([3, Lemma 3]). *Let N be a 3-prime near-ring.*

(i) *If $z \in Z - \{0\}$, then z is not a zero divisor.*
(ii) *If $Z - \{0\}$ contains an element z for which $z + z \in Z$, then $(N, +)$ is abelian.*

Lemma 2 ([1, Lemma 1.3]). *Let N be 3-prime, and let d be a nonzero derivation on N.*

(i) *If U is a nonzero semi-group right ideal (resp. semi-group left ideal) and x is an element of N such that $Ux = \{0\}$ (resp. $xU = \{0\}$), then $x = 0$.*
(ii) *If U is nonzero semi-group right ideal or semi-group left ideal, then $d(U) \neq \{0\}$.*
(iii) *If U is a nonzero semi-group right ideal and x is an element of N which centralizes U, then $x \in Z$.*

Lemma 3 ([1, Lemma 1.4]). *Let N be 3-prime, and U a nonzero semi-group ideal of N. Let d be a nonzero derivation on N.*

(i) *If $x, y \in N$ and $xUy = \{0\}$, then $x = 0$ or $y = 0$.*
(ii) *If $x \in N$ and $d(U)x = \{0\}$, then $x = 0$.*
(iii) *If $x \in N$ and $xd(U) = \{0\}$, then $x = 0$.*

Lemma 4 ([1, Lemma 1.5]). *If N is 3-prime and Z contains a nonzero semigroup left ideal or semigroup right ideal, N is a commutative ring.*

Lemma 5 ([2, Lemma 3]). *Let N be a 2-torsion free 3-prime near-ring, D a symmetric bi-(σ,τ)-derivation of N and d the trace of D. If $xd(N) = \{0\}$ for all $x \in N$, then $x = 0$ or $D = 0$.*

Lemma 6 ([2, Lemma 4]). *Let N be a near-ring. D is a symmetric bi-(σ,τ)-derivation of N if and only if $D(xy,z) = \tau(x)D(y,z) + D(x,z)\sigma(y)$ for all $x, y, z \in N$.*

Lemma 7 ([2, Lemma 5]). *Let N be a near-ring, D a symmetric bi-(σ,τ)-derivation of N. Then, for all $x,y,z,w \in N$,*

(i) $[D(x,z)\sigma(y)+\tau(x)D(y,z)]w = D(x,z)\sigma(y)w+\tau(x)D(y,z)w,$

(ii) $[\tau(x)D(y,z)+D(x,z)\sigma(y)]w = \tau(x)D(y,z)w+D(x,z)\sigma(y)w.$

Theorem 1 ([2, Theorem 1]). *Let N be a 3-prime near-ring, D a nonzero symmetric bi-(σ,τ)-derivation of N. If N is 2-torsion free and $D(N,N)\subset Z$, then N is commutative ring.*

Lemma 8. *Let N be a 2-torsion free 3-prime near-ring, U a nonzero semigroup ideal of N and D symmetric bi-(σ,τ)-derivation of N. If $D(U,U)=\{0\}$, then $D=0$.*

Proof. Suppose $D(x,y)=0$ for all $x,y\in U$. Then taking xz, $z\in N$ instead of x, we have

$$\begin{aligned}0 &= D(xz,y) = D(x,y)\sigma(z)+\tau(x)D(z,y)\\ &= \tau(x)D(y,z)\end{aligned}$$

Replacing yw by y, $w\in N$ in last relation, we get

$$\begin{aligned}0 &= \tau(x)D(y,z)\sigma(w)+\tau(x)\tau(y)D(w,z)\\ &= \tau(x)\tau(y)D(w,z)\end{aligned}$$

for all $x,y\in U$, $w,z\in N$. Since τ is an automorphism of N and, $U\neq\{0\}$, we get $D(N,N)=\{0\}$ by Lemma 3. That is, $D=0$. □

Lemma 9. *Let N be a 2-torsion free 3-prime near-ring, U a nonzero semigroup ideal of N, D a nonzero symmetric bi-(σ,τ)-derivation of N and $a\in N$. If $D(U,U)a=\{0\}$, then $a=0$. (Similarly, if $D(U,U)a=0$, then $a=0$.)*

Proof. Suppose $D(x,y)a=0$ for all $x,y\in U$. Taking wy, $w\in N$ instead of y and using Lemma 7(i), we get

$$\begin{aligned}0 &= D(x,wy)a = [D(w,x)\sigma(y)+\tau(w)D(y,x)]a\\ &= D(w,x)\sigma(y)a+\tau(w)D(y,x)a\\ &= D(x,w)\sigma(y)a\end{aligned}$$

Since σ is an automorphism of N and $U\neq\{0\}$, we get $D(N,U)=\{0\}$ or $a=0$ by Lemma 3(i). Assume $D(N,U)=\{0\}$. Hence $D(U,U)=\{0\}$, and so $D=0$ by Lemma 8. This completes the proof.

The other case ($aD(U,U)=\{0\}$) can be treated similarly. □

Theorem 2. *Let N be a 3-prime near-ring, U a nonzero semigroup ideal of N and D a nonzero symmetric bi-(σ,τ)-derivation of N. If N is 2-torsion free and $D(U,U)\subset Z$, then N is commutative ring.*

Proof. Since $D(x,y) \in Z$ for all $x,y \in U$, we have $D(x,y)z = zD(x,y)$, for all $x,y \in U$ and $z \in N$. Relacing xw by w, $w \in U$, we get

$$\begin{aligned} D(xw,y)z &= zD(xw,y) \\ D(x,y)\sigma(w)z + \tau(x)D(w,y)z & \\ &= zD(x,y)\sigma(w) + z\tau(x)D(w,y) \end{aligned} \tag{3.1}$$

Taking $z = \sigma(w)$ and using $D(U,U) \subset Z$, we obtain that

$$\begin{aligned} &D(x,y)\sigma(w)\sigma(w) + \tau(x)D(w,y)\sigma(w) \\ &= \sigma(w)D(x,y)\sigma(w) + \sigma(w)\tau(x)D(w,y) \end{aligned}$$

and so

$$D(w,y)[\sigma(w),\tau(x)] = 0, \text{ for all } x,y,w \in U$$

Since Z contains no nonzero divisors of zero, we see that for each $w \in U$, either $D(w,y) = 0$ for all $y \in U$ or $[\tau(x),\sigma(w)] = 0$, for all $x \in U$. In the first case, equation (3.1) yields

$$D(x,y)[\sigma(w),z] = 0, \quad \text{for all } x,y,w \in U, z \in N.$$

By the same argument as above, we have, we have $[\sigma(w),z] = 0$, for all $z \in N$. In particular, we obtain for all $x \in U$, $[\tau(x),\sigma(w)] = 0$, in each case. That is, $U \subset Z$ by Lemma 2(iii). Hence N is a commutative ring by Lemma 4. This completes the proof. □

Theorem 3. *Let N be a 2-torsion free 3-prime near-ring. Let D_1 and D_2 be nonzero symmetric bi-(σ,τ)-derivations of N and d_1 and let d_2 be nonzero trace of D_1 and D_2, respectively. If $d_2(y)$, $d_2(y)+d_2(y) \in C\ (D_1(x,z))$ for all $x,y,z \in N$, then $(N,+)$ is abelian and $d_2(N) \subseteq Z$.*

Proof. Since $d_2(y)$, $d_2(y)+d_2(y) \in C\ (D_1(x,z))$ for all $x,y,z \in N$, if both w and $w+w$ commute elementwise with $D_1(x,z)$ for all $s \in N$, then

$$[D_1(x,z)+D_1(s,z)](w+w) = [D_1(x,z)+D_1(x,z)+D_1(s,z)+D_1(s,z)]w$$

On the other hand,

$$[D_1(x,z)+D_1(s,z)](w+w) = [D_1(x,z)+D_1(s,z)+D_1(x,z)+D_1(s,z)]w$$

Comparing these expressions we get, for all $x,s \in N$,

$$D_1((x,s),z)w = 0 \tag{3.2}$$

Thus, let $w = d_2(y)$ in (3.2), we get $D_1((x,s),z)d_2(y) = 0$ for all $x,z,s,y \in N$, so $D_1((x,s),z) = 0$. Since (x,s) is also an additive commutator for any $z \in N$, we have

$$\begin{aligned} 0 = D_1((x,s)z,z) &= D_1((x,s),z)\sigma(z) + \tau((x,s))d_1(z) \\ &= \tau((x,s))d_1(z) \end{aligned}$$

Since τ is an automorphism, we get $(x,s)=0$. That is, $(N,+)$ is abelian.

Now, replacing xw by x in $D_1(x,z)d_2(y)=d_2(y)D_1(x,z)$, we get

$$\begin{aligned}0&=D_1(xw,z)d_2(y)-d_2(y)D_1(xw,z)\\&=D_1(x,z)\sigma(w)d_2(y)+\tau(x)D_1(w,z)d_2(y)\\&\quad-d_2(y)\tau(x)D_1(w,z)-d_2(y)D_1(x,z)\sigma(w)\end{aligned}$$

Taking $d_1(t)$ for $\sigma(w)$ in the last relation, we get

$$\tau(x)D_1(w,z)d_2(y)=d_2(y)\tau(x)D_1(w,z) \tag{3.3}$$

Replacing wu by w in (3.3) and using (3.3), we have

$$\begin{aligned}\tau(x)D_1(w,z)\sigma(u)d_2(y)&=d_2(y)\tau(x)D_1(w,z)\sigma(u)\\&=\tau(x)D_1(w,z)d_2(y)\sigma(u)\end{aligned}$$

$$\tau(x)D_1(w,z)[\sigma(u),d_2(y)]=0$$

Since U is a nonzero semi-group ideal and D_1 is nonzero, we get $[\sigma(u),d_2(y)]=0$. Since σ is an automorphism, we have $d_2(N)\subseteq Z$. □

Acknowledgement: The authors are deeply grateful to the referee for the valuable suggestions.

REFERENCES

[1] H. E. Bell, *On derivations in near-rings. II.*, ser. Mathematics and its Applications. Dordrecht: Kluwer Acad. Publ., 1997, vol. 426.
[2] Y. Çeven and M. A. Öztürk, "Some properties of symmetric bi-(σ,τ)-derivations in near-rings," *Commun. Korean Math. Soc.*, vol. 22, no. 4, pp. 487–491, 2007.
[3] G. Maksa, "On the trace of symmetric bi-derivations," *C. R. Math. Rep. Acad. Sci. Canada*, vol. 9, no. 6, pp. 303–307, 1987.
[4] M. A. Öztürk and Y. B. Jun, "On the trace of symmetric bi-derivations in near-rings," *Inter. J. Pure and Appl. Math.*, vol. 17, no. 1, pp. 9–102, 2004.
[5] G. Pilz, *Near-rings*, ser. North-Holland Mathematics Studies. Amsterdam: North-Holland Publishing Co., 1983, vol. 23.

Authors' addresses

Mehmet Ali Öztürk
Adıyaman University, Faculty of Arts and Sciences, Department of Mathematics, 02040 Adıyaman, TURKEY
E-mail address: maozturk@adiyaman.edu.tr

Hasret Yazarlı
Cumhuriyet University, Faculty of Arts and Sciences, Department of Mathematics, 58140 Sivas, TURKEY
E-mail address: hyazarli@cumhuriyet.edu.tr