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AN OVERDETERMINED SYSTEM WITH TWO SUPER-SINGULAR LINES CONTAINING SECOND ORDER HYPERBOLIC EQUATIONS

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Abstract. The paper treats a linear overdetermined system which contains second order hyperbolic equations with two linear super singularities. The authors obtain some new integral representations of the solutions.

1. INTRODUCTION

We derive the general solutions of the system (2.1) that contains arbitrary constants subject to the conditions $\alpha < 1, \beta < 1, \gamma < 1$ or $\alpha > 1, \beta > 1, \gamma > 1$.

Hyperbolic differential equations with singular coefficients or singular surfaces possess importance in diverse areas of mathematical physics and mathematical engineering, including elasticity, hydrodynamics, thermodynamics and other problems. Furthermore, it is also well known that hyperbolic differential equations with one or more singular lines occur in engineering and physical processes. For example, the non-standard hyperbolic equation of second order with two singular lines is employed to describe the transformation spectrum of electric signals on long lines with variable parameters in the theory of the electric flail [3, 4, 9].

The problem of obtaining the solutions of equations of hyperbolic type with singular coefficients is described in a number of works [1, 3–5, 9]. In [9], for the equation

$$\frac{\partial^2 U}{\partial t^2} = \sum_{j=1}^{n} [a_j \frac{\partial^2 U}{\partial x_j^2} + \frac{\eta_j}{X_j} \frac{\partial U}{\partial x_j}] + g(x,t),$$

(here $a_j = const. > o, \eta_j = const. > o$) a representation of the solutions was obtained by using the solutions of the corresponding regularity equation. On this basis a solution of the Cauchy problem was received, when the boundary conditions are

given on the $t = o$ surface. In [3], the equation

$$\frac{\partial^2 U}{\partial t^2} + \frac{\eta}{t}\frac{\partial U}{\partial t} - (\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}) = f(x,y,t).$$

(in the case when $\eta > 0$ in region $x > 0$, $t > 0$, $-\infty < y < \infty$) was solved

$$U(x,0) = 0, \frac{\partial U(x,y,t)}{\partial t}\Big|_{t=0} = 0, U(0,y,t) = g(y,t).$$

Investigations of first and second order overdetermined systems with singular and super-singular coefficients are described in [6, 10]. A study of a nonlinear system of ordinary differential equations with singular and super-singular coefficient (in a case which is connected to the theory of P-Laplacians and psedo-Laplacians) is described in [8].

2. MAIN RESULTS

Let D be the following region: $D = \{0 < x < \lambda_1, 0 < y < \lambda_2\}$, which is bounded by

$$\Gamma_1 = \{0 < x < \lambda_1, y = 0\}, \quad \Gamma_2 = \{x = 0, 0 < y < \lambda_2\}.$$

In the region D we consider the following system:

$$\begin{cases} \frac{\partial u(x,y)}{\partial x \partial y} + \frac{a_1(x,y)}{y^\beta}\frac{\partial u(x,y)}{\partial x} + \frac{b_1(x,y)}{x^\alpha}\frac{\partial u(x,y)}{\partial y} + \frac{c_1(x,y)}{x^\alpha y^\beta}u(x,y) = \frac{f_1(x,y)}{x^\alpha y^\beta} \\ \frac{\partial u(x,y)}{\partial y} + \frac{b_2(x,y)}{y^\gamma}u(x,y) = \frac{f_2(x,y)}{y^\gamma} \end{cases} \quad (2.1)$$

where the coefficients $a_1(x,y)$, $b_j(x,y)$, $c_1(x,y)$, $f_j(x,y)$, $j = 1,\ 2$, are given functions in the domain $\overline{D}$. Moreover $\alpha = const. > 0$, $\beta = const. > 0$, $\gamma = const. > 0$.

2.1. *The case of $\alpha < 1, \beta < 1, \gamma < 1$.*

We first determine an integral representation of the solution of the given system in the case when the coefficients of the first equation are related together in a certain way. Then we investigate the case when the coefficients of the first equation of system (2.1) are not related to each other.

Let $C_y^1(\overline{D})$ denote the set functions which are continuous on $\overline{D}$ together with their derivatives with respect to the variable y.

We always assume that $b_1(x,y) \in C_y^1(\overline{D})$ and that the coefficients $a_1(x,y)$, $b_1(x,y)$, $b_2(x,y)$, $c_1(x,y)$ are continuous functions in the domain D.

According to [9], the first equation of system (2.1) can be written in the following form

$$(\frac{\partial}{\partial y} + \frac{a_1(x,y)}{y^\beta})(\frac{\partial}{\partial x} + \frac{b_1(x,y)}{x^\alpha})u(x,y) = \frac{f_1(x,y) + c_2(x,y)u(x,y)}{x^\alpha y^\beta}, \quad (2.2)$$

where

$$c_2(x,y) = y^\beta \frac{\partial b_1(x,y)}{\partial y} + a_1(x,y) \cdot b_1(x,y) - c_1(x,y).$$

2.1.1. *The case of $c_2(x, y) = 0$*

Suppose that the functions $a_1(x, y)$, $b_1(x, y)$ and $c_1(x, y)$ of the system (2.1) satisfy the following relation

$$c_1(x, y) = y^\beta \frac{\partial b_1(x, y)}{\partial y} + a_1(x, y) \cdot b_1(x, y)$$

for all $(x, y) \in D$.

Under this condition, the first equation of (2.1) can be written in the following form:

$$\left(\frac{\partial}{\partial y} + \frac{a_1(x, y)}{y^\beta}\right)\left(\frac{\partial}{\partial x} + \frac{b_1(x, y)}{x^\alpha}\right)u(x, y) = \frac{f_1(x, y)}{x^\alpha y^\beta}.$$

The solution of this equation is

$$u(x, y) = \exp[-\Omega_{b_1}^\alpha(x, y)][\psi_1(y) + \int_0^x \exp[\Omega_{b_1}^\alpha(t, y) - \Omega_{a_1}^\beta(t, y)] \cdot (\varphi_1(t) +$$

$$+ \int_0^y \exp[\Omega_{a_1}^\beta(t, s)] \cdot t^{-\alpha} s^{-\beta} \cdot f_1(t, s) ds) dt] \equiv K_1[\varphi_1(x), \psi_1(y), f_1(x, y)],$$

$$(2.3)$$

where $\varphi_1(x)$ and $\psi_1(y)$ are arbitrary continuous functions belonging to the class $C^1(\Gamma_1)$ and $C^1(\Gamma_2)$, respectively. In fact, these functions will be determined in the sequel,

$$\Omega_{b_1}^\alpha(x, y) = \int_0^x \frac{b_1(t, y)}{t^\alpha} dt, \qquad \Omega_{a_1}^\beta(x, y) = \int_0^y \frac{a_1(x, s)}{s^\beta} ds.$$

Now, we write the second equation of (2.1) in the following form:

$$\frac{\partial}{\partial y}[\exp[\Omega_{b_2}^\gamma(x, y)] \cdot u(x, y)] = \exp[\Omega_{b_2}^\gamma(x, y)] y^{-\gamma} f_2(x, y), \qquad (2.4)$$

where

$$\Omega_{b_2}^\gamma(x, y) = \int_0^y \frac{b_2(x, s)}{s^\gamma} ds.$$

By plugging $u(x, y)$ from (2.3) into (2.4) we obtain

$$\frac{\partial}{\partial y}\{\exp[\Omega_{b_2}^\gamma(x, y) - \Omega_{b_1}^\alpha(x, y)][\psi_1(y) + \int_0^x \exp[\Omega_{b_1}^\alpha(t, y) - \Omega_{a_1}^\beta(t, y)] \cdot (\varphi_1(t) +$$

$$+ \int_0^y \exp[\Omega_{a_1}^\beta(t, s)] t^{-\alpha} s^{-\beta} f_1(t, s) ds) dt]\} = f_2(x, y) y^{-\gamma} \exp[\Omega_{b_2}^\gamma(x, y)].$$

$$(2.5)$$

Suppose that the functions $b_1(x, y)$ and $b_2(x, y)$ satisfy the following relation

$$y^\gamma \frac{\partial b_1(x, y)}{\partial y} = x^\alpha \frac{\partial b_2(x, y)}{\partial x}. \qquad (2.6)$$

By differentiating and using the relation (2.6) , the equality (2.5) can be expressed in the form

$$\psi'_1(y) + \frac{b_2(0,y)}{y^\gamma}\psi_1(y) = f_2(x,y)y^{-\gamma}\cdot\exp[\Omega^\alpha_{b_1}(x,y)] - \frac{b_2(0,y)}{y^\gamma}$$

$$\cdot\int_0^x \exp[\Omega^\alpha_{b_1}(t,y) - \Omega^\beta_{a_1}(t,y)]\cdot(\varphi_1(t) + \int_0^y \exp[\Omega^\beta_{a_1}(t,s)]t^{-\alpha}s^{-\beta}f_1(t,s)ds)dt -$$

$$-\int_0^x \frac{\partial}{\partial y}\{\exp[\Omega^\alpha_{b_1}(t,y) - \Omega^\beta_{a_1}(t,y)]\cdot(\varphi_1(t) + \int_0^y \exp[\Omega^\beta_{a_1}(t,s)]t^{-\alpha}s^{-\beta}f_1(t,s)ds)\}dt$$

Since the left hand side does not depend on x, we have

$$\frac{\partial}{\partial x}[f_2(x,y)y^{-\gamma}\exp[\Omega^\alpha_{b_1}(x,y)]] - \frac{\partial}{\partial y}\{\exp[\Omega^\alpha_{b_1}(x,y) - \Omega^\beta_{a_1}(x,y)]\cdot(\varphi_1(x) +$$

$$+ \int_0^y \exp[\Omega^\beta_{a_1}(x,s)]x^{-\alpha}s^{-\beta}f_1(x,s)ds)\} = b_2(0,y)y^{-\gamma}\exp[\Omega^\alpha_{b_1}(x,y) - \Omega^\beta_{a_1}(x,y)]$$

$$\cdot(\varphi_1(x) + \int_0^y \exp[\Omega^\beta_{a_1}(x,s)]x^{-\alpha}s^{-\beta}f_1(x,s)ds),$$

$$(2.7)$$

or in a simpler form

$$x^\alpha\frac{\partial f_2(x,y)}{\partial x} + b_1(x,y)f_2(x,y) = (b_2(x,y) - a_1(x,y)y^{\gamma-\beta})\exp[-\Omega^\beta_{a_1}(x,y)]\cdot$$

$$\cdot[x^\alpha\varphi_1(x) + \int_0^y \exp[\Omega^\beta_{a_1}(x,s)]s^{-\beta}f_1(x,s)ds] + f_1(x,y)y^{\gamma-\beta}, (x,y) \in D,$$

$$(2.8)$$

Making use of (2.7), we can define $\psi_1(y)$ by the following ordinary differential equation

$$\psi'_1(y) + \frac{b_2(0,y)}{y^\gamma}\psi_1(y) = \frac{f_2(0,y)}{y^\gamma}.$$

According to [6,9], the solution of this equation is represented by the following form

$$\psi_1(y) = \exp[-\Omega^\gamma_{b_2}(0,y)][C + \int_0^y \exp[\Omega^\gamma_{b_2}(0,s)]s^{-\gamma}f_2(0,s)ds], \qquad (2.9)$$

where C is an arbitrary constant.

From expression (2.8), the function $x^\alpha\varphi_1(x)$ can be expressed in the form

$$x^\alpha\varphi_1(x) = \frac{x^\alpha\frac{\partial f_2(x,y)}{\partial x} + b_1(x,y)f_2(x,y) - f_1(x,y)y^{\gamma-\beta}}{b_2(x,y) - a_1(x,y)y^{\gamma-\beta}}\cdot\exp[\Omega^\beta_{a_1}(x,y)] -$$

$$-\int_0^y \exp[\Omega^\beta_{a_1}(x,s)]s^{-\beta}f_1(x,s)ds, (b_2(x,y) \neq a_1(x,y)y^{\gamma-\beta}).$$

$$(2.10)$$

We note that the left part of this expression is independent of the variable y, so we have

$$\frac{\partial}{\partial y}\{\frac{x^{\alpha}\frac{\partial f_2(x,y)}{\partial x}+b_1(x,y)f_2(x,y)-f_1(x,y)y^{\gamma-\beta}}{b_2(x,y)-a_1(x,y)y^{\gamma-\beta}}\cdot\exp[\Omega_{a_1}^{\beta}(x,y)]\}=$$
$$=\exp[\Omega_{a_1}^{\beta}(x,y)]y^{-\beta}f_1(x,y).$$
(2.11)

By using (2.11) and assuming that the following limit exists,

$$\lim_{y\to 0}\{\frac{x^{\alpha}\frac{\partial f_2(x,y)}{\partial x}+b_1(x,y)f_2(x,y)-f_1(x,y)y^{\gamma-\beta}}{b_2(x,y)-a_1(x,y)y^{\gamma-\beta}}\cdot\exp[\Omega_{a_1}^{\beta}(x,y)]\}=F_1(x),$$
(2.12)

the function $\varphi_1(x)$ can be determined by the following formula:

$$\varphi_1(x)=F_1(x)x^{-\alpha},\quad \alpha<1,\quad F_1(x)\in C(\overline{\Gamma_1}).$$
(2.13)

In the particular case of $\gamma=\beta$, if $y\to 0$, then

$$\varphi_1(x)=x^{-\alpha}[\frac{x^{\alpha}\frac{\partial f_2(x,0)}{\partial x}+b_1(x,0)f_2(x,0)-f_1(x,0)}{b_2(x,0)-a_1(x,0)}]$$

from equality (2.10).

After simplification, condition (2.11) can be written in the following form:

$$x^{\alpha}[b_2(x,y)-a_1(x,y)y^{\gamma-\beta}]\frac{\partial^2 f_2(x,y)}{\partial x\partial y}$$
$$+\{\frac{a_1(x,y)}{y^{\beta}}[b_2(x,y)-a_1(x,y)y^{\gamma-\beta}]x^{\alpha}-$$
$$x^{\alpha}\frac{\partial}{\partial y}[b_2(x,y)-a_1(x,y)y^{\gamma-\beta}]\}\frac{\partial f_2(x,y)}{\partial x}+$$
$$b_1(x,y)[b_2(x,y)-a_1(x,y)y^{\gamma-\beta}]\frac{\partial f_2(x,y)}{\partial y}+$$
$$+\{\frac{a_1(x,y)}{y^{\beta}}b_1(x,y)\cdot[b_2(x,y)-a_1(x,y)y^{\gamma-\beta}]-$$
$$b_1(x,y)\frac{\partial}{\partial y}[b_2(x,y)-a_1(x,y)y^{\gamma-\beta}]+$$
$$+[b_2(x,y)-a_1(x,y)y^{\gamma-\beta}]\cdot\frac{\partial b_1(x,y)}{\partial y}\}f_2(x,y)$$
$$=y^{\gamma-\beta}[b_2(x,y)-a_1(x,y)y^{\gamma-\beta}]\frac{\partial f_1(x,y)}{\partial y}+$$
$$+\{[b_2(x,y)-a_1(x,y)y^{\gamma-\beta}]^2y^{-\beta}$$

$$+\frac{a_1(x,y)}{y^\beta}[b_2(x,y)-a_1(x,y)y^{\gamma-\beta}]y^{\gamma-\beta}-y^{\gamma-\beta}\frac{\partial}{\partial y}[b_2(x,y)-$$

$$-a_1(x,y)y^{\gamma-\beta}]+(\gamma-\beta)y^{\gamma-\beta-1}\cdot[b_2(x,y)-a_1(x,y)y^{\gamma-\beta}]\}f_1(x,y), \quad (2.14)$$

where $(x,y)\in D$.

In what the follows, $C_{xy}^2(D)$ will denote the set of all functions $u(x,y)$ that are continuous on D together with their first derivatives, and $u_{xy}\in C(D)$.

Henceforth, we have the following conclusion:

Theorem 1. *Let $\alpha<1$, $\beta<1$, $\gamma<1$ in system (2.1), and assume that the coefficients and right hand sides of this system satisfy the following conditions:*

1) $a_1(x,y),b_2(x,y),f_2(x,y)\in C_x^1(\overline{D}),b_1(x,y),f_1(x,y)\in C_y^1(\overline{D}),f_2(x,y)\in$
   $C'_{xy}(D),\frac{\partial}{\partial y}[b_2(x,y)-a_1(x,y)y^{\gamma-\beta}]\in C(D),$
2) $c_1(x,y)=y^\beta\frac{\partial b_1(x,y)}{\partial y}+a_1(x,y)\cdot b_1(x,y),$
3) $y^\gamma\frac{\partial b_1(x,y)}{\partial y}=x^\alpha\frac{\partial b_2(x,y)}{\partial x},$
4) *The functions $f_1(x,y)$ and $f_2(x,y)$ satisfy conditions (2.8), (2.1.1),*
5) *The limit (2.12) exists.*

Then the desired solution of system (2.1) of class $C_{xy}^2(D)$ is determined by (2.3), in which $\varphi_1(x)$ which is determined by formula (2.13), where the value of $\psi_1(y)$ is defined by formula (2.9), provided that C is an arbitrary constant.

Example 1. Assume that in (2.1) $\alpha<1,\beta<1,\gamma<1,\beta=\gamma$ and the coefficients are constants, moreover, suppose that the right parts are defined by the following functions:

$$f_1(x,y)=x^\alpha\exp[b_1\omega_\alpha(x)+a_1\omega_\beta(y)],$$
$$f_2(x,y)=x\exp[b_1\omega_\alpha(x)+a_1\omega_\beta(y)]E(y).$$

where

$$E(y)=1-(b_2-a_1)\omega_\beta(y),\quad \omega_\beta(y)=\frac{1}{(\beta-1)y^{\beta-1}},\quad \omega_\alpha(x)=\frac{1}{(\alpha-1)x^{\alpha-1}}.$$

Then the function of the form

$$u(x,y)=c\exp[b_2\omega_\gamma(y)+b_1\omega_\alpha(x)]-x\omega_\beta(y)\exp[a_1\omega_\beta(y)+b_1\omega_\alpha(x)]$$

is the solution of system (2.1), where c is an arbitrary constant .

2.1.2. $c_2(x,y)\neq 0$

Suppose that the functions $a_1(x,y)$, $b_1(x,y)$, $c_1(x,y)$ satisfy the following relation

$$c_1(x,y)\neq y^\beta\frac{\partial b_1(x,y)}{\partial y}+a_1(x,y)\cdot b_1(x,y), \quad (2.15)$$

for all $(x, y) \in D$ in the system (2.1).

In this case, the first equation of system (2.1), can be represented in the form (2.2). We denote $f_3(x, y) = f_1(x, y) + c_2(x, y)u(x, y)$ where $f_3(x, y)$ is a given function. Then the solution of equation (2.2) can be written as

$$u(x, y) = K_1[\varphi_1(x), \psi_1(y), f_3(x, y)]. \tag{2.16}$$

Using these items that are dependent on $u(x, y)$, we obtain the following integral equation:

$$u(x, y) - \int_0^x dt \int_0^y [\exp[\Omega_{b_1}^\alpha(t, y) - \Omega_{b_1}^\alpha(x, y) + \Omega_{a_1}^\beta(t, s) - \Omega_{a_1}^\beta(t, y)] \cdot$$
$$\cdot \frac{c_2(t, s)}{t^\alpha s^\beta} u(t, s) ds = K_1[\varphi_1(x), \psi_1(y), f_1(x, y)]. \tag{2.17}$$

In fact, the problem of finding the solution of the overdetermined system (2.1), in the case of $\alpha < 1, \beta < 1, \gamma < 1$, is reduced to the solution of a two dimensional Volterra integral equation with two weak singular lines in the kernel (2.17).

According to [7,11], the solution of the integral equation (2.17) can be represented in the following form

$$u(x, y) = K_1[\varphi_1(x), \psi_1(y), f_1(x, y)] +$$
$$\int_0^x dt \int_0^y \Gamma(x, y; t, s) K_1[\varphi_1(t), \psi_1(s), f_1(t, s)] ds$$
$$\equiv T_{1,1}[\varphi_1(x), \psi_1(y), f_1(x, y)], \tag{2.18}$$

where $\Gamma(x, y; t, s)$ is the resolvent of the integral equation (2.17).

We now return to the problem of finding the arbitrary functions $\varphi_1(x), \psi_1(y)$. We repeat the scheme of 2.1.1 in the case when $\gamma = \beta$, and $y \to 0$. We obtain

$$\varphi_1(x) = x^{-\alpha}\left[\frac{x^\alpha \frac{\partial f_2(x,0)}{\partial x} + b_1(x, 0) f_2(x, 0) - f_1(x, 0) - c_2(x, 0)u(x, 0)}{b_2(x, 0) - a_1(x, 0)}\right], \tag{2.19}$$

$(b_2(x, 0) \neq a_1(x, 0))$,

$$\psi_1(y) = \exp[-\Omega_{b_2}^\gamma(0, y)][C + \int_0^y \exp[\Omega_{b_2}^\gamma(0, s)]s^{-\gamma} f_2(0, s) ds], \tag{2.20}$$

$\psi_1(0) = C$,

$$u(x, 0) = \exp[-\Omega_{b_1}^\alpha(x, 0)][\psi_1(0) + \int_0^x \exp[\Omega_{b_1}^\alpha(t, 0)]\varphi_1(t) dt]. \tag{2.21}$$

By substituting the value of $\varphi_1(x)$ from (2.19) into (2.21), we obtain the following one dimensional Volterra type integral equation:

$$u(x,0) + \int_0^x \exp[\Omega_{b_1}^\alpha(t,0) - \Omega_{b_1}^\alpha(x,0)] \cdot \frac{c_2(t,0)u(t,0)}{b_2(t,0) - a_1(t,0)} \frac{dt}{t^\alpha} = \exp[-\Omega_{b_1}^\alpha(x,0)]$$

$$\cdot \{\psi_1(0) + \int_0^x \exp[\Omega_{b_1}^\alpha(t,0)][\frac{t^\alpha \frac{\partial f_2(t,0)}{\partial t} + b_1(t,0)f_2(t,0) - f_1(t,0)}{b_2(t,0) - a_1(t,0)}]\frac{dt}{t^\alpha}\}, \quad (2.22)$$

Take a new unknown function

$$W(x) = u(x,0)\exp[\Omega_{b_1}^\alpha(x,0)]. \quad (2.23)$$

Then it follows that the integral equation (2.22) can be represented in the following form

$$W(x) + \int_0^x \frac{c_2(t,0)W(t)}{b_2(t,0) - a_1(t,0)} \frac{dt}{t^\alpha} dt = E(x), \ \alpha < 1, \quad (2.24)$$

where

$$E(x) = \{\psi_1(0) + \int_0^x \exp[\Omega_{b_1}^\alpha(t,0)][\frac{t^\alpha \frac{\partial f_2(t,0)}{\partial t} + b_1(t,0)f_2(t,0) - f_1(t,0)}{b_2(t,0) - a_1(t,0)}]\frac{dt}{t^\alpha}\},$$

and $\alpha < 1$. According to [2], the solution of equation (2.24) is given by the following formula

$$W(x) = E(x) - \int_0^x \exp[\Omega_{c_3}^\alpha(t,0) - W_{c_3}^\alpha(x,0)]\frac{c_3(t)}{t^\alpha}E(t)dt \equiv T_2[E(x)],$$

where

$$c_3(x) = \frac{c_2(x,0)}{b_2(x,0) - a_1(x,0)}, \ W_{c_3}^\alpha(x,0) = \int_0^x \frac{c_3(t)}{t^\alpha}dt.$$

Using formula (2.23), the solution of integral equation (2.22) can be written as

$$u(x,0) = \exp[-\Omega_{b_1}^\alpha(x,0)T_2[E(x)].$$

By substituting the value of $u(x,0)$ into (2.19), we obtain the value of $\phi_1(x)$:

$$\varphi_1(x) =$$

$$x^{-\alpha}[\frac{x^\alpha \frac{\partial f_2(x,0)}{\partial x} + b_1(x,0)f_2(x,0) - f_1(x,0) - c_2(x,0)\exp[-\Omega_{b_1}^\alpha(x,0)]T_2[E(x)]}{b_2(x,0) - a_1(x,0)}],$$

$$(2.25)$$

Then we substitute the values of $\varphi_1(x)$, $\psi_1(y)$ which were obtained from (2.1.2) and (2.20), respectively, in formula (2.17).

Henceforth, we deduce the following conclusion

Theorem 2. *Let $\alpha < 1, \beta < 1, \gamma < 1, \gamma = \beta$ in system* (2.1). *Assume the followings: the coefficients and right hand sides of this system satisfies the conditions: 1, 3, 5, from theorem 1, and $c_2(x,y) \neq 0$. Moreover, suppose that the functions $f_1(x,y), f_2(x,y)$ satisfy the following conditions:*

(1)

$$x^\alpha \frac{\partial f_2(x,y)}{\partial x} + b_1(x,y) f_2(x,y)$$

$$= (b_2(x,y) - a_1(x,y)y^{\gamma-\beta}) \exp[-W_{a_1}^\beta(x,y)][x^\alpha \varphi_1(x) +$$

$$+ \int_0^y \exp[\Omega_{a_1}^\beta(x,s)]s^{-\beta}$$

$$\{f_1(x,s) + c_2(x,s)T_{1,1}[\varphi_1(x), \psi_1(y), f_1(x,y)]\}ds] +$$

$$+ y^{\gamma-\beta}[f_1(x,y) + c_2(x,y)T_{1,1}[\varphi_1(x), \psi_1(y), f_1(x,y)],$$

(2)

$$x^\alpha[b_2(x,y) - a_1(x,y)y^{\gamma-\beta}]\frac{\partial^2 f_2(x,y)}{\partial x \partial y}$$

$$+ \{\frac{a_1(x,y)}{y^\beta}[b_2(x,y) - a_1(x,y)y^{\gamma-\beta}]x^\alpha - x^\alpha \frac{\partial}{\partial y}[b_2(x,y) -$$

$$- a_1(x,y)y^{\gamma-\beta}]\}\frac{\partial f_2(x,y)}{\partial x} + b_1(x,y)[b_2(x,y) - a_1(x,y)y^{\gamma-\beta}]\frac{\partial f_2(x,y)}{\partial y} +$$

$$\{\frac{a_1(x,y)}{y^\beta}b_1(x,y)[b_2(x,y) - a_1(x,y)y^{\gamma-\beta}] - b_1(x,y)\frac{\partial}{\partial y}[b_2(x,y) - a_1(x,y)y^{\gamma-\beta}]$$

$$+ [b_2(x,y) - a_1(x,y)y^{\gamma-\beta}]\frac{\partial b_1(x,y)}{\partial y}\}f_2(x,y)$$

$$= y^{\gamma-\beta}[b_2(x,y) - a_1(x,y)y^{\gamma-\beta}]$$

$$\frac{\partial}{\partial y}[f_1(x,y) + c_2(x,y) \cdot T_{1,1}[\varphi_1(x), \psi_1(y), f_1(x,y)]] +$$

$$\{[b_2(x,y) - a_1(x,y)y^{\gamma-\beta}]^2 y^{-\beta} + \frac{a_1(x,y)}{y^\beta}$$

$$\cdot [b_2(x,y) - a_1(x,y)y^{\gamma-\beta}]y^{\gamma-\beta}$$

$$- y^{\gamma-\beta}\frac{\partial}{\partial y}[b_2(x,y) - a_1(x,y)y^{\gamma-\beta}] + (\gamma-\beta)y^{\gamma-\beta-1}$$

$$\cdot [b_2(x,y) - a_1(x,y)y^{\gamma-\beta}]\}[f_1(x,y) + c_2(x,y)T_{1,1}[\varphi_1(x), \psi_1(y), f_1(x,y)]].$$

Then the desired solution of system (2.1) *in the class $C_{xy}^2(D)$ is determined by* (2.3), *in which $\varphi_1(x)$ is defined by* (2.1.2) *and $\psi_1(y)$ is determined by formula* (2.20) *, where C is an arbitrary constant.*

2.2. *The case of* $b_1(x,y) \in C_y^1(\overline{D})$, $\alpha > 1$, $\beta > 1$, $\gamma > 1$.

In system (2.1) let $\alpha > 1$, $\beta > 1$, $\gamma > 1$, $c_2(x,y) = 0$. By repeating the method that was explained in section 2.1.1, we obtain the following result:

Theorem 3. *Let* $\alpha > 1$, $\beta > 1$, $\gamma > 1$, *in system* (2.1) *and assume that the coefficients and right hand sides of this system satisfy the conditions 1, 2, 3 of Theorem 1. Suppose also that* $a_1(0,0) > 0$, $b_1(0,0) > 0$, $b_2(0,0) > 0$. *Moreover, assume the existence of the following limits:*

$$E_1(x) = \lim_{y \to 0} \{ \frac{x^\alpha \frac{\partial f_2(x,y)}{\partial x} + b_1(x,y) f_2(x,y) - f_1(x,y) y^{\gamma-\beta}}{b_2(x,y) - a_1(x,y) y^{\gamma-\beta}}$$
$$\exp[W_{a_1}^\beta(x,y) - a_1(0,0)\omega_\beta(y)] \},$$

$$E_2(y) = \lim_{x \to 0} (\exp[-b_1(0,0)\omega_\alpha(x)] f_2(x,y)),$$

where $E_1(x) \in C(\overline{\Gamma_1})$, $E_2(y) \in C(\overline{\Gamma_2})$.

Besides, suppose that the functions $f_1(x,y)$, $f_2(x,y)$ *satisfy the condition* (2.11) *and it is also true that*

$$x^\alpha \frac{\partial f_2(x,y)}{\partial x} + b_1(x,y) f_2(x,y)$$
$$= (b_2(x,y) - a_1(x,y) y^{\gamma-\beta}) \exp[a_1(0,0)\omega_\beta(y) - W_{a_1}^\beta(x,y)]$$
$$(x^\alpha \varphi_2(x) + \int_0^y \exp[W_{a_1}^\beta(x,s) - a_1(0,0)\omega_\beta(s)] s^{-\beta} f_1(x,s) ds) + f_1(x,y) y^{\gamma-\beta},$$

$(x,y) \in D$, $\varphi_2(x) = E_1(x) x^{-\alpha}$, $\alpha > 1$, $\beta > 1$.

Moreover, assume that the functions, $a_1(x,y)$, $b_1(x,y)$, $b_2(x,y)$ *in he neighborhoods* Γ_1 *and* Γ_2 *satisfy the following conditions:*

$$|b_1(x,y) - b_1(0,0)| \leq H_1 x^{\delta_3}, \quad \delta_3 > \alpha - 1, \quad at \quad x \to 0;$$
$$|a_1(x,y) - a_1(0,0)| \leq H_2 y^{\delta_4}, \quad \delta_4 > \beta - 1, \quad at \quad y \to 0;$$
$$|b_2(x,y) - b_2(0,0)| \leq H_3 x^{\delta_5}, \quad \delta_5 > \gamma - 1, \quad at \quad y \to 0;$$

Then the desired solution of system (2.1) *in the class* $C_{xy}^2(D)$ *is determined by:*

$$u(x,y) = \exp[-W_{b_1}^\alpha(x,y) + b_1(0,0)\omega_\alpha(x)]\{\psi_2(y) +$$
$$+ \int_0^x \exp[W_{b_1}^\alpha(t,y) - b_1(0,0)\omega_\alpha(t) + a_1(0,0)\omega_\beta(y) - W_{a_1}^\beta(t,y)] \cdot (\varphi_2(t) +$$
$$+ \int_0^y \exp[W_{a_1}^\beta(t,s) - a_1(0,0)\omega_\beta(s)] \cdot t^{-\alpha} s^{-\beta} \cdot f_1(t,s) ds) dt, \} \qquad (2.26)$$

where

$$\psi_2(y) = \exp[b_2(0,0)\omega_\gamma(y) - W_{b_2}^\gamma(0,y)][C$$

$$+ \int_0^y \exp[W_{b_2}^\gamma(0,s) - b_2(0,0)\omega_\gamma(s)]s^{-\gamma} E_2(s)ds],$$

(here C is an arbitrary constant), and

$$W_{b_1}^\alpha(x,y) = \int_0^x \frac{b_1(t,y) - b_1(0,0)}{t^\alpha}dt, \quad W_{a_1}^\beta(x,y) = \int_0^y \frac{a_1(x,s) - a_1(0,0)}{s^\beta}ds,$$

$$W_{b_2}^\gamma(x,y) =$$

$$\int_0^y \frac{b_2(x,s) - b_2(0,0)}{s^\gamma}ds, \quad \omega_\gamma(y) = \frac{1}{(\gamma-1)y^{\gamma-1}}, \quad \omega_\alpha(x) = \frac{1}{(\alpha-1)x^{\alpha-1}}.$$

Remark 2. The integral representation (3) also satisfies the system (2.1) when $b_2(0,0) < 0$, $a_1(0,0) < 0$, $b_1(0,0) < 0$, and the functions $E_2(y)$, $E_1(x)$ are such that $E_2(0) = 0$, $E_1(0) = 0$, and they have the following asymptotic behaviors:

$$E_2(y) = 0[\exp[-|b_2(0,0)|\omega_\gamma(y)]y^{\delta_1}], \quad \delta_1 > \gamma - 1, \quad as \quad y \to 0,$$

$$E_1(x) = 0[\exp[-|b_1(0,0)|\omega_\alpha(x)] \cdot x^{\delta_2}], \quad \delta_2 < \alpha - 1, \quad as \quad x \to 0.$$

Moreover, the function $f_1(x,y)$ satisfies that $f_1(x,0) = 0$, $f_1(0,y) = 0$, and it has the following asymptotic behavior:

$$f_1(x,y) = 0[\exp[-|a_1(0,0)|\omega_\beta(y)] \cdot y^{-\delta_4}], \quad y \to 0,$$

$$f_1(x,y) = 0[\exp[-|b_1(0,0)|\omega_\alpha(x)] \cdot x^{-\delta_3}], \quad x \to 0.$$

Remark 1. If the coefficients of system (2.1) satisfy all of the conditions of Theorem 3, then the solution of the system (2.1) is unbounded in Γ_1, Γ_2, and its behavior is determined by the following asymptotic conditions:

$$u(x,y) = O[\exp(b_1(0,0)\omega_\alpha(x))] \text{ at } x \to 0,$$

and

$$u(x,y) = O[\exp(b_2(0,0)\omega_\gamma(y))] \text{ at } y \to 0, \quad (\gamma > \beta \quad \text{or} \quad \gamma = \beta, \, b_2(0,0) > a_1(0,0)).$$

Remark 2. For all possible cases of the parameters α, β and γ, integral representations of the solutions of the system (2.1) are obtained not only in the case when $c_1(x,y) = y^\beta \frac{\partial b_1(x,y)}{\partial y} + a_1(x,y) \cdot b_1(x,y)$, but also when $c_1(x,y) \neq y^\beta \frac{\partial b_1(x,y)}{\partial y} + a_1(x,y) \cdot b_1(x,y)$.

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