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METHOD OF LOWER AND UPPER FUNCTIONS AND THE EXISTENCE OF SOLUTIONS TO SINGULAR PERIODIC PROBLEMS FOR NONLINEAR DIFFERENTIAL EQUATIONS OF ORDER TWO

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Abstract. We construct nonconstant lower and upper functions for the periodic boundary value problem $u'' = f(t, u)$, $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and find their estimates. By means of these results we prove existence criteria for the problems $u'' \pm g(u) = e(t)$, $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$, where $\limsup_{x \rightarrow 0^+} g(x) = \infty$ is allowed and $e \in \mathbb{L}[0, 2\pi]$ need not be essentially bounded.

Mathematical Subject Classification: 34B15, 34C25

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1. Introduction

In this paper we construct lower and upper functions to the periodic boundary value problem

$$u'' = f(t, u), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \quad (1.1)$$

By means of these results, we prove existence criteria for the problems

$$u'' \pm g(u) = e(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$

where $\limsup_{x \rightarrow 0^+} g(x) = \infty$ is allowed and $e \in \mathbb{L}[0, 2\pi]$ need not be essentially bounded. We assume that $f : [0, 2\pi] \times \mathbb{R} \mapsto \mathbb{R}$ fulfills the Carathéodory conditions on $[0, 2\pi] \times \mathbb{R}$, i.e. f has the following properties: (i) for each $x \in \mathbb{R}$ the function $f(\cdot, x)$ is measurable on $[0, 2\pi]$; (ii) for almost every $t \in [0, 2\pi]$ the function $f(t, \cdot)$ is continuous on \mathbb{R} ; (iii) for each compact set $K \subset \mathbb{R}$ the function $m_K(t) = \sup_{x \in K} |f(t, x)|$ is Lebesgue integrable on $[0, 2\pi]$.

For a given subinterval J of \mathbb{R} (possibly unbounded), $\mathbb{C}(J)$ denotes the set of functions continuous on J . Furthermore, $\mathbb{L}[0, 2\pi]$ stands for the set of functions Lebesgue integrable on $[0, 2\pi]$, $\mathbb{L}_2[0, 2\pi]$ is the set of functions square Lebesgue integrable on $[0, 2\pi]$ and $\mathbb{AC}[0, 2\pi]$ denotes the set of functions absolutely continuous on $[0, 2\pi]$. For x bounded on $[0, 2\pi]$, $y \in \mathbb{L}[0, 2\pi]$ and $z \in \mathbb{L}_2[0, 2\pi]$ we denote

$$\|x\|_{\mathbb{C}} = \sup_{t \in [0, 2\pi]} |x(t)|, \quad \bar{y} = \frac{1}{2\pi} \int_0^{2\pi} y(s) ds,$$

$$\|y\|_1 = \int_0^{2\pi} |y(t)| dt \quad \text{and} \quad \|z\|_2 = \left(\int_0^{2\pi} z^2(t) dt \right)^{\frac{1}{2}}.$$

By a *solution of (1.1)* we mean a function $u : [0, 2\pi] \mapsto \mathbb{R}$ such that $u' \in \mathbb{AC}[0, 2\pi]$, $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and

$$u''(t) = f(t, u(t)) \quad \text{for a.e. } t \in [0, 2\pi].$$

Definition 1. A function σ_1 is said to be a *lower function of the problem (1.1)* if $\sigma_1' \in \mathbb{AC}[0, 2\pi]$,

$$\begin{aligned} \sigma_1''(t) &\geq f(t, \sigma_1(t)) \quad \text{for a.e. } t \in [0, 2\pi], \\ \sigma_1(0) &= \sigma_1(2\pi), \quad \sigma_1'(0) \geq \sigma_1'(2\pi). \end{aligned}$$

Similarly, a function σ_2 is said to be an *upper function of the problem (1.1)* if $\sigma_2' \in \mathbb{AC}[0, 2\pi]$,

$$\begin{aligned} \sigma_2''(t) &\leq f(t, \sigma_2(t)) \quad \text{for a.e. } t \in [0, 2\pi], \\ \sigma_2(0) &= \sigma_2(2\pi), \quad \sigma_2'(0) \leq \sigma_2'(2\pi). \end{aligned}$$

The lower and upper functions approach we will use here is based on the following theorem which is contained in [8, Theorems 4.1 and 4.2].

Theorem 2. *Let σ_1 and σ_2 be a lower and an upper function of the problem (1.1), respectively.*

(I) *Suppose $\sigma_1(t) \leq \sigma_2(t)$ on $[0, 2\pi]$. Then there is a solution u of the problem (1.1) such that $\sigma_1(t) \leq u(t) \leq \sigma_2(t)$ on $[0, 2\pi]$.*

(II) *Suppose $\sigma_1(t) \geq \sigma_2(t)$ on $[0, 2\pi]$ and there is $m \in \mathbb{L}[0, 2\pi]$ such that*

$$\begin{aligned} f(t, x) &\geq m(t) \quad \text{for a.e. } t \in [0, 2\pi] \text{ and all } x \in \mathbb{R} \\ \text{(or } f(t, x) &\leq m(t) \quad \text{for a.e. } t \in [0, 2\pi] \text{ and all } x \in \mathbb{R}.) \end{aligned}$$

Then there is a solution u of the problem (1.1) such that $\|u'\|_{\mathbb{C}} \leq \|m\|_1$ and

$$\sigma_2(t_u) \leq u(t_u) \leq \sigma_1(t_u) \quad \text{for some } t_u \in [0, 2\pi].$$

2. Construction of lower and upper functions

Proposition 1. *Assume that there are $A \in \mathbb{R}$ and $b \in L[0, 2\pi]$ such that*

$$\bar{b} = 0, \quad (2.1)$$

$$f(t, x) \leq b(t) \text{ for a.e. } t \in [0, 2\pi] \text{ and all } x \in [A, B], \quad (2.2)$$

where

$$B = A + \frac{\pi}{3} \|b\|_1. \quad (2.3)$$

Then there exists a lower function σ of the problem (1.1) such that

$$A \leq \sigma(t) \leq B \text{ on } [0, 2\pi]. \quad (2.4)$$

Proof. Define

$$\sigma_0(t) = c_0 + \int_0^{2\pi} g(t, s)b(s)ds \text{ for } t \in [0, 2\pi],$$

where

$$g(t, s) = \begin{cases} \frac{t(s-2\pi)}{2\pi} & \text{if } 0 \leq t \leq s \leq 2\pi, \\ \frac{(t-2\pi)s}{2\pi} & \text{if } 0 \leq s < t \leq 2\pi \end{cases}$$

is the Green function of the problem $v'' = 0$, $v(0) = v(2\pi) = 0$ and

$$c_0 = -\frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^{2\pi} g(t, s)b(s)ds \right) dt.$$

Then

$$\sigma_0''(t) = b(t) \text{ a.e. on } [0, 2\pi] \quad (2.5)$$

and

$$\sigma_0(0) = \sigma_0(2\pi). \quad (2.6)$$

Furthermore, by virtue of (2.1) we have also

$$\sigma_0'(0) = \sigma_0'(2\pi). \quad (2.7)$$

Multiplying the relation (2.5) by σ_0 , integrating it over $[0, 2\pi]$ and using the Hölder inequality we get

$$\|\sigma_0'\|_2^2 \leq \|b\|_1 \|\sigma_0\|_C.$$

Further, as $\bar{\sigma}_0 = 0$, the Sobolev inequality (see [5, Proposition 1.3]) yields

$$\|\sigma_0'\|_2^2 \leq \sqrt{\frac{\pi}{6}} \|b\|_1 \|\sigma_0'\|_2,$$

and so

$$\|\sigma'_0\|_2 \leq \sqrt{\frac{\pi}{6}} \|b\|_1,$$

wherefrom using again the Sobolev inequality we get

$$\|\sigma_0\|_C \leq \frac{\pi}{6} \|b\|_1.$$

Thus, the function σ given by

$$\sigma(t) = \frac{\pi}{6} \|b\|_1 + A + \sigma_0(t) \text{ for } t \in [0, 2\pi] \quad (2.8)$$

satisfies (2.4). Furthermore, according to (2.1), (2.2) and (2.6)-(2.7) we have

$$\sigma''(t) = \sigma''_0(t) = b(t) \geq f(t, \sigma(t)) \text{ for a.e. } t \in [0, 2\pi] \quad (2.9)$$

and

$$\sigma(0) = \sigma(2\pi), \quad \sigma'(0) = \sigma'(2\pi), \quad (2.10)$$

i.e. σ is a lower function of (1.1). \square

The following assertion is dual to Proposition 1 and its proof will be omitted.

Proposition 2. *Assume that there are $A \in \mathbb{R}$ and $b \in \mathbb{L}[0, 2\pi]$ such that*

$$\bar{b} = 0$$

and

$$f(t, x) \geq b(t) \text{ for a.e. } t \in [0, 2\pi] \text{ and all } x \in [A, B]$$

where B is given by (2.3). Then there exists an upper function σ of the problem (1.1) with the property (2.4).

3. Applications to Lazer-Solimini singular problems

In this section we will consider possible singular problems of the attractive type

$$u'' + g(u) = e(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \quad (3.1)$$

and of the repulsive type

$$u'' - g(u) = e(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \quad (3.2)$$

where

$$g \in \mathbb{C}(0, \infty) \text{ and } e \in \mathbb{L}[0, 2\pi] \quad (3.3)$$

and it is allowed that $\limsup_{x \rightarrow 0^+} g(x) = \infty$.

The problem (3.1) has been studied by Lazer and Solimini in [6] for $e \in \mathbb{C}[0, 2\pi]$ and g positive. In [9, Corollary 3.3], their existence result has been extended to $e \in \mathbb{L}[0, 2\pi]$ essentially bounded from above. Here, we bring conditions for the existence of solutions to (3.1) without assuming boundedness of e .

Theorem 1. *Assume (3.3) and let there exist $A_1, A_2 \in (0, \infty)$ such that*

$$g(x) \geq \bar{e} \text{ for all } x \in [A_1, B_1], \quad (3.4)$$

$$g(x) \leq \bar{e} \text{ for all } x \in [A_2, B_2], \quad (3.5)$$

where

$$B_1 - A_1 = B_2 - A_2 = \frac{\pi}{3} \|e - \bar{e}\|_1 \quad (3.6)$$

and $A_2 \geq B_1$.

Then the problem (3.1) has a solution u such that $A_1 \leq u(t) \leq B_2$ on $[0, 2\pi]$.

Proof. Define, for a.e. $t \in [0, 2\pi]$,

$$f(t, x) = e(t) - \begin{cases} g(A_1) & \text{if } x < A_1, \\ g(x) & \text{if } x \geq A_1. \end{cases}$$

Then f satisfies the Carathéodory conditions on $[0, 2\pi] \times \mathbb{R}$. Furthermore, by (3.4) and (3.6), f satisfies (2.1)-(2.3) with $b(t) = e(t) - \bar{e}$ a.e. on $[0, 2\pi]$ and $[A, B] = [A_1, B_1]$. Hence, by Proposition 1 there exists a lower function σ_1 of (1.1) such that $\sigma_1(t) \in [A_1, B_1]$ for all $t \in [0, 2\pi]$. Similarly, (3.5), (3.6) and Proposition 2 yield the existence of an upper function σ_2 of (1.1) such that $\sigma_2(t) \in [A_2, B_2]$ on $[0, 2\pi]$. Now, since $A_2 \geq B_1$, we have $\sigma_1(t) \leq \sigma_2(t)$ on $[0, 2\pi]$ and the assertion (I) of Theorem 2 gives the existence of the desired solution u to (1.1) which is naturally also a solution to (3.1). \square

Classical Lazer and Solimini's considerations [6] of the repulsive problem (3.2) have been extended by several authors (see e.g. [1]-[4], [7] and [10]). Here we present a related result, where e need not be essentially bounded.

Theorem 2. *Assume (3.3),*

$$\lim_{x \rightarrow 0^+} \int_x^1 g(\xi) d\xi = \infty, \quad (3.7)$$

and

$$g_* := \inf_{x \in (0, \infty)} g(x) > -\infty. \quad (3.8)$$

Furthermore, let there exist $A_1, A_2 \in (0, \infty)$ such that

$$g(x) \leq -\bar{e} \text{ for all } x \in [A_1, B_1], \quad (3.9)$$

$$g(x) \geq -\bar{e} \text{ for all } x \in [A_2, B_2], \quad (3.10)$$

where (3.6) is true and $A_1 \geq B_2$.

Then the problem (3.2) has a positive solution.

Proof. Denote

$$K = \|e\|_1 + |g_*|, \quad B = B_1 + 2\pi K \text{ and } K^* = K\|e\|_1 + \int_{A_2}^B |g(x)|dx.$$

It follows from (3.7) that $\limsup_{x \rightarrow 0^+} g(x) = \infty$ and there exists $\varepsilon \in (0, A_2)$ such that

$$\int_{\varepsilon}^{A_2} g(x)dx > K^* \text{ and } g(\varepsilon) > 0. \quad (3.11)$$

Define

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \geq \varepsilon, \\ g(\varepsilon) & \text{if } x < \varepsilon, \end{cases}$$

and

$$f(t, x) = e(t) + \tilde{g}(x) \text{ for a.e. } t \in [0, 2\pi] \text{ and all } x \in \mathbb{R}.$$

Now, we can argue as in the proof of Theorem 1 obtaining a lower function σ_1 and an upper function σ_2 of (1.1) such that $\sigma_1(t) \geq \sigma_2(t)$ on $[0, 2\pi]$. The assertion (II) of Theorem 2 (with $m(t) = g_* + e(t)$ a.e. on $[0, 2\pi]$) implies that (1.1) has a solution u such that $u(t_u) \in [A_2, B_1]$ for some $t_u \in [0, 2\pi]$ and $\|u'\|_{\mathbb{C}} \leq K$. It remains to show that $u(t) \geq \varepsilon$ holds on $[0, 2\pi]$.

Let t_0 and $t_1 \in [0, 2\pi]$ be such that

$$u(t_0) = \min_{t \in [0, 2\pi]} u(t) \text{ and } u(t_1) = \max_{t \in [0, 2\pi]} u(t).$$

Clearly, $A_2 \leq u(t_1) \leq B$. With respect to the periodic boundary conditions we have $u'(t_0) = u'(t_1) = 0$. Now, multiplying the differential relation $u''(t) = e(t) + \tilde{g}(u(t))$ by $u'(t)$ and integrating over $[t_0, t_1]$ we get

$$0 = \int_{t_0}^{t_1} u''(t)u'(t)dt = \int_{t_0}^{t_1} e(t)u'(t)dt + \int_{t_0}^{t_1} \tilde{g}(u(t))u'(t)dt,$$

i.e.

$$\int_{u(t_0)}^{u(t_1)} \tilde{g}(x)dx = - \int_{t_0}^{t_1} e(t)u'(t)dt \leq K\|e\|_1.$$

Further,

$$\int_{u(t_0)}^{A_2} \tilde{g}(x)dx \leq K\|e\|_1 + \int_{A_2}^B |\tilde{g}(x)|dx = K^*$$

which, with respect to (3.11), is possible only if $u(t_0) \geq \varepsilon$. Thus, u is a solution to (3.2). \square

Example 3. Let $g(x) = \frac{1}{x^\gamma}$ on $(0, \infty)$. If $\gamma > 0$, then Theorem 1 ensures the existence of a positive solution to (3.1) for any $e \in \mathbb{L}[0, 2\pi]$ such that

$$\bar{e} > 0 \text{ and } \frac{\pi}{3} \bar{e}^{\frac{1}{\gamma}} \|e - \bar{e}\|_{\mathbb{L}} < 1. \quad (3.12)$$

The function $e(t) = c + \frac{1}{\sqrt{2\pi t}} - \frac{1}{\pi}$ with $c \in \mathbb{R}$ is not essentially bounded from above on $[0, 2\pi]$. However, it satisfies (3.12) if

$$0 < c < \left(\frac{3}{\pi}\right)^\gamma.$$

We should mention that provided $e \in \mathbb{C}[0, 2\pi]$ or e is essentially bounded from above, the condition $\bar{e} > 0$ is sufficient for the existence of a solution to (3.1) (cf. [6] or [9], respectively).

Example 4. Let $e \in \mathbb{L}[0, 2\pi]$ be essentially unbounded from below and let

$$g(x) = \frac{1 + \sin\left(\frac{\pi}{x}\right)}{x} - \arctan(x), \quad x \in (0, \infty).$$

Then g verifies the assumptions (3.3), (3.7) and (3.8) of Theorem 2. Let us suppose that $\bar{e} = -5$. Then the equation $g(x) = 5$ has exactly 5 roots in the interval $[0.125, \infty)$. In particular, we have (see Figures 1 and 2)

$$\begin{aligned} x_1 &\approx 0.126804, \quad x_2 \approx 0.141071, \quad x_3 \approx 0.167853, \quad x_4 \approx 0.200541, \quad x_5 \approx 0.244461, \\ g(x) &> 5 \text{ on } (x_2, x_3) \cup (x_4, x_5) \text{ and } g(x) < 5 \text{ on } (x_1, x_2) \cup (x_3, x_4) \cup (x_5, \infty). \end{aligned}$$

Therefore, by Theorem 2, if

$$\|e - \bar{e}\|_{\mathbb{L}} \leq \frac{3}{\pi}(x_5 - x_4) \approx 0.0419392,$$

the problem

$$u'' = \frac{1 + \sin\left(\frac{\pi}{u}\right)}{u} - \arctan(u) + e(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \quad (3.13)$$

has a solution u_1 such that $u_1(t^*) \in [x_4, x_5 + d_1]$ for some $t^* \in [0, 2\pi]$, where $d_1 = x_5 - x_4$ (see Figure 3).

Similarly, by Theorems 1 and 2, if

$$\|e - \bar{e}\|_{\mathbb{L}} < \frac{3}{2\pi}(x_5 - x_4) \approx 0.0209699,$$

the problem (3.13) has at least 2 different solutions u_1 and u_2 , where $u_1(t^*) \in (x_5 - d_2, x_5 + d_2)$ for some $t^* \in [0, 2\pi]$ and $u_2(t) \in (x_4 - d_2, x_4 + d_2)$ for all $t \in [0, 2\pi]$, where $d_2 = \frac{1}{2}(x_5 - x_4)$ (see Figure 4).

Finally, if

$$\|e - \bar{e}\|_{\mathbb{L}} \leq \frac{3}{\pi}(x_2 - x_1) \approx 0.0136238,$$

the problem (3.13) has at least 3 different solutions u_1 , u_2 and u_3 , where $u_1(t^*) \in [x_5 - d_3, x_5 + d_3]$ for some $t^* \in [0, 2\pi]$, $u_2(t) \in [x_4 - d_3, x_4 + d_3]$ for all $t \in [0, 2\pi]$ and $u_3(t) \in [x_1, x_2]$ for all $t \in [0, 2\pi]$, where $d_3 = x_2 - x_1$ (see Figure 5).

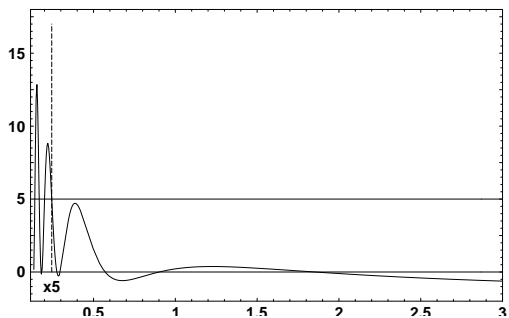


Figure 1.

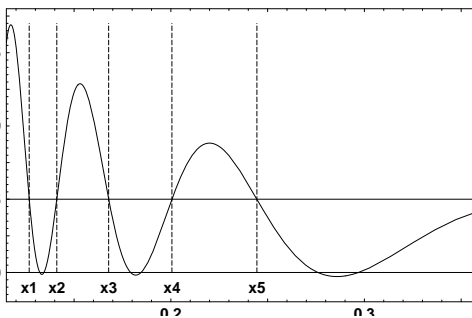


Figure 2.

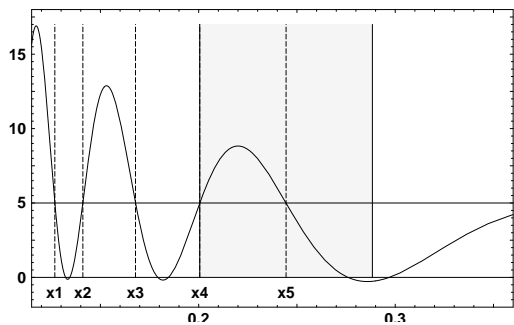


Figure 3.

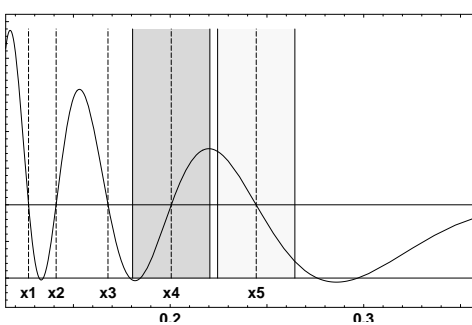


Figure 4.

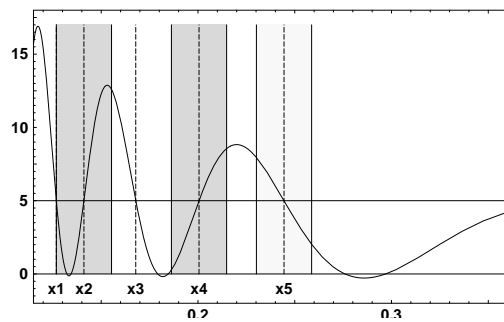


Figure 5.

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