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ON THE INVESTIGATION OF SOME NON-LINEAR BOUNDARY VALUE PROBLEMS WITH PARAMETERS

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Abstract. A scheme of the numerical-analytic method based upon successive approximations for the investigation of non-linear two-point boundary value problems containing parameters both in the differential equation and in the boundary condition is given.

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1. Introduction

The so-called numerical-analytic method (shortly NAM) based upon successive approximations was introduced by the first author jointly with Professor A. Samoilenko [1,2] for the purpose of studying the existence of solutions of non-linear boundary value problems (BVP) and finding approximations to them. For a survey of the further application and development of the NAM to various types of BVPs, including periodic, two-point, multipoint, impulsive, and parametrised ones, one can consult our series of papers in the *Ukrainian Mathematical Journal* joint with Samoilenko and Trofimchuk. The most recently published paper [3] from this series contains the seventh part of the survey. Extentions of NAM to some types of parametrised boundary value problems (PBVPS) can be found in [4,5].

2. Main results

We consider the following two-point non-linear boundary value problem containing parameters both in the given differential equation and in the boundary condition:

$$dx/dt = f(t, x, \lambda_1), \tag{2.1}$$

$$Ax(0) + C(\lambda_1)x(\lambda_2) = d(\lambda_1, \lambda_2), \tag{2.2}$$

$$x_1(0) = x_{10}, \quad x_2(0) = x_{20}. \tag{2.3}$$

Here, we suppose that $x : [0,T] \to \mathbb{R}^n$, $T > 0$ is fixed, the functions $f : \Omega := [0,T] \times D \times [a_1, b_1] \to \mathbb{R}^n$ and $d : I_1 \times I_2 \to \mathbb{R}^n$ are continuous in their domains of definition, $D \subset \mathbb{R}^n (n \geq 3)$ is a closed, connected, and bounded domain, and $\lambda_1 \in I_1 := [a_1, b_1], \lambda_2 \in I_2 := (0, T]$ are unknown scalar parameters. The $n \times n$ matrices A and $C(\lambda_1)$ are supposed to be such that $\det h(\lambda_1) \neq 0$ and $\operatorname{rank} \left[r_{11}(\lambda_1),\ r_{12}(\lambda_1) \right] = 2$ for some real k_1 and k_2 ($k_1 \neq k_2$) and all $\lambda_1 \in I_1$, where $h(\lambda_1) := k_1 A + k_2 C(\lambda_1)$, $H(\lambda_1) := h(\lambda_1)^{-1}$,

$$\begin{bmatrix} r_{11}(\lambda_1) & r_{12}(\lambda_1) \\ r_{21}(\lambda_1) & r_{22}(\lambda_1) \end{bmatrix} = E - k_1 H(\lambda_1) \left[A + C(\lambda_1) \right].$$

(In the equality above, the matrices $r_{11}(\lambda_1)$, $r_{12}(\lambda_1)$, $r_{21}(\lambda_1)$, and $r_{22}(\lambda_1)$ have dimension 2×2, $2 \times (n-2)$, $(n-2) \times 2$, $(n-2) \times (n-2)$, respectively.)

We aim at obtaining the values $\lambda_1^* \in I_1$ and $\lambda_2^* \in I_2$ for which the BVP (2.1), (2.2) has a solution x^* satisfying the additional condition (2.3) for its first and second components. By a solution of (2.1)–(2.3), we thus mean the pair (λ, x), where $\lambda = (\lambda_1,\ \lambda_2)$.

It is obvious that the right-hand side boundary in BVP (2.1)–(2.3) should also be regarded as a parameter.

Let us denote by $|f|$ the column $(|f_1|, |f_2|, \ldots, |f_n|)$. The inequalities between the vectors will be understood component-wise.

With this conventions adopted, we set

$$D_\beta := \{x \in \mathbb{R}^n : B(x, \beta(x)) \subset D\},$$

where $\beta : \mathbb{R}^n \to \mathbb{R}^n$, and $B(x, \beta(x))$ is the $\beta(x)$–neighbourhood of an $x \in \mathbb{R}^n$.

We also assume that the following three conditions hold for the BVP (2.1)–(2.3):

(i) *f is continuous on Ω and bounded by some vector $M \in \mathbb{R}^n_+$:*

$$|f(t, x, \lambda_1)| \leq M \quad \textit{for all } (t, x, \lambda_1) \in \Omega,$$

and is Lipschitzian in the last two variables, i.e.,

$$|f(t, x', \lambda_1') - f(t, x'', \lambda_1'')| \leq K|x' - x''| + |\lambda_1' - \lambda_1''| M_1,$$

where K and M_1 are non-negative matrices of dimension $n \times n$ and $n \times 1$, respectively;

(ii) *The set D_β, where*

$$\beta(x, \lambda) := \frac{1}{2} T M' + \beta_1(x, \lambda),$$

$$d_1(x,\lambda) := d(\lambda) - [A + C(\lambda_1)]x,$$

$$\beta_1(x,\lambda) := |(k_1 - k_2)H(\lambda_1)\,[d(\lambda_1,\lambda_2) - (A + C(\lambda_1))x]\,| + |k_1 H(\lambda_1) d_1(x,\lambda)|,$$

and

$$M' := \frac{1}{2}\left[\max_{(t,x,\lambda_1)\in\Omega} f(t,x,\lambda_1) - \min_{(t,x,\lambda_1)\in\Omega} f(t,x,\lambda_1)\right],$$

is not empty:

$$D_\beta \neq \emptyset;$$

(iii) The greatest eigen-value $\lambda_{\max}(K)$ of the matrix K satisfies the inequality

$$\lambda_{\max}(K) < \frac{q}{T},$$

where $q = \frac{3}{10}$.

Let us introduce the sequence of functions

$$\begin{aligned} x_{m+1}(t,y,\lambda) &= z(y) + k_1 H(\lambda_1) d_1(z(y),\lambda) + \int_0^t f(s, x_m(s,y,\lambda),\lambda_1)ds \\ &- \frac{t}{\lambda_2}\int_0^{\lambda_2} f(\tau, x_m(\tau,y,\lambda),\lambda_1)d\tau \\ &+ \frac{t}{\lambda_2}(k_2 - k_1)H(\lambda_1) d_1(z(y),\lambda), \end{aligned} \tag{2.4}$$

where

$$z = \mathrm{col}(z_1, z_2, \dots, z_i, \dots, z_j, \dots, z_n) = \mathrm{col}(y_1, y_2, \dots, y_i(y), \dots, y_j(y), \dots, y_{n-2}) = z(y),$$

$y = \mathrm{col}(y_1, y_2, \dots, y_{n-2})$, and $y_i(y)$, $y_j(y)$ are solutions of the first two equations in the system

$$x_{m+1}(0,y,\lambda) = \mathrm{col}(x_{10}, x_{20}, x_3(0), \dots, x_n(0)),$$

i.e., the system

$$[E - k_1 H(\lambda_1)\{A + C(\lambda_1)\}]z = \mathrm{col}(x_{10}, x_{20}, x_3(0), \dots, x_n(0)) - d(\lambda_1,\lambda_2). \tag{2.5}$$

(Here and above, i and j denote the numbers of components of the vector z with respect to which system (2.5) is solvable.)

We set $G = \{y \in \mathbb{R}^{n-2} : z(y) \in D_\beta\}$. One can verify by direct computation that sequence (2.4) depending on the parameters λ_1, λ_2 and on the additional $(n-2)$–dimensional vector y, satisfies the boundary conditions (2.2), (2.3) for arbitrary $\lambda_1 \in I_1$, $\lambda_2 \in I_2$, and $y \in G$.

Theorem 1 *Assume that the conditions (i)–(iii) hold.*

Then:

1. *The sequence (2.4) converges to the function* $x^* = x^*(t, y, \lambda)$ *as* $m \to \infty$ *uniformly in* $(t, y, \lambda) \in [0, T] \times G \times I_1 \times I_2$*;*

2. *The limit function* x^* *is a solution of the "perturbed" BVP (2.6), (2.2), (2.3),*

$$dx/dt = f(t, x, \lambda_1) + \Delta(y, \lambda), \tag{2.6}$$

with the initial value $x^*(0, y, \lambda) = z(y) + k_1 H(\lambda_1) d_1(z(y), \lambda)$*, where*

$$\Delta(y, \lambda) := \frac{1}{\lambda_2}(k_2 - k_1) H(\lambda_1) d_1(z(y), \lambda) - \frac{1}{\lambda_2} \int_0^{\lambda_2} f(t, x^*(t, y, \lambda), \lambda_1) dt;$$

3. *The following error estimation holds:*

$$\begin{aligned} |x_m(t, y, \lambda) - x^*(t, y, \lambda)| \le \overline{\alpha}_1(t, \lambda_2)[Q^m(\lambda_2)(E - Q(\lambda_2))^{-1} M' \\ + K Q(\lambda_2)^{m-1}(E - Q(\lambda_2))^{-1} \beta_1(z(y), \lambda)], \end{aligned} \tag{2.7}$$

where $\overline{\alpha}_1(t, \lambda_2) := \frac{10}{9}\alpha_1(t, \lambda_2) \le \frac{5}{9}\lambda_2$, $\alpha_1(t, \lambda_2) := 2t\left(1 - t\lambda_2^{-1}\right)$, $Q(\lambda_2) := \frac{3\lambda_2}{10} K$.

The *proof* of Theorem 1 can be carried out by using the techniques from [2] (Theorems 16.1, 18.1, and 20.1) and Theorem 1 of [4].

The following statement establishes the relation of the limit function x^* to the solution of the original BVP (2.1)–(2.3).

Theorem 2 *Under the conditions of Theorem 1, the pair* $(x^*(\cdot, y^*, \lambda^*), \lambda^*)$ *is a solution of the BVP (2.1)–(2.3) if, and only if* (y^*, λ^*) *satisfies the determining equation*

$$\Delta(y, \lambda) = \frac{1}{\lambda_2}(k_2 - k_1) H(\lambda_1) d_1(z(y), \lambda) - \frac{1}{\lambda_2} \int_0^{\lambda_2} f(t, x^*(t, y, \lambda), \lambda_1) dt = 0. \tag{2.8}$$

The proof of Theorem 2 is analogous to the corresponding statements from [2] (Theorems 16.3 and 18.3).

3. Sufficient existence conditions

In what follows, we need to consider the mth approximation to the determining equation (2.8):

$$\Delta_m(y, \lambda) := \frac{1}{\lambda_2}(k_2 - k_1) H(\lambda_1) d_1(z(y), \lambda) - \frac{1}{\lambda_2} \int_0^{\lambda_2} f(t, x_m(t, y, \lambda), \lambda_1) dt = 0. \tag{3.1}$$

Theorem 3 *Suppose that, for PBVP* (2.1)–(2.3)*, conditions (i)–(iii) hold and, furthermore,* *(iv) There exists a closed, convex subset*

$$\Omega_1 = G_1 \times I_1' \times I_2' \subset G \times I_1 \times I_2,$$

where, for some $m \geq 1$ fixed, the approximate determining equation (3.1) *has only one solution $(\tilde{y}, \tilde{\lambda})$, which has non-zero topological index;*

(v) The inequality

$$\inf_{(y,\lambda)\in\partial\Omega} |\Delta_m(y,\lambda)| > \frac{10}{27} \sup_{\lambda \in I_1' \times I_2'} \{\lambda_2 K W(y,\lambda)\} \tag{3.2}$$

is satisfied on the boundary $\partial\Omega_1$ of the subset Ω_1, where

$$W(x,y) := Q^m(\lambda_2)(E - Q(\lambda_2))^{-1} M' + K Q(\lambda_2)^{m-1}(E - Q(\lambda_2))^{-1} \beta_1(z(y),\lambda).$$

Then, there exists a solution (x^*, λ^*) of PBVP (2.1)–(2.3), *and the initial value $x^*(0)$ of this solution at $t = 0$ is equal to*

$$z(y^*) + k_1 H d_1(z(y^*), \lambda^*),$$

where $y^* \in G_1$, $\lambda_1^* \in I_1'$, and $\lambda_2^* \in I_2'$.

Proof. Based on inequalities (2.7) and (3.2), similarly to Theorems 3.1 and 17.1 of [2], one can show that the vector fields $\Delta(\cdot,\lambda)$ and $\Delta_m(\cdot,\lambda)$ are homotopic for all λ, which, by the well-known result of degree theory, immediately implies the assertion of Theorem 3. ∎

4. Necessary Existence Conditions

The following subsidiary statements will be used in the sequel.

Lemma 4 *Under conditions (i)–(iii), for an arbitrary pair*

$$\{(z',\lambda'),\ (z'',\lambda'')\} \subset D_\beta \times I_1 \times I_2, \tag{4.1}$$

the inequality

$$\begin{aligned} |x^*(t,y',\lambda') - x^*(t,y'',\lambda'')| \leq \left[E + \overline{\alpha}_1(t,\gamma_2) K [E - Q(\gamma_2)]^{-1}\right] \{|z(y') - z(y'')| \\ + b_1(y',y'',\lambda',\lambda'')\} \\ + \overline{\alpha}_1(t,\gamma_2) K [E - Q(\gamma_2)]^{-1} |\lambda_1' - \lambda_1''| M_1 \end{aligned} \tag{4.2}$$

holds, where

$$\begin{aligned} b_1(y',y'',\lambda',\lambda'') := |k_1[H(\lambda_1')d_1(z(y'),\lambda') - H(\lambda_1'')d_1(z(y''),\lambda'')]| \\ + T|k_2 - k_1| \left| \frac{1}{\lambda_2'} H(\lambda_1')d_1(z(y'),\lambda') - \frac{1}{\lambda_2'} H(\lambda_1'')d_1(z(y''),\lambda'') \right| + 2TM, \end{aligned}$$

$$z(y') = \mathrm{col}(y_1', y_2', \ldots, y_{i-1}', y_i'(y'), \ldots, y_j'(y'), \ldots, y_{n-2}'),$$

$$z(y'') = \mathrm{col}(y_1'', y_2'', \ldots, y_{i-1}'', y_i''(y''), \ldots, y_j''(y''), \ldots, y_{n-2}''),$$

and $\gamma_2 = \max\{\lambda_2', \lambda_2''\}$.

Proof. By virtue of (2.4), we have

$$\begin{aligned}
x_1(t,y',\lambda') - x_1(t,y'',\lambda'') &= z(y') - z(y'') \\
&+ k_1 \left[H(\lambda_1')d_1(z(y'),\lambda') - H(\lambda''{}_1)d_1(z(y''),\lambda'')\right] \\
&+ \int_0^t \left[f(s,z(y'),\lambda_1') - f(s,z(y''),\lambda_1'')\right] ds \\
&- \frac{t}{\lambda_2'} \int_0^{\lambda_2'} f(\tau,z(y'),\lambda_1')d\tau + \int_0^{\lambda_2''} f(\tau,z(y''),\lambda_1'')d\tau \\
&+ \frac{t}{\lambda_2'}(k_2-k_1)H(\lambda_1')d_1(z(y'),\lambda') \\
&- \frac{t}{\lambda_2''}(k_2-k_1)H(\lambda_1'')d_1(z(y''),\lambda'') \\
&= z(y') - z(y'') \\
&+ k_1 \left[H(\lambda_1')d_1(z(y'),\lambda') - H(\lambda''{}_1)d_1(z(y''),\lambda'')\right] \\
&+ \int_0^t \Bigg\{ f(s,z(y'),\lambda_1') - f(s,z(y''),\lambda_1'') \\
&- \frac{1}{\lambda_2'} \int_0^{\lambda_2'} \left[f(\tau,z(y'),\lambda_1') - f(\tau,z(y''),\lambda_1'')\right] d\tau \Bigg\} ds \\
&+ \frac{t}{\lambda_2''} \int_0^{\lambda_2''} f(\tau,z(y''),\lambda_1'')d\tau - + \frac{t}{\lambda_2'} \int_0^{\lambda_2'} f(\tau,z(y''),\lambda_1'')d\tau \\
&- t(k_2-k_1)\left[\frac{1}{\lambda_2'}H(\lambda_1')d_1(z(y'),\lambda') - \right. \\
&\left. \frac{1}{\lambda_2''}H(\lambda_1'')d_1(z(y'),\lambda'')\right].
\end{aligned}$$

By using the Lipschitz condition on f, similarly to Lemma 19.1 from [2, p. 154], we obtain

$$\begin{aligned}
|x_1(t,y',\lambda') - x_1(t,y'',\lambda'')| &\le [E + \alpha_1(t,\gamma_2)K]|z(y') - z(y'')| \\
&+ \alpha_1(t,\gamma_2)|\lambda_1' - \lambda_1''|M_1 \\
&+ b_1(y',y'',\lambda',\lambda'').
\end{aligned}$$

One can prove by induction that

$$\begin{aligned}|x_m(t,y',\lambda') - x_m(t,y'',\lambda'')| \leq & \sum_{i=0}^{m} \alpha_i(t,\gamma_2) K^i |z(y') - z(y'')| \\ & + \sum_{i=0}^{m} \alpha_i(t,\gamma_2) K^{i-1} |\lambda_1' - \lambda_1''| M_1 \\ & + \sum_{i=0}^{m-1} \alpha_i(t,\gamma_2) K^i b_1(y',y'',\lambda',\lambda''), \end{aligned} \tag{4.3}$$

where (*see,* e.g., [2, p. 148] or [5])

$$\alpha_{m+1}(t,\gamma) := \left(1 - \frac{t}{\gamma}\right) \int_0^t \alpha_m(s,\gamma) ds + \frac{t}{\gamma} \int_t^{\gamma} \alpha_m(s,\gamma) ds,$$

and $\alpha_0(t,\gamma) \equiv 1$.

Taking into account estimate (*see* Lemma 4 in [6])

$$\alpha_{m+1}(t,\gamma) \leq \left(\frac{3}{10}\gamma\right)^m \overline{\alpha}_1(t,\gamma),$$
$$\alpha_1(t,\gamma) = \frac{10}{9} \overline{\alpha}_1(t,\gamma) \leq \frac{5}{9}\gamma$$

and passing to the limit as $m \to \infty$ in (4.3), we obtain the required inequality (4.2). ■

Lemma 5 *Let us suppose that BVP* (2.1)–(2.3) *satisfies conditions (i)–(iii).*

Then the determining function Δ *is continuous in the domain* $G \times I_1 \times I_2$ *and, for arbitrary pairs* (4.1)*, the following relation holds:*

$$\begin{aligned}|\Delta(y',\lambda') - \Delta(y'',\lambda'')| \leq & \; b_2(y',y'',\lambda',\lambda'') + \frac{\gamma_2}{\gamma_1} |\lambda_1' - \lambda_1''| M_1 \\ & + \frac{\gamma_2}{\gamma_1} K \left[E + \frac{10}{27}\gamma_2 K (E - Q(\gamma_2))^{-1}\right] \Big(|z(y') - z(y'')| \\ & + b_1(y',y'',\lambda',\lambda'') \Big) =: \epsilon(\Delta(y',\lambda'), \Delta(y'',\lambda'')), \end{aligned} \tag{4.4}$$

where

$$b_2(y',y'',\lambda',\lambda'') := |k_2 - k_1| \left| \frac{1}{\lambda_2'} H(\lambda_1') d_1(z(y'),\lambda') - \frac{1}{\lambda_2'} H(\lambda_1') d_1(z(y'),\lambda') \right| + 2M$$

and $\gamma_1 := \min\{\lambda_2', \lambda_2''\}$.

Proof. For every $\{y',y''\} \subset G$ such that $\{z(y'), z(y'')\} \subset D_\beta$, there exists a continuous limit function of the uniformly convergent function sequence (2.4). The determining function is thus also continous and bounded in the domain $G \times I_1 \times I_2$. Due to the form of the function Δ in (2.8), we have

$$\Delta(y', \lambda') - \Delta(y'', \lambda'') = \frac{1}{\lambda_2'}(k_2 - k_1)H(\lambda_1)d_1(z(y'), \lambda') - \frac{1}{\lambda_2''}(k_2 - k_1)H(\lambda_1)d_1(z(y''), \lambda'') - \frac{1}{\lambda_2'}\int_0^{\lambda_2'} f(t, x^*(t, y', \lambda'), \lambda_1')dt + \frac{1}{\lambda_2''}\int_0^{\lambda_2''} f(t, x^*(t, y'', \lambda''), \lambda_1'')dt.$$

By direct computation, using the Lipschitz condition on f and estimate (4.2), we obtain

$$\begin{aligned}
|\Delta(y', \lambda') - \Delta(y'', \lambda'')| &\le b_2(y', y'', \lambda', \lambda'') \\
&\quad + \left| \frac{1}{\lambda_2'}\int_0^{\lambda_2'} [f(t, x^*(t, y', \lambda'), \lambda_1') - f(t, x^*(t, y'', \lambda''), \lambda_1'')]\, dt \right| \\
&\le b_2(y', y'', \lambda', \lambda'') + \frac{1}{\lambda_2'}\int_0^{\lambda_2'} \left\{ K|x^*(t, y', \lambda') - x^*(t, y'', \lambda'')| + |\lambda_1' - \lambda_1''|M_1 \right\} dt \\
&\le b_2(y', y'', \lambda', \lambda'') + \frac{1}{\lambda_2'}\int_0^{\lambda_2'} \Big\{ K\Big[\left(E + \overline{\alpha}_1(t, \gamma_2)K(E - Q(\gamma_2))^{-1}\right) |z(y') - z(y'')| \\
&\quad + \left(E + \overline{\alpha}_1(t, \gamma_2)K(E - Q(\gamma_2))^{-1}\right) b_1(y', y'', \lambda', \lambda'') \Big] + |\lambda_1' - \lambda_1''|M_1 \Big\} dt \\
&\le b_2(y', y'', \lambda', \lambda'') \\
&\quad + \frac{\gamma_2}{\gamma_1}K\left[E + \frac{10}{27}\gamma_2 K(E - Q(\gamma_2))^{-1}\right] \Big(|z(y') - z(y'')| + b_1(y', y'', \lambda', \lambda'') \Big) \\
&\quad + \frac{\gamma_2}{\gamma_1}|\lambda_1' - \lambda_1''|M_1 \le \epsilon(\Delta(y', \lambda'), \Delta(y'', \lambda'')),
\end{aligned}$$

as required. ■

The following statement gives a necessary condition for the existence of solutions of PBVP (2.1)–(2.3).

Theorem 6 *Assume that conditions (i)–(iii) hold. Then the subset*

$$\Omega_2 = G_2 \times I_1'' \times I_2'' \subset G \times I_1 \times I_2$$

may contain a pair (y^*, λ^*) *generating a solution*

$$x^*(t, y^*, \lambda^*) = \lim_{m\to\infty} x_m(t, y^*, \lambda^*)$$

of PBVP (2.1)–(2.3) *only if, for every* $m \ge 1$ *and every pair* $(\tilde{y}, \tilde{\lambda})$, *the following* *relation holds true:*

$$\Delta_m(\tilde{y},\tilde{\lambda}) \le \sup_{(y,\lambda)\in\Omega_2}\Bigg\{ b_2(\tilde{y},y,\tilde{\lambda},\lambda) + \frac{\gamma_2}{\gamma_1}|\tilde{\lambda}_1-\lambda_1|M_1 + \frac{\gamma_2}{\gamma_1}K\left[E+\frac{10}{27}\gamma_2 K(E-Q(\gamma_2))^{-1}\right]\Big(|z(\tilde{y})-z(y)| + b_1(\tilde{y},y,\tilde{\lambda},\lambda)\Big)\Bigg\} + \epsilon(\Delta(\tilde{y},\tilde{\lambda}),\Delta_m(\tilde{y},\tilde{\lambda})). \tag{4.5}$$

Proof. Let the determining function Δ vanish at $y = y^*$, $\lambda = \lambda^*$, i.e., that $x^*(\cdot, y^*, \lambda^*)$ is a solution of the PBVP (2.1)–(2.3). Rewriting inequality (4.4) for the pairs $(y',\lambda') = (\tilde{y},\tilde{\lambda})$ and $(y'',\lambda'') = (y^*,\lambda^*)$, we obtain

$$|\Delta(\tilde{y},\tilde{\lambda})| \le b_2(\tilde{y},y^*,\tilde{\lambda},\lambda^*) + \frac{\gamma_2}{\gamma_1}|\tilde{\lambda}_1-\lambda_1^*|M_1 + \frac{\gamma_2}{\gamma_1}K\left[E+\frac{10}{27}\gamma_2 K(E-Q(\gamma_2))^{-1}\right]\Big(|z(\tilde{y})-z(y^*)| + b_1(\tilde{y},y^*,\tilde{\lambda},\lambda^*)\Big).$$

Relations (2.8) and (3.1) yield

$$|\Delta(y,\lambda)-\Delta_m(y,\lambda)| = \left|\frac{1}{\lambda_2}\int_0^{\lambda_2}[f(t,x^*(t,y,\lambda),\lambda_1) - f(t,x_m(t,y,\lambda),\lambda_1)]\,dt\right| \le \frac{1}{\lambda_2}KW(y,\lambda)\int_0^{\lambda_2}\overline{\alpha}_1(t,\lambda_2)dt = \frac{10}{27}\lambda_2 KW(y,\lambda) = \epsilon(\Delta_m(y,\lambda),\Delta_m(y,\lambda)). \tag{4.6}$$

Relation (4.6) with $(y,\lambda) = (\tilde{y},\tilde{\lambda})$ implies

$$|\Delta_m(\tilde{y},\tilde{\lambda})| \le |\Delta(\tilde{y},\tilde{\lambda})| + \epsilon(\Delta_m(\tilde{y},\tilde{\lambda}),\Delta_m(\tilde{y},\tilde{\lambda})). \tag{4.7}$$

Combining (4.6) and (4.7), we obtain the desired necessary condition (4.5). ∎

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