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INVESTIGATION OF A CLASS OF TWO-DIMENSIONAL CONJUGATE INTEGRAL EQUATION WITH FIXED SUPER-SINGULAR KERNELS

NUSRAT RAJABOV AND MIKLÓS RONTÓ

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Abstract. In this paper, two-dimensional linear conjugate Volterra integral equations containing super-singularities in the kernels are considered. The existence of a unique solution in a certain function class is established. Formulas representing the solution are given.

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Keywords: conjugate Volterra integral equation, super-singularity in kernel, two-dimensional integral equation

1. INTRODUCTION

Let D denote the rectangle $D = \{a < x < a_0,\ b_0 < y < b\}$ and introduce the sets $\Gamma_1 = \{a < x < a_0,\ y = b\}$ and $\Gamma_2 = \{x = a,\ b_0 < y < b\}$. In D, we consider the two-dimensional integral equation

$$u(x,y) + \lambda \int_a^x \frac{u(t,y)}{(t-a)^\alpha} dt - \mu \int_y^b \frac{u(x,s)}{(b-s)^\beta} ds + \delta \int_a^x \frac{dt}{(t-a)^\alpha} \int_y^b \frac{u(t,s)}{(b-s)^\beta} ds = f(x,y), \quad (1.1)$$

and the integral equation conjugate to equation (1.1)

$$T^{\alpha,\beta}_{\lambda,\mu}(v) \equiv v(x,y) + \frac{\lambda}{(x-a)^\alpha} \int_x^{a_0} v(t,y)dt - \frac{\mu}{(b-y)^\beta} \int_{b_0}^y v(x,s)ds + \frac{\delta}{(x-a)^\alpha (b-y)^\beta} \int_x^{a_0} dt \int_{b_0}^y v(t,s)ds = g(x,y), \quad (1.2)$$

where $\alpha > 1,\ \beta > 1,\ \{\lambda, \mu, \delta\} \subset \mathbb{R},\ f(x,y),\ g(x,y)$ are the given functions and $u(x,y)$, $v(x,y)$, are the unknown functions.

By the study of the solutions of the integral equations (1.1) and (1.2), the problem can be reduced to the determination of the continuous solution of a hyperbolic equation with two super-singular lines and its conjugate equation in D.

We seek a solution of equation (1.1) in the class of functions $u(x,y) \in C(\overline{D})$ that vanish on the singular lines Γ_1 and Γ_2. Moreover, we will assume that the unknown function $u(x,y)$ vanishes as $x \to a$ by an order higher than $\alpha - 1$, and it vanishes as $y \to b$ by an order higher than $\beta - 1$.

We note that the one-dimensional integral equations of types (1.1) and (1.2) are studied in [2,4–6]. Two-dimensional , three-dimensional and some many-dimensional Volterra type integral equations of type (1.1) are studied in [2,3,7–9]. One-dimensional singular integral equations with Cauchy kernels are considered in [1].

2. THE CASE WHERE $\delta = -\lambda\mu$

In this case, the integral equation (1.2) can be represented in the following form:

$$v(x,y) - \frac{\mu}{(b-y)^{\beta}} \int_{b_0}^{y} v(x,s)ds + \frac{\lambda}{(x-a)^{\alpha}} \int_{x}^{a_0} [v(t,y) - \frac{\mu}{(b-y)^{\beta}} \int_{b_0}^{y} v(t,s)ds]dt = g(x,y).$$

If we introduce a new unknown function

$$W(x,y) = v(x,y) - \frac{\mu}{(b-y)^{\beta}} \int_{b_0}^{y} v(x,s)ds, \tag{2.1}$$

we arrive to a one-dimensional conjugate Volterra type integral equation

$$W(x,y) + \frac{\lambda}{(x-a)^{\alpha}} \int_{x}^{a_0} W(t,y)dt = g(x,y). \tag{2.2}$$

In the case where $a < x < a_0$, according to [3], the integral equation (2.2) has a unique solution which is given by the formula

$$W(x,y) = g(x,y) - \frac{\lambda}{(x-a)^{\alpha}} \int_{x}^{a_0} \exp[\lambda(\omega_a^{\alpha}(t) - \omega_a^{\alpha}(x))]g(t,y)dt, \tag{2.3}$$

where $\omega_a^{\alpha}(x) = [(\alpha-1)(x-a)^{\alpha-1}]^{-1}$.

Analogously, the solution of the integral equation (2.1), for $b_0 < y < b$, is given by the formula

$$v(x,y) = W(x,y) + \frac{\mu}{(b-y)^{\beta}} \int_{b_0}^{y} \exp[\mu(\omega_b^{\beta}(y) - \omega_b^{\beta}(s))]W(t,y)ds, \tag{2.4}$$

where $\omega_b^{\beta}(y) = [(\beta-1)(b-y)^{\beta-1}]^{-1}$.

By substituting the value of $W(x,y)$ into (2.4), we obtain a general solution of the integral equation (1.2) in the form

$$
\begin{aligned}
v(x,y) = g(x,y) &- \frac{\lambda}{(x-a)^\alpha}\int_x^{a_0} \exp[\lambda(\omega_a^\alpha(t)-\omega_a^\alpha(x))]g(t,y)dt \\
&+ \frac{\mu}{(b-y)^\beta}\int_{b_0}^{y} \exp[\mu(\omega_b^\beta(y)-\omega_b^\beta(s))]g(x,s)ds \\
&- \frac{\lambda\mu}{(x-a)^\alpha(b-y)^\beta}\int_x^{a_0} \exp[\lambda(\omega_a^\alpha(t)-\omega_a^\alpha(x))]dt \\
&\times \int_{b_0}^{y} \exp[\mu(\omega_b^\beta(y)-\omega_b^\beta(s))]g(t,s)ds \equiv (T_{\lambda,\mu}^{\alpha,\beta})^{-1}g. \quad (2.5)
\end{aligned}
$$

We thus obtain the following

Theorem 1. *Assume that in equation (1.2) the parameters are related by the equality $\delta = -\lambda\mu$, and $g(x,y) \in C(D)$. Then the non-homogeneous integral equation (1.2) has a unique solution in class $C(D)$, which is given by formula (2.5).*

3. The case where $\delta \neq -\lambda\mu$

In this case, the integral equation (1.2) can be represented in the following form:

$$
\begin{aligned}
T_{\lambda,\mu}^{\alpha,\beta}(v) \equiv v(x,y) &+ \frac{\lambda}{(x-a)^\alpha}\int_x^{a_0} v(t,y)dt \\
&- \frac{\mu}{(b-y)^\beta}\int_{b_0}^{y} v(x,s)ds - \frac{\lambda\mu}{(x-a)^\alpha(b-y)^\beta}\int_x^{a_0} dt \int_{b_0}^{y} v(t,s)ds \\
&= g(x,y) - \frac{\delta_1}{(x-a)^\alpha(b-y)^\beta}\int_x^{a_0} dt \int_{b_0}^{y} v(t,s)ds.
\end{aligned}
$$

Let us introduce the function

$$
g_1(x,y) = g(x,y) - \frac{\delta_1}{(x-a)^\alpha(b-y)^\beta}\int_x^{a_0} dt \int_{b_0}^{y} v(t,s)ds,
$$

where $\delta_1 = \delta + \lambda\mu$. Clearly, $g_1(x,y) \in C(D)$. Then the solution of the integral equation $T_{\lambda,\mu}^{\alpha,\beta}(v) = g_1(x,y)$ is as follows:

$$
\begin{aligned}
v(x,y) = g_1(x,y) &- \frac{\lambda}{(x-a)^\alpha}\int_x^{a_0} \exp[\lambda(\omega_a^\alpha(t)-\omega_a^\alpha(x))]g_1(t,y)dt \\
&+ \frac{\mu}{(b-y)^\beta}\int_{b_0}^{y} \exp[\mu(\omega_b^\beta(y)-\omega_b^\beta(s))]g_1(x,s)ds \\
&- \frac{\lambda\mu}{(x-a)^\alpha(b-y)^\beta}\int_x^{a_0} \exp[\lambda(\omega_a^\alpha(t)-\omega_a^\alpha(x))]dt \\
&\times \int_{b_0}^{y} \exp[\mu(\omega_b^\beta(y)-\omega_b^\beta(s))]g_1(t,s)ds \equiv (T_{\lambda,\mu}^{\alpha,\beta})^{-1}g_1(x,y). \quad (3.1)
\end{aligned}
$$

In formula (3.1), by substituting the value of $g_1(x,y)$ and rearranging the appropriate terms, we arrive to the solution of the integral equation

$$\Phi(x,y)+\frac{\delta_1}{(x-a)^{\alpha}(b-y)^{\beta}}\int_x^{a_0}dt\int_{b_0}^{y}\Phi(t,s)ds=E_{\lambda,\mu}^{\alpha,\beta}[g(x,y)], \tag{3.2}$$

where

$$\begin{aligned}E_{\lambda,\mu}^{\alpha,\beta}[g(x,y)]&=\exp[\lambda\omega_a^{\alpha}(x)-\mu\omega_b^{\beta}(y)](T_{\lambda,\mu}^{\alpha,\beta})^{-1}g(x,y)\\&=\exp[\lambda\omega_a^{\alpha}(x)-\mu\omega_b^{\beta}(y)]g(x,y)\\&-\frac{\lambda}{(x-a)^{\alpha}}\exp[-\mu\omega_b^{\beta}(y)]\int_x^{a_0}\exp[\lambda\omega_a^{\alpha}(t)]g(t,y)dt\\&+\frac{\mu}{(b-y)^{\beta}}\exp[\lambda\omega_a^{\alpha}(x)]\int_{b_0}^{y}\exp[-\mu\omega_b^{\beta}(s)]g(x,s)ds\\&-\frac{\lambda\mu}{(x-a)^{\alpha}(b-y)^{\beta}}\int_x^{a_0}\exp[\lambda\omega_a^{\alpha}(t)]dt\int_{b_0}^{y}\exp[-\mu\omega_b^{\beta}(s)]g(t,s)ds\end{aligned} \tag{3.3}$$

and

$$\Phi(x,y)=\exp[\lambda\omega_a^{\alpha}(x)-\mu\omega_a^{\alpha}(y)]v(x,y).$$

4. Representation of a solution by functional series of $\exp(-\omega_a^{\alpha}(x))$

We seek a solution for integral equation (3.2) in the class of functions that can be represented in the form

$$\Phi(x,y)=\sum_{n=1}^{\infty}(\exp(-\omega_a^{\alpha}(x)))^n\Phi_n(y)(x-a)^{-\alpha}, \tag{4.1}$$

where $\Phi_n(y)$ are unknown functions.

We assume that function $g(x,y)$ admits representation in the form

$$g(x,y)=\exp[-\lambda\omega_a^{\alpha}(x)+\mu\omega_b^{\beta}(y)]\sum_{n=1}^{\infty}[\exp(-\omega_a^{\alpha}(x))^n](x-a)^{-\alpha}g_n(y), \tag{4.2}$$

where $g_n(y)$ are known functions. Moreover, assume that the series (4.2) converges absolutely and uniformly. By substituting this value $g(x,y)$ into (3.3), we have

$$\begin{aligned}E_{\lambda,\mu}^{\alpha,\beta}[g(x,y)]&=\sum_{n=1}^{\infty}[\exp(-\omega_a^{\alpha}(x))]^n(x-a)^{-\alpha}\\&\times\left(\frac{n+\lambda}{n}\right)\left[g_n(y)+\mu(b-y)^{-\mu}\int_{b_0}^{y}g_n(s)ds\right]\\&-(x-a)^{-\alpha}\lambda\left[\sum_{n=1}^{\infty}n^{-1}\exp(-\omega_a^{\alpha}(a_0))^n\left(g_n(y)+\mu(b-y)^{-\mu}\int_{b_0}^{y}g_n(s)ds\right)\right].\end{aligned}$$

By substituting the values of $\Phi(x,y)$ and $E^{\alpha,\beta}_{\lambda,\mu}[g(x,y)]$ into the integral equation (3.2) and equating the coefficients at $[\exp(-\omega^{\alpha}_{a}(x))]^{k}$, $k=0,1,2,\dots$, we obtain the following relations between the functions $\Phi_n(y)$ and $g_n(y)$, $n=0,1,2,\dots$:

$$\begin{aligned}\delta_1(b-y)^{-\beta}\sum_{n=1}^{\infty}n^{-1}[\exp(-\omega^{\alpha}_{a}(a_0))]^{n}\int_{b_0}^{y}\Phi_n(s)ds\\ =-\sum_{n=1}^{\infty}n^{-1}[\exp(-\omega^{\alpha}_{a}(a_0))]^{n}\lambda[g_n(y)+(b-y)^{-\beta}\mu\int_{b_0}^{y}g_n(s)ds]\end{aligned}\tag{4.3}$$

and

$$\Phi_n(y)-\frac{\delta_1}{n(b-y)^{\beta}}\int_{b_0}^{y}\Phi_n(s)ds=\left(\frac{n+\lambda}{n}\right)g_n(y)+\frac{\mu(n+\lambda)}{n(b-y)^{\mu}}\int_{b_0}^{y}g_n(s)ds.\tag{4.4}$$

According to [3], if the system of integral equation (4.4) has a solution, then it can be represented in the form

$$\begin{aligned}\Phi_n(y)=\frac{n+\lambda}{n}\Bigg[g_n(y)\\ +\left(\frac{\mu n-\delta_1}{n}\right)\frac{1}{(b-y)^{\beta}}\int_{b_0}^{y}\exp\left[\frac{\delta_1}{n}(\omega^{\beta}_{b}(s)-\omega^{\beta}_{b}(y))\right]g_n(s)ds\Bigg],\end{aligned}\tag{4.5}$$

where $n=0,1,2,\dots$, $\omega^{\beta}_{b}(y)=[(\beta-1)(b-y)^{\beta-1}]^{-1}$.

Furthermore, it follows from equality (4.3) that

$$\delta_1(b-y)^{-\beta}\int_{b_0}^{y}\Phi_n(s)ds=-\lambda[g_n(y)+\mu(b-y)^{-\beta}\int_{b_0}^{y}g_n(s)ds]\tag{4.6}$$

for $n=0,1,2,\dots$. From expression (4.6), by substituting the values of $\Phi_n(s)$ according to formula (4.5), we obtain the equality

$$\begin{aligned}\left(\frac{n+\lambda}{n}\right)\Bigg[\frac{\mu n}{(b-y)^{\beta}}\int_{b_0}^{y}g_n(s)ds\\ -\frac{(\mu n-\delta_1)}{(b-y)^{\beta}}\int_{b_0}^{y}\exp\left(\frac{\delta_1}{n}(\omega^{\beta}_{b}(s)-\omega^{\beta}_{b}(y))\right)g_n(s)ds\Bigg]\\ =-\lambda\left[g_n(y)+\frac{\mu}{(b-y)^{\beta}}\int_{b_0}^{y}g_n(s)ds\right],\quad n=0,1,2,3\dots.\end{aligned}\tag{4.7}$$

From formula (4.1), by substituting the value $\Phi_n(y)$ from equality (4.5), where $\Phi(x,y)=v(x,y)\exp[\lambda\omega^{\alpha}_{a}(x)-\mu\omega^{\beta}_{b}(y)]$, we find

$$v(x,y)=\exp[-\lambda\omega_a^\alpha(x)+\mu\omega_b^\beta(y)]\sum_{n=1}^{\infty}\frac{\exp(-\omega_a^\alpha(x))^n}{(x-a)^\alpha}\left(\frac{n+\lambda}{n}\right)$$
$$\times\left[g_n(y)+\frac{\mu n-\delta_1}{n(b-y)^\beta}\int_{b_0}^{y}(\exp(\omega_b^\beta(s)-\omega_b^\beta(y)))^{\frac{\delta_1}{n}}g_n(s)ds\right]. \quad (4.8)$$

Thus, we arrive at the following conclusion.

Theorem 2. *Assume that in the integral equation (1.2) $\delta\neq-\lambda\mu$, and that the function $g(x,y)$ is represented by series (4.2), which converges absolutely and uniformly. Then the integral equation (1.2) has a solution in the class of functions $v(x,y)$ that are representable in the form*

$$v(x,y)=\exp[-\lambda\omega_a^\alpha(x)+\mu\omega_b^\beta(y)]\sum_{n=1}^{\infty}\frac{\Phi_n(y)}{(x-a)^\alpha\exp(\omega_a^\alpha(x))^n}.$$

Moreover, if the functions $g_k(y)$, $k=1,2,\ldots,$ in (4.2) satisfy the infinite system of solvability conditions (4.7), then that solution is unique and can be represented by by formula (4.8).

Remark 1. In the case where $\delta\neq-\lambda\mu$, the solution of the integral equation (3.2) could be sought in the class of functions that are representable by a functional series of $\exp(-\omega_b^\beta(y))$, i. e.,

$$\Phi(x,y)=\sum_{n=1}^{\infty}\exp(-n\omega_b^\beta(y))(b-y)^{-\beta}W_n(x)$$

where $W_n(x)$ are unknown functions. Then one assumes that the function $g(x,y)$ is represented in the form

$$g(x,y)=\exp[-\lambda\omega_a^\alpha(x)+\mu\omega_b^\beta(y)]\sum_{n=1}^{\infty}\frac{\exp(-n\omega_b^\beta(y))}{(b-y)^\beta}g_n(x).$$

By modifying suitably the argument above, in that case, one can also obtain a statement similar to Theorem 2.

5. Remarks on a non-model integral equation

In the domain D, we consider the two-dimensional integral equation

$$u(x,y)+\int_a^x\frac{A(t)u(t,y)}{(t-a)^\alpha}dt-\int_y^b\frac{B(s)u(x,s)}{(b-s)^\beta}ds$$
$$+\int_a^x\frac{dt}{(t-a)^\alpha}\int_y^b\frac{c(t,s)u(t,s)}{(b-s)^\beta}ds=f(x,y), \quad (5.1)$$

and its conjugate equation

$$v(x,y) + \frac{A(x)}{(x-a)^\alpha}\int_x^{a_0} v(t,y)dt - \frac{B(y)}{(b-y)^\beta}\int_{b_0}^{y} v(x,s)ds + \frac{c(x,y)}{(x-a)^\alpha (b-y)^\beta}\int_x^{a_0} dt \int_{b_0}^{y} v(t,s)ds = g(x,y). \quad (5.2)$$

Integral equations of form (5.1) are studied in [8].

Remark 2. One can find a solution of the integral equation (5.2) if $c(x,y) \equiv -A(x)B(y)$. In that case, as is shown in [6], the question is reduced to finding a solution of two split systems of one-dimensional conjugate integral equations of type (5.2).

Remark 3. In the case where $c(x,y) \not\equiv -A(x)B(y)$, the problem of finding solution for integral equation (5.2) is reduced to the problem of the determination of a solution of the integral equation

$$\begin{aligned} v(x,y) &+ \frac{c_1(x,y)}{(x-a)^\alpha (b-y)^\beta}\int_x^{a_0} \exp[A(a)(\omega_\alpha(t)-\omega_\alpha(x)) - W^-_{A,\alpha}(t) - W^-_{A,\alpha}(x)]dt \\ &\times \int_{b_0}^{y} \exp[B(b)(\omega_b^\beta(y)-\omega_b^\beta(s)) + W^-_{b,\beta}(s) - W^-_{b,\beta}(y)]v(t,s)ds \\ &\equiv (T^{\alpha,\beta}_{A(x),B(y)})^{-1}(g(x,y)), \end{aligned}$$

for any $(x,y) \in D$, where $(T^{\alpha,\beta}_{A(x),B(y)})^{-1}$ is a known integral operator.

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Authors' addresses

Nusrat Rajabov
Tajik National University, 17 Rudaki Ave., 734025 Dushanbe, Tajikistan
E-mail address: nusrat38@mail.ru

Miklós Rontó
Institute of Mathematics, University of Miskolc, H-3515 Miskolc-Egyetemváros, Hungary
E-mail address: matronto@gold.uni-miskolc.hu