Miskolc Mathematical Notes
Vol. 13 (2012), No 2, pp. 581-603

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2012.318

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APPROXIMATION PROPERTIES OF REAL AND COMPLEX POST-WIDDER OPERATORS BASED ON q-INTEGERS

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Received 18 December, 2010

Abstract. In this paper, we study the statistical approximation properties of real and complex Post-Widder operators based on q-integers. We also obtain a Voronovskaya-type formula in statistical sense for these operators.

2000 *Mathematics Subject Classification:* 41A36, 40A35, 30E10

Keywords: A-statistical convergence, statistical approximation, Post-Widder operators, q-integers

1. INTRODUCTION

Positive linear operators and their q-modifications have recently become an active research area in the approximation theory (see, i.e., [1,2,6,7,10,12,13,16,18–26,28]). In this paper, we improve the classical Post-Widder operators via the techniques from the q-calculus. We obtain various statistical approximation theorems including some Korovkin-type results and Voronovskaya-type formulas in statistical and ordinary sense for these operators. Our investigations cover both real and complex cases.

Throughout the paper we employ the standard notations of q-calculus. As usual, by a q-integer and a q-factorial we mean respectively:

$$[n]_q := 1 + q + ... + q^n \text{ for } n \in \mathbb{N} \text{ with } [0]_q := 0$$

and

$$[n]_q! := [1]_q [2]_q ... [n]_q \text{ for } n \in \mathbb{N} \text{ with } [0]_q! := 1.$$

Also, a q-binomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!} \text{ for } 0 \le k \le n.$$

Now let $0 < q < 1$. For simplicity, we use the following standard notations:

$$(1+t)_q^\infty := \prod_{j=0}^{\infty} \left(1 + q^j t\right).$$

Then, we know [14, 15] that the q-Jackson definite integral of a function f is defined by

$$\int_0^a f(t)\, d_q t := a(1-q) \sum_{n=0}^{\infty} f\left(aq^n\right) q^n \text{ for } a > 0.$$

Now we recall the q-analogue of the exponential function as follows (see [3, 17]):

$$E_q(x) = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q!} = (1 + (1-q)x)_q^{\infty}.$$

Notice that the above series converges for every x whenever $q \in (0,1)$. Then, the corresponding q-Gamma function is defined by (see [17])

$$\Gamma_q(x) := \int_0^{1/(1-q)} t^{x-1} E_q(-qt)\, d_q t.$$

As usual, let $C_B[0,\infty)$ denote the space of all bounded and continuous functions on $[0,\infty)$ endowed with the norm

$$\|f\| = \sup_{x \in [0,\infty)} |f(x)|.$$

In the present paper, we improve the classical Post-Widder operators by considering the above terminology as follows:

$$P_{n,q}(f;x) = \frac{1}{[n]_q!} \int_0^{1/(1-q)} f\left(\frac{xt}{[n]_q}\right) t^n E_q(-qt)\, d_q t, \tag{1.1}$$

where $x \in [0,\infty)$, $q \in (0,1)$ and $f \in C_B[0,\infty)$. We use the following test functions

$$e_m(y) = y^m \text{ for } m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Then, we first observe that these operators are positive and linear and that

$$P_{n,q}(e_0;x) = e_0(x) = 1. \tag{1.2}$$

Notice that the operators $P_{n,q}(f)$ map $C_B[0,+\infty)$ into itself. Also, it follows from (1.1) that

$$\lim_{q \to 1^-} P_{n,q}(f;x) = \frac{1}{n!} \int_0^{\infty} f\left(\frac{xt}{n}\right) t^n e^{-t} dt =: P_n(f;x),$$

which is the classical Post-Widder operator (see, for instance, [5, 27]).

2. Local Approximation Results for q-Post-Widder Operators

Let $f \in C_B[0,\infty)$ and $\delta > 0$. Then, we consider the first modulus of continuity, the Peetre's K-functional, and the second modulus of smoothness, which are defined respectively by:

$$\omega(f,\delta) = \sup\{|f(y)-f(x)| : |y-x| \le \delta;\ x,y \in [0,\infty)\},$$

$$K(f,\delta) = \inf\left\{\|f-g\| + \delta\left\|g''\right\| : g \in C_B^2[0,\infty)\right\},$$

$$\omega_2(f,\delta) = \sup_{0<h\le\delta}\ \sup_{x\in[0,\infty)} |f(x+2h)-2f(x+h)+f(x)|,$$

where $C_B^2[0,\infty) := \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. Then, it is well-known that (see [4])

$$K(f,\delta) \le A\omega_2(f,\sqrt{\delta})$$

for some positive constant A. With this terminology, in order to get a local approximation result for q-Post-Widder Operators we first need the following two results.

Lemma 1. *For every $m,n \in \mathbb{N}$ and $x \in [0,+\infty)$, we have*

$$P_{n,q}(e_m;x) = \frac{[n+1]_q[n+2]_q...[n+m]_q}{[n]_q^m}x^m.$$

Proof. Let $m,n \in N$ and $x \in [0,+\infty)$. Then, by (1.1) and (1.2), we get

$$\begin{aligned} P_{n,q}(e_m;x) &= \frac{1}{[n]_q!}\int_0^{1/(1-q)} \left(\frac{xt}{[n]_q}\right)^m t^n E_q(-qt)\,d_qt \\ &= \frac{1}{[n]_q!}\frac{x^m}{[n]_q^m}\int_0^{1/(1-q)} t^{n+m}E_q(-qt)\,d_qt \\ &= \frac{1}{[n]_q!}\frac{x^m}{[n]_q^m}\Gamma_q(n+m+1) \\ &= \frac{1}{[n]_q^m}\frac{[n+m]_q!}{[n]_q!}x^m \\ &= \frac{[n+1]_q[n+2]_q...[n+m]_q}{[n]_q^m}x^m, \end{aligned}$$

which completes the proof. □

If we use the moment function given by

$$\varphi(y) := \varphi_x(y) = y-x,$$

then the following result is an immediate consequence of Lemma 1.

Corollary 1. *For every $n \in \mathbb{N}$ and $x \in [0,+\infty)$, we have*

$$P_{n,q}(\varphi;x) = \left(\frac{[n+1]_q}{[n]_q} - 1\right)x,$$

$$P_{n,q}(\varphi^2;x) = \left(\frac{[n+1]_q[n+2]_q}{[n]_q^2} - \frac{2[n+1]_q}{[n]_q} + 1\right)x^2.$$

Now we are ready to give our approximation result.

Theorem 1. *Let $q \in (0,1)$. Then, for every $n \in \mathbb{N}$, $x \in [0,+\infty)$ and $f \in C_B[0,\infty)$, we have*

$$\left|P_{n,q}(f;x) - f(x)\right| \le C\omega_2(f, x\delta_n) + \omega(f, x\alpha_n)$$

for some positive constant C, where

$$\delta_n := \left(\frac{[n+1]_q^2}{[n]_q^2} + \frac{[n+1]_q[n+2]_q}{[n]_q^2} - \frac{4[n+1]_q}{[n]_q} + 2\right)^{1/2} \tag{2.1}$$

and

$$\alpha_n := \frac{[n+1]_q}{[n]_q} - 1 \tag{2.2}$$

Proof. Define an auxiliary operator $P_{n,q}^* : C_B[0,\infty) \to C_B[0,\infty)$ by

$$P_{n,q}^*(f;x) := P_{n,q}(f;x) - f\left(\frac{[n+1]_q}{[n]_q}x\right) + f(x). \tag{2.3}$$

Then, by Corollary 1, we get

$$P_{n,q}^*(\varphi;x) = 0. \tag{2.4}$$

Now, for a given $g \in C_B^2[0,\infty)$, it follows from the Taylor formula that

$$g(y) - g(x) = (y-x)g'(x) + \int_x^y (y-u)g''(u)du, \quad y \in [0,\infty).$$

Taking into account (2.3) and using (2.4) we get, for every $x \in [0,\infty)$, that

$$\begin{aligned}\left|P_{n,q}^*(g;x) - g(x)\right| &= \left|P_{n,q}^*(g(y)-g(x);x)\right| \\ &= \left|g'(x)P_{n,q}^*(\varphi;x) + P_{n,q}^*\left(\int_x^y (y-u)g''(u)du;x\right)\right| \\ &= \left|P_{n,q}^*\left(\int_x^y (y-u)g''(u)du;x\right)\right|\end{aligned}$$

$$= \left| P_{n,q}\left(\int_x^y (y-u) g''(u)du; x \right) - \int_x^{x[n+1]_q/[n]_q} \left(\frac{[n+1]_q}{[n]_q} x - u \right) g''(u)du \right|.$$

Since

$$\left| P_{n,q}\left(\int_x^y (y-u) g''(u)du; x \right) \right| \le \frac{\|g''\|}{2} P_{n,q}\left(\varphi^2; x\right)$$

and

$$\left| \int_x^{x[n+1]_q/[n]_q} \left(\frac{[n+1]_q}{[n]_q} x - u \right) g''(u)du \right| \le \frac{\|g''\|}{2} \left(\frac{[n+1]_q}{[n]_q} - 1 \right)^2 x^2$$

we get

$$\left| P^*_{n,q}(g;x) - g(x) \right| \le \frac{\|g''\|}{2} P_{n,q}\left(\varphi^2; x\right) + \frac{\|g''\|}{2} \left(\frac{[n+1]_q}{[n]_q} - 1 \right)^2 x^2.$$

Hence, Corollary 1 implies that

$$\left| P^*_{n,q}(g;x) - g(x) \right| \le \frac{\|g''\|}{2} \left(\left(\frac{[n+1]_q [n+2]_q}{[n]_q^2} - \frac{2[n+1]_q}{[n]_q} + 1 \right) + \left(\frac{[n+1]_q}{[n]_q} - 1 \right)^2 \right) x^2,$$

which gives

$$\left| P^*_{n,q}(g;x) - g(x) \right| \le \frac{\|g''\|}{2} \left(\frac{[n+1]_q^2}{[n]_q^2} + \frac{[n+1]_q [n+2]_q}{[n]_q^2} - \frac{4[n+1]_q}{[n]_q} + 2 \right) x^2. \tag{2.5}$$

Now, considering (2.1) and (2.2), if $f \in C_B[0,\infty)$ and $g \in C_B^2[0,\infty)$, we may write from (2.5) that

$$\begin{aligned}
\left| P_{n,q}(f;x) - f(x) \right| &\le \left| P^*_{n,q}(f-g;x) - (f-g)(x) \right| \\
&\quad + \left| P^*_{n,q}(g;x) - g(x) \right| + \left| f\left(\frac{[n+1]_q}{[n]_q} x \right) - f(x) \right| \\
&\le 2\|f-g\| + x^2 \delta_n \frac{\|g''\|}{2} + \left| f\left(\frac{[n+1]_q}{[n]_q} x \right) - f(x) \right| \\
&\le 2\left(\|f-g\| + x^2 \delta_n \left\|g''\right\| \right) + \omega\left(f, x\alpha_n\right),
\end{aligned}$$

which yields that

$$\begin{aligned}\left|P_{n,q}(f;x)-f(x)\right| &\leq 2K\left(f,x^2\delta_n^2\right)+\omega\left(f,x\alpha_n\right)\\ &\leq C\omega_2\left(f,x\delta_n\right)+\omega\left(f,x\alpha_n\right).\end{aligned}$$

Therefore, the proof is completed. □

Now we consider the elements of locally Lipschitz functions. Let $0<\alpha\leq 1$ and let D be a subset of the interval $[0,\infty)$. Then, by $Lip_M(D,\alpha)$ we denote the space of all functions satisfying the condition

$$|f(y)-f(x)|\leq M\,|y-x|^{\alpha} \quad \text{for} \quad y\in\overline{D} \text{ and } x\in[0,\infty), \tag{2.6}$$

where $\overline{D}$ denotes the closure of D in $[0,\infty)$.

Then we can give our second local approximation result.

Theorem 2. *Let D be a subset of $[0,\infty)$, $q\in(0,1)$ and $\alpha\in(0,1]$. Then, for every $n\in\mathbb{N}$, $x\in[0,+\infty)$ and $f\in C_B[0,\infty)\cap Lip_M(D,\alpha)$, we have*

$$\left|P_{n,q}(f;x)-f(x)\right|\leq M\left\{\gamma_n^{\alpha}x^{\alpha}+(d(x,D))^{\alpha}\right\},$$

where M is a positive constant; $d(x,D)$ is the distance between x and E defined as $d(x,D):=\inf\{|y-x|: y\in D\}$; and γ_n is given by

$$\gamma_n:=\left(\frac{[n+1]_q[n+2]_q}{[n]_q^2}-\frac{2[n+1]_q}{[n]_q}+1\right)^{1/2}. \tag{2.7}$$

Proof. Let $\overline{D}$ denote the closure of D in $[0,\infty)$. Let $x\in[0,\infty)$ be fixed. Then, there exists a point $x_0\in\overline{D}$ such that $|x-x_0|=d(x,D)$. Using the triangle inequality, we may write that

$$|f(y)-f(x)|\leq|f(y)-f(x_0)|+|f(x)-f(x_0)|.$$

Hence, it follows from (2.6) that

$$\begin{aligned}\left|P_{n,q}(f;x)-f(x)\right| &\leq P_{n,q}(|f(y)-f(x)|;x)\\ &\leq P_{n,q}(|f(y)-f(x_0)|;x)+|f(x)-f(x_0)|\\ &\leq M\left\{P_{n,q}(|y-x_0|^{\alpha};x)+|x-x_0|^{\alpha}\right\}.\\ &= M\left\{P_{n,q}(|y-x_0|^{\alpha};x)+(d(x,D))^{\alpha}\right\}.\end{aligned}$$

Using the Cauchy-Bunyakowsky-Schwarz inequality for positive linear operators we find that

$$\left|P_{n,q}(f;x)-f(x)\right|\leq M\left\{P_{n,q}(\varphi^2;x)^{\alpha/2}+(d(x,D))^{\alpha}\right\}.$$

Thus, the proof follows from (2.7) and Corollary 1. □

3. Rates of convergence of the operators $P_{n,q}$

Consider the following weighted space

$$E := \left\{ f \in C[0,\infty) : |f(x)| \le M(1+x^2) \text{ for some } M > 0 \right\}. \tag{3.1}$$

Now, for a given $b > 0$, by $\omega_b(f,\delta)$ we denote the usual modulus of continuity of f on the closed interval $[0,b]$, which is defined to be

$$\omega(f,\delta)_{[0,b]} := \sup\{|f(y) - f(x)| : |y-x| \le \delta;\ x,y \in [0,b]\}.$$

The next theorem gives the rate of convergence of the operators $P_{n,q}(f)$ to f for all $f \in E$.

Theorem 3. *For every $f \in E$, $b > 0$ and $q \in (0,1)$, we have*

$$\left\| P_{n,q}(f) - f \right\|_{[0,b]} \le C\left\{\gamma_n^2 + \omega(f,\gamma_n)_{[0,b+1]}\right\},$$

where γ_n is given by (2.7); C is a positive constant depending on f,b; and the symbol $\|\cdot\|_{[0,b]}$ denotes the classical sup-norm on the space $C[0,b]$.

Proof. Let $x \in [0,b]$ be fixed. Assume that $f \in E$, $b > 0$ and $q \in (0,1)$. If $y \le b+1$, then one can write, for any $\delta > 0$, that

$$|f(y) - f(x)| \le \omega(f,|y-x|)_{[0,b+1]} \le \left(1 + \frac{|y-x|^2}{\delta^2}\right)\omega(f,\delta)_{[0,b+1]}. \tag{3.2}$$

On the other hand, if $y > b+1$, since $y - x > 1$, we get, for some positive constant M depending on $f \in E$, that

$$\begin{aligned} |f(y) - f(x)| &\le M\left\{2 + x^2 + y^2\right\} \\ &= M\left\{2 + x^2 + ((y-x)+x)^2\right\} \\ &= M\left\{2 + 2x^2 + 2x(y-x) + (y-x)^2\right\} \\ &\le M\left\{2 + 2x^2 + (2x+1)(y-x)^2\right\} \\ &\le M\left(2x^2 + 2x + 3\right)(y-x)^2, \end{aligned}$$

which implies that

$$|f(y) - f(x)| \le N(y-x)^2, \tag{3.3}$$

where $N := M\left(2b^2 + 2b + 3\right)$. From (3.2) and (3.3), we get, for all $x \in [0,b]$ and $y \ge 0$, that

$$|f(y) - f(x)| \le N(y-x)^2 + \left(1 + \frac{(y-x)^2}{\delta^2}\right)\omega(f,\delta)_{[0,b+1]}$$

and therefore

$$\left|P_{n,q}(f;x) - f(x)\right| \le N P_{n,q}(\varphi^2;x) + \omega(f,\delta)_{[0,b+1]}\left(1 + \frac{1}{\delta^2}P_{n,q}(\varphi^2;x)\right).$$

Now applying Corollary 1, we obtain that

$$\left|P_{n,q}(f;x)-f(x)\right| \leq N\gamma_n^2 x^2+\omega(f,\delta)_{[0,b+1]}\left(1+\frac{\gamma_n^2 x^2}{\delta^2}\right)$$
$$\leq Nb^2\gamma_n^2+\omega_{b+1}(f,\delta)_{[0,b+1]}\left(1+\frac{\gamma_n^2 b^2}{\delta^2}\right),$$

where γ_n is given by (2.7). If we take $\delta=\gamma_n$ on the right hand side of the last inequality, then we immediately see that

$$\left|P_{n,q}(f;x)-f(x)\right| \leq C\left\{\gamma_n^2+\omega(f,\gamma_n)_{[0,b+1]}\right\},$$

where $C:=\max\{Nb^2,1+b^2\}$. Finally, taking supremum over $x\in[0,b]$, the proof is completed. □

4. Statistical approximation results

In this section, we study the statistical approximation properties of the q-Post-Widder operators. We first recall the concept of statistical convergence.

Let $A:=[a_{jn}]$, $j,n\in\mathbb{N}$, be an infinite matrix. The A-transform of a sequence $x:=(x_n)$ is given by $Ax:=((Ax)_n)$ with $(Ax)_n=\sum_{n=1}^{\infty}a_{jn}$ provided that the series converges for each $n\in\mathbb{N}$. Then, we say that a matrix A is regular if $\lim Ax=L$ whenever $\lim x=L$. Now let $A=[a_{jn}]$ be nonnegative regular summability matrix. Then, a sequence $x=(x_n)$ is said to be A-statistically convergent to L if $\lim_n\sum_{n:|x_n-L|\geq\varepsilon}a_{jn}=0$ holds for every $\varepsilon>0$. In this case, we write $st_A-\lim x=L$ (see [9]). It is well-known that if we take $A=C_1=[c_{jn}]$, the Cesáro matrix of order one defined to be $c_{jn}=1/j$ if $1\leq n\leq j$, and $c_{jn}=0$ otherwise, then we get the classical definition of statistical convergence (see [8]). In this case, we use the notation $st-\lim$ instead of $st_{C_1}-\lim$. Also, if $A=I$, the identity matrix, then it reduces to the ordinary ordinary convergence. Notice that every convergent sequence is A-statistically convergent to the same value, but the converse does not hold true. As we can see the following example, there exists an A-statistically convergent sequence but non-convergent in the usual sense.

Now let $A=[a_{jn}]$ be a nonnegative regular summability matrix. Assume that (q_n) is sequence from $(0,1)$ such that

$$st_A-\lim_n\frac{1}{[n]_{q_n}}=0,\;\; st_A-\lim_n q_n=1 \text{ and } st_A-\lim_n q_n^n=\beta\in(0,1). \tag{4.1}$$

Indeed, such a sequence (q_n) can be constructed by the following way. Take $A=C_1=[c_{jn}]$ and define the sequence (q_n) by

$$q_n:=\begin{cases} e^{-n}, & \text{if } n=m^2 \ (m\in\mathbb{N}) \\ \dfrac{n}{n+1}, & \text{otherwise.}\end{cases} \tag{4.2}$$

Then observe that $q_n \in (0,1)$ for each $n \in \mathbb{N}$. It is easy to check that although (q_n) is non-convergent in the usual sense, it satisfies the conditions in (4.1) with the choice of $A = C_1$ and $\beta = 1/e$. On the other hand, since

$$\frac{[n+1]_{q_n}}{[n]_{q_n}} = \frac{1-q_n^n}{1-q_n^{n-1}} \quad \text{and} \quad \frac{[n+1]_{q_n}[n+2]_{q_n}}{[n]_{q_n}^2} = \frac{(1-q_n^n)(1-q_n^{n+1})}{\left(1-q_n^{n-1}\right)^2},$$

we obtain from (4.1) that

$$st_A - \lim_n \frac{[n+1]_{q_n}}{[n]_{q_n}} = st_A - \lim_n \frac{[n+1]_{q_n}^2}{[n]_{q_n}^2} = st_A - \lim_n \frac{[n+1]_{q_n}[n+2]_{q_n}}{[n]_{q_n}^2} = 1. \quad (4.3)$$

Then it follows from (4.3) that

$$st_A - \lim_n \delta_n = st_A - \lim \alpha_n = st_A - \lim \gamma_n = 0, \quad (4.4)$$

where δ_n, α_n and γ_n are given by (2.1), (2.2) and (2.7), respectively.

Now taking a sequence (q_n) from $(0,1)$ satisfying (4.1) instead of a fixed number $q \in (0,1)$ in the definition of operators (1.1), we obtain the following (pointwise) statistical approximation result.

Theorem 4. *Let $A = [a_{jn}]$ be a nonnegative regular summability matrix, and let (q_n) be a sequence from $(0,1)$ satisfying (4.1). Then, for every $x \in [0,+\infty)$ and $f \in C_B[0,\infty)$, we have*

$$st_A - \lim_n \left|P_{n,q_n}(f;x) - f(x)\right| = 0.$$

Proof. Let $x \in [0,\infty)$ and $f \in C_B[0,\infty)$ be fixed. By the right continuity of ω and ω_2 at zero, one can get from (4.4) that

$$st_A - \lim_n \omega(f, x\alpha_n) = st_A - \lim_n \omega_2(f, x\delta_n) = 0. \quad (4.5)$$

Now, for a given $\varepsilon > 0$, define the following sets:

$$U := \left\{n \in \mathbf{N} : \left|P_{n,q_n}(f;x) - f(x)\right| \geq \varepsilon\right\},$$
$$U_1 := \left\{n \in \mathbf{N} : \omega(f;x\alpha_n) \geq \frac{\varepsilon}{2}\right\},$$
$$U_2 := \left\{n \in \mathbf{N} : \omega_2(f;x\delta_n) \geq \frac{\varepsilon}{2C}\right\},$$

where C is a positive constant as in Theorem 1. Then, we may write from the inequality in Theorem 1 that

$$U \subseteq U_1 \cup U_2,$$

which gives, for every $j \in \mathbb{N}$,

$$\sum_{n \in U} a_{jn} \leq \sum_{n \in U_1} a_{jn} + \sum_{n \in U_2} a_{jn}. \quad (4.6)$$

Letting $j \to \infty$ in (4.6) and also using (4.5), we have

$$\lim_j \sum_{n \in U} a_{jn} = 0,$$

which guarantees that

$$st_A - \lim_n \left| P_{n,q_n}(f;x) - f(x) \right| = 0.$$

Thus, the proof is completed. □

If we take $A = I$, the identity matrix, by Theorem 4, we obtain the classical approximation result.

Corollary 2. *Let (q_n) be a sequence from $(0,1)$ satisfying the conditions:*

$$\lim_n \frac{1}{[n]_{q_n}} = 0, \ \lim_n q_n = 1 \text{ and } \lim_n q_n^n = \eta \in (0,1). \tag{4.7}$$

Then, for every $x \in [0,+\infty)$ and $f \in C_B[0,\infty)$, the sequence $\{P_{n,q_n}(f;x)\}$ converges (pointwise) to $f(x)$.

We should note that if a sequence (q_n) satisfies (4.7), then it also satisfies (4.1); and hence the classical (pointwise) approximation result in Corollary 2 implies the statistical one in Theorem 4. However, if we choose the sequence (q_n) as in (4.2), then we can easily see that (4.7) does not hold true while (4.1) is still valid. This example shows that our statistical approximation result in Theorem 4 is more applicable than the classical one.

Now we obtain a uniform approximation theorem in statistical sense.

Theorem 5. *Let $A = [a_{jn}]$ be a nonnegative regular summability matrix, and let (q_n) be a sequence from $(0,1)$ satisfying (4.1). Assume that D is a compact set in $[0,\infty)$. Then, for every $f \in C_B[0,\infty) \cap Lip_M(D,\alpha)$ with $M > 0$ and $\alpha \in (0,1]$, we have*

$$st_A - \lim_n \left\| P_{n,q_n}(f) - f \right\|_D = 0.$$

Proof. Let $x \in D$ be fixed. In this case, we have $d(x,D) = 0$. Using this and Theorem 2, we immediately get that, for each $x \in D$,

$$\left| P_{n,q_n}(f;x) - f(x) \right| \le M \gamma_n^{\alpha} x^{\alpha}. \tag{4.8}$$

Since D is compact, the number $L := \sup_{x \in D}\{x^{\alpha}\}$ is finite. Hence, taking supremum over $x \in D$ on the both sides of (4.8), we obtain that

$$\left\| P_{n,q_n}(f) - f \right\|_D \le LM\gamma_n^{\alpha},$$

which yields, for every $\varepsilon > 0$,

$$\left\{ n \in \mathbf{N} : \left\| P_{n,q_n}(f) - f \right\|_D \ge \varepsilon \right\} \subseteq \left\{ n \in \mathbf{N} : \gamma_n \ge \left(\frac{\varepsilon}{LM} \right)^{1/\alpha} \right\}$$

and hence

$$\sum_{n:\ \|P_{n,q_n}(f)-f\|_D \geq \varepsilon} a_{jn} \leq \sum_{n:\ \gamma_n \geq \left(\frac{\varepsilon}{LM}\right)^{1/\alpha}} a_{jn}$$

holds for every $j \in \mathbb{N}$. Now taking limit as $j \to \infty$ on both sides of the last inequality, and also considering (4.4), we deduce that

$$\lim_n \sum_{n:\ \|P_{n,q_n}(f)-f\|_D \geq \varepsilon} a_{jn} = 0,$$

which is the desired result. □

If $A = I$ in Theorem 5, then we easily get the uniform approximation result in the ordinary sense.

Corollary 3. *Let (q_n) be a sequence from $(0,1)$ satisfying (4.7), and let D be a compact set in $[0,\infty)$. Then, for every $f \in C_B[0,\infty) \cap Lip_M(D,\alpha)$ with $M > 0$ and $\alpha \in (0,1]$, the sequence $\{P_{n,q_n}(f)\}$ is uniformly convergent to f on D.*

For a uniform approximation process, we also get the next result.

Theorem 6. *Let $A = [a_{jn}]$ be a nonnegative regular summability matrix, and let (q_n) be a sequence from $(0,1)$ satisfying (4.1). Then, for every $f \in E$ and $b > 0$, we have*

$$st_A - \lim_n \|P_{n,q_n}(f) - f\|_{[0,b]} = 0.$$

Proof. Assume that $x \in [0,b]$ is fixed. Then, for a given $\varepsilon > 0$, consider the following sets:

$$V := \left\{ n \in \mathbf{N} : \|P_{n,q_n}(f) - f\|_{[0,b]} \geq \varepsilon \right\},$$

$$V_1 := \left\{ n \in \mathbf{N} : \gamma_n \geq \left(\frac{\varepsilon}{2C}\right)^{1/2} \right\},$$

$$V_2 := \left\{ n \in \mathbf{N} : \omega(f,\gamma_n)_{[0,b+1]} \geq \frac{\varepsilon}{2C} \right\},$$

where γ_n is given by (2.7), and C is a positive constant as in Theorem 3. Then, it follows from Theorem 3 that

$$V \subseteq V_1 \cup V_2.$$

The last inclusion implies that

$$\sum_{n \in V} a_{jn} \leq \sum_{n \in V_1} a_{jn} + \sum_{n \in V_2} a_{jn}. \tag{4.9}$$

On the other hand, by the right continuity of $\omega(f,\cdot)_{[0,b+1]}$ at zero, one can get from (4.4) that

$$st_A - \lim_n \omega(f,\gamma_n)_{[0,b+1]} = 0. \tag{4.10}$$

Then, letting $j \to \infty$ in (4.9) and using (4.4), (4.10), we easily get that

$$\lim_{n} \sum_{n \in V} a_{jn} = 0,$$

which completes the proof. □

Of course, if we take $A = I$ in Theorem 6, we obtain the next result.

Corollary 4. *Let (q_n) be a sequence from $(0,1)$ satisfying (4.7). Then, for every $f \in E$ and $b > 0$, the sequence $\{P_{n,q_n}(f)\}$ is uniformly convergent to f on $[0,b]$.*

5. Voronovskaya-type results

In this section, we prove a Voronovskaya-type theorem for the q-Post-Widder operators. Throughout this section we consider a sequence (q_n) satisfying (4.1) instead of a fixed $q \in (0,1)$ in the definition (1.1).

We first need the following lemma.

Lemma 2. *Let $A = [a_{jn}]$ be a nonnegative regular summability matrix, and let (q_n) be a sequence from $(0,1)$ satisfying (4.1). Then, we have*

$$st_A - \lim_{n\to\infty} [n]_{q_n} P_{n,q_n}(\varphi;x) = \beta x,$$
$$st_A - \lim_{n\to\infty} [n]_{q_n} P_{n,q_n}(\varphi^2;x) = \beta x^2,$$
$$st_A - \lim_{n\to\infty} [n]_{q_n}^2 P_{n,q_n}\left(\varphi^4;x\right) = 3\beta^2(1-\beta)x^4$$

uniformly with respect to $x \in [0,b]$, $(b > 0)$, where $\beta := st_A - \lim q_n^n$ as stated in (4.1).

Proof. It follows from Corollary 1 that

$$[n]_{q_n} P_{n,q_n}(\varphi;x) = \left([n+1]_{q_n} - [n]_{q_n}\right)x$$

and

$$[n]_{q_n} P_{n,q_n}(\varphi^2;x) = \left(\frac{[n+1]_{q_n}[n+2]_{q_n}}{[n]_{q_n}} - 2[n+1]_{q_n} + [n]_{q_n}\right)x^2.$$

Then, using (4.1) we easily get

$$st_A - \lim_{n\to\infty} [n]_{q_n} P_{n,q_n}(\varphi;x) = \beta x$$

and

$$st_A - \lim_{n\to\infty} [n]_{q_n} P_{n,q_n}(\varphi^2;x) = \beta x^2$$

uniformly with respect to $x \in [0,b]$. On the other hand, by Lemma 1, we obtain that

$$P_{n,q_n}\left(\varphi^4;x\right) = P_{n,q_n}(e_4;x) - 4xP_{n,q_n}(e_3;x) + 6x^2 P_{n,q_n}(e_2;x)$$
$$- 4x^3 P_{n,q_n}(e_1;x) + x^4$$

$$
\begin{aligned}
&= \frac{[n+1]_{q_n}[n+2]_{q_n}[n+3]_{q_n}[n+4]_{q_n}}{[n]_{q_n}^4} x^4 \\
&\quad -4\frac{[n+1]_{q_n}[n+2]_{q_n}[n+3]_{q_n}}{[n]_{q_n}^3} x^4 \\
&\quad +6\frac{[n+1]_{q_n}[n+2]_{q_n}}{[n]_{q_n}^2} x^4 - 4\frac{[n+1]_{q_n}}{[n]_{q_n}} x^4 + x^4.
\end{aligned}
$$

Then, we get

$$
\begin{aligned}
[n]_{q_n}^2 P_{n,q_n}\left(\varphi^4;x\right) = \Bigg(& \frac{[n+1]_{q_n}[n+2]_{q_n}[n+3]_{q_n}[n+4]_{q_n}}{[n]_{q_n}^2} \\
& -4\frac{[n+1]_{q_n}[n+2]_{q_n}[n+3]_{q_n}}{[n]_{q_n}} \\
& +6[n+1]_{q_n}[n+2]_{q_n} - 4[n]_{q_n}[n+1]_{q_n} + [n]_{q_n}^2 \Bigg) x^4.
\end{aligned}
$$

Now using the fact that

$$
[n]_{q_n} = \frac{1-q_n^{n-1}}{1-q_n}
$$

and also considering (4.1), after some simple calculations, for each $x \in [0,\infty)$, we have

$$
st_A - \lim_{n\to\infty} [n]_{q_n}^2 P_{n,q_n}\left(\varphi^4;x\right) = 3\beta^2(1-\beta)x^4,
$$

uniformly with respect to $x \in [0,b]$, which is the desired result. □

Then we get our Voronovskaya-type theorem as follows.

Theorem 7. *Let $A = [a_{jn}]$ be a nonnegative regular summability matrix, and let (q_n) be a sequence from $(0,1)$ satisfying (4.1). Then, for every $f \in E$ such that f', $f'' \in E$, we have*

$$
st_A - \lim_{n\to\infty} [n]_{q_n} \left\{P_{n,q_n}(f;x) - f(x)\right\} = \beta x f'(x) + \frac{\beta x^2 f''(x)}{2}
$$

uniformly with respect to $x \in [0,b]$, $(b > 0)$, where $\beta := st_A - \lim q_n^n$ as stated in (4.1).

Proof. Let f, f', $f'' \in E$. For each $x \geq 0$, define a function $\Psi(y) := \Psi(y,x)$ by

$$
\Psi(y) = \begin{cases} \dfrac{f(y) - f(x) - (y-x)f'(x) - \frac{1}{2}(y-x)^2 f''(x)}{(y-x)^2}, & \text{if } y \neq x \\ 0, & \text{if } y = x. \end{cases}
$$

Then by assumption we have $\Psi(x,x) = 0$ and the function $\Psi(\cdot,x)$ belongs to E. Hence, by Taylor's theorem we get

$$f(y) = f(x) + (y-x)f'(x) + \frac{(y-x)^2}{2}f''(x) + (y-x)^2\Psi(y,x).$$

Hence we easily observe that

$$\begin{aligned}[n]_{q_n}\{P_{n,q_n}(f;x) - f(x)\} &= f'(x)[n]_{q_n}P_{n,q_n}(\varphi;x) \\ &+ \frac{f''(x)}{2}[n]_{q_n}P_{n,q_n}(\varphi^2;x) \\ &+ [n]_{q_n}P_{n,q_n}(\varphi^2\Psi;x).\end{aligned}$$

If we apply the Cauchy-Schwarz inequality for the last term on the right-hand side of the last equality, then we conclude that

$$[n]_{q_n}\left|P_{n,q_n}\left(\varphi^2\Psi;x\right)\right| \le \left([n]_{q_n}^2 P_{n,q_n}(\varphi^4;x)\right)^{1/2}\left(P_{n,q_n}(\Psi^2;x)\right)^{1/2}. \tag{5.1}$$

Let $\eta(y,x) := \Psi^2(y,x)$. In this case, observe that $\eta(x,x) = 0$ and $\eta(\cdot,x) \in E$. Then, it follows from Theorem 6 that

$$st_A - \lim_{n\to\infty} P_{n,q_n}\left(\Psi^2;x\right) = st_A - \lim_{n\to\infty} P_{n,q_n}(\eta(y,x);x) = \eta(x,x) = 0 \tag{5.2}$$

uniformly with respect to $x \in [0,b]$, $(b > 0)$. Now, by (5.1), (5.2), and Lemma 2, we obtain that

$$st_A - \lim_{n\to\infty}[n]_{q_n}P_{n,q_n}(\varphi^2\Psi;x) = 0$$

uniformly with respect to $x \in [0,b]$. Combining the above facts with Lemma 2, we observe that

$$st_A - \lim_{n\to\infty}[n]_{q_n}\left\{P_{n,q_n}(f;x) - f(x)\right\} = \beta x f'(x) + \frac{\beta x^2 f''(x)}{2}$$

uniformly with respect to $x \in [0,b]$. Thus, the proof is completed. □

The next result is an immediate consequence of Theorem 7.

Corollary 5. *Let (q_n) be a sequence from $(0,1)$ satisfying (4.7). Then, for every $f \in E$ such that f', $f'' \in E$, we have*

$$\lim_{n\to\infty}[n]_{q_n}\left\{P_{n,q_n}(f;x) - f(x)\right\} = \eta x f'(x) + \frac{\eta x^2 f''(x)}{2}$$

uniformly with respect to $x \in [0,b]$, $(b > 0)$, where $\eta = \lim q_n^n$ as stated in (4.1).

6. Approximation properties of complex q-Post-Widder operators

In this section, we will introduce the complex q-Post-Widder operators and study their approximation properties.

Now, let

$$D := \{z \in \mathbb{C} : |z| < 1\}, \ \bar{D} := \{z \in \mathbb{C} : |z| \leq 1\}$$

and

$$C\left(\bar{D}\right) : \left\{f : \bar{D} \to \mathbb{C} : f \text{ is continuous on } D\right\}.$$

Then, we propose the complex q-Post-Widder operators as follows:

$$P^{*}_{n,q}(f;z) = \frac{1}{[n]_q!} \int_0^{1/(1-q)} f\left(\frac{ze^{it}}{[n]_q}\right) t^n E_q(-qt)\, d_q t, \tag{6.1}$$

where $z \in \bar{D}$, $n \in \mathbb{N}$, $f \in C\left(\bar{D}\right)$ and $q \in (0,1)$.

It is not hard to see that if f is a constant function on $\bar{D}$, say $f(z) \equiv c$, then we have, for every $n \in \mathbb{N}$ that $P^{*}_{n,q}(c;z) = c$. Hence, the operators $P^{*}_{n,q}$ preserve the constant functions. In order to get some geometric properties of the operators $P^{*}_{n,q}$ in (6.1) we first need the following concepts.

Now define the subspace of $C\left(\bar{D}\right)$ by

$$A\left(\bar{D}\right) := \left\{f \in C\left(\bar{D}\right) : f \text{ is analytic on } D \text{ with } f(0) = 0\right\}.$$

If $f \in C\left(\bar{D}\right)$, then, as in Section 3, we consider the (first) modulus of continuity of f on $\bar{D}$, denoted by $\omega(f,\delta)_{\bar{D}}$, $\delta > 0$, as follows (see [11]):

$$\omega(f;\delta)_{\bar{D}} := \sup\left\{|f(z) - f(w)| : |z-w| \leq \delta, \ z,w \in \bar{D}\right\}.$$

It is easy to see that, for any $c,\delta > 0$

$$\omega(f;c\delta)_{\bar{D}} \leq (1+c)\omega(f;\delta)_{\bar{D}}. \tag{6.2}$$

We obtain the following result.

Theorem 8. *For each fixed $n \in \mathbb{N}$ and $q \in (0,1)$, $P^{*}_{n,q}\left(A\left(\bar{D}\right)\right) \subset A\left(\bar{D}\right)$.*

Proof. Let $n \in \mathbb{N}$, $f \in A\left(\bar{D}\right)$ and $q \in (0,1)$ be fixed. Since $f(0) = 0$, it follows from (6.1) that

$$P^{*}_{n,q}(f;0) = \frac{1}{[n]_q!} \int_0^{1/(1-q)} f(0)\, t^n E_q(-qt)\, d_q t = 0.$$

Now we show that $P_{n,q}(f)$ is continuous on $\bar{D}$. To see this assume that z, $(z_m) \in \bar{D}$ and that $\lim_m z_m = z$. Hence, we get from the definition of ω that

$$\left|P^{*}_{n,q}(f;z_m) - P^{*}_{n,q}(f;z)\right|$$

$$\begin{aligned}
&\le \frac{1}{[n]_q!}\int_0^{1/(1-q)}\left| f\left(\frac{z_m e^{it}}{[n]_q}\right) - f\left(\frac{z e^{it}}{[n]_q}\right)\right| t^n E_q(-qt)\, d_q t \\
&\le \frac{\omega\left(f, \frac{|z_m - z|}{[n]_q}\right)_{\bar{D}}}{[n]_q!}\int_0^{1/(1-q)} t^n E_q(-qt)\, d_q t \\
&= \omega\left(f, \frac{|z_m - z|}{[n]_q}\right)_{\bar{D}} \\
&\le \omega(f, |z_m - z|)_{\bar{D}}
\end{aligned}$$

due to $[n]_q \ge 1$ for every $n \in \mathbb{N}$. Since $\lim_m z_m = z$, we may write that

$$\lim_m \omega(f, |z_m - z|)_{\bar{D}} = 0.$$

Due to the right continuity of $\omega(f, \cdot)$ at zero. Hence, we get

$$\lim_m P^*_{n,q}(f; z_m) = P_{n,q}(f; z),$$

which gives the continuity of $P^*_{n,q}(f)$ at the point $z \in \bar{D}$.

Finally, since $f \in A(\bar{D})$, we can write $f(z) = \sum_{k=1}^{\infty} a_k z^k$ for $z \in D$. Then, we get

$$f\left(\frac{z e^{it}}{[n]_q}\right) = \sum_{k=1}^{\infty} \frac{a_k z^k e^{ikt}}{[n]_q^k}. \tag{6.3}$$

Since $\left|\frac{a_k e^{ikt}}{[n]_q^k}\right| \le |a_k|$ for every $t \in \mathbb{R}$, the series in (6.3) is uniformly convergent with respect to $t \in \mathbb{R}$. Hence, we conclude that

$$\begin{aligned}
P^*_{n,q}(f; z) &= \frac{1}{[n]_q!}\int_0^{1/(1-q)}\left(\sum_{k=1}^{\infty} \frac{a_k z^k e^{ikt}}{[n]_q^k}\right) t^n E_q(-qt)\, d_q t \\
&= \frac{1}{[n]_q!}\sum_{k=1}^{\infty} \frac{a_k z^k}{[n]_q^k}\left(\int_0^{1/(1-q)} e^{ikt} t^n E_q(-qt)\, d_q t\right) \\
&= \sum_{k=1}^{\infty} \ell_{n,k} z^k,
\end{aligned}$$

where $\ell_{n,k}$ is given by

$$\ell_{n,k} := \frac{a_k}{[n]_q! [n]_q^k}\int_0^{1/(1-q)} e^{ikt} t^n E_q(-qt)\, d_q t \quad (k, n \in \mathbb{N}). \tag{6.4}$$

We should remark that

$$\left|\ell_{n,k}\right| \leq |a_k| \text{ for every } k,n \in \mathbb{N}.$$

Therefore, for each $n \in \mathbb{N}$ and $f \in A(\bar{D})$, the function $P^*_{n,q}(f)$ has a Taylor series expansion whose Taylor coefficients are $\ell_{n,k}$ given by (6.4). Combining the above facts we obtain the desired result. □

Now consider the following space:

$$B(\bar{D}) := \{f \in C(\bar{D}) : f \text{ is analytic on } D \text{ with } f(0) = 1 \text{ and } Re[f(z)] > 0 \text{ for every } z \in D\}.$$

Then we have the next result.

Theorem 9. *For each fixed $n \in \mathbb{N}$ and $q \in (0,1)$, $P^*_{n,q}(B(\bar{D})) \subset B(\bar{D})$.*

Proof. Let $n \in \mathbb{N}$, $f \in B(\bar{D})$ and $q \in (0,1)$ be fixed. As in the proof of Theorem 8, we see that $P^*_{n,q}(f)$ is analytic on D and continuous on $\bar{D}$. Since $f(0) = 1$, we easily get that

$$P^*_{n,q}(f;0) = \frac{1}{[n]_q!} \int_0^{1/(1-q)} f(0)\, t^n E_q(-qt)\, d_q t = 1.$$

Finally, we may write that, for every $z \in D$,

$$Re\left[P^*_{n,q}(f;z)\right] = \frac{1}{[n]_q!} \int_0^{1/(1-q)} Re\left[f\left(\frac{ze^{it}}{[n]_q}\right)\right] t^n E_q(-qt)\, d_q t > 0$$

since $Re\left[f(\frac{ze^{it}}{[n]_q})\right] > 0$. Thus, the proof is completed. □

If we consider the following space of Lipschitz class functions:

$$Lip^*_M(\bar{D};\alpha) := \{f : \bar{D} \to \mathbb{C}; |f(z) - f(w)| \leq M|z-w|^{\alpha} \text{ for every } z,w \in \bar{D}\},\quad (\alpha \in (0,1] \text{ and } M > 0),$$

we can get the next result.

Theorem 10. *For each fixed $n \in \mathbb{N}$ and $q \in (0,1)$, $P^*_{n,q}\left(Lip^*_M(\bar{D};\alpha)\right) \subset Lip^*_M(\bar{D};\alpha)$.*

Proof. Let $f \in Lip^*_M(\bar{D};\alpha)$. Then, for every $z,w \in \bar{D}$, we observe that

$$\left|P^*_{n,q_n}(f;z) - P^*_{n,q}(f;w)\right| \leq \frac{1}{[n]_q!} \int_0^{1/(1-q)} \left|f\left(\frac{ze^{it}}{[n]_q}\right) - f\left(\frac{we^{it}}{[n]_q}\right)\right| t^n E_q(-qt) d_q t$$

$$
\begin{aligned}
&\leq \frac{M}{[n]_q!} \int_0^{1/(1-q)} \left| \frac{ze^{it}}{[n]_q} - \frac{we^{it}}{[n]_q} \right|^\alpha t^n E_q(-qt) d_q t \\
&= \frac{M |z-w|^\alpha}{[n]_q^\alpha} \\
&\leq M |z-w|^\alpha,
\end{aligned}
$$

which completes the proof. □

Using the definition of $\omega(f;\delta)_{\bar{D}}$ for $f \in C\left(\bar{D}\right)$ and $\delta > 0$, we obtain the following global smoothness preservation for the operators $P_{n,q}^*$ given by (6.1).

Theorem 11. *For each fixed $n \in \mathbb{N}$, $q \in (0,1)$ and $f \in C\left(\bar{D}\right)$, we have*

$$\omega(P_{n,q}^*(f);\delta)_{\bar{D}} \leq \omega(f;\delta)_{\bar{D}}.$$

Proof. Let $\delta > 0$, $n \in \mathbb{N}$, $q \in (0,1)$ and $f \in C\left(\bar{D}\right)$ be given. Assume that $z, w \in \bar{D}$ and $|z-w| \leq \delta$. Then, we have

$$
\begin{aligned}
\left|P_{n,q}^*(f;z) - P_{n,q}^*(f;w)\right| &\leq \frac{1}{[n]_q!} \int_0^{1/(1-q)} \left| f\left(\frac{ze^{it}}{[n]_q}\right) - f\left(\frac{we^{it}}{[n]_q}\right) \right| t^n E_q(-qt) d_q t \\
&\leq \omega\left(f; \frac{|z-w|}{[n]_q}\right)_{\bar{D}} \leq \omega\left(f; \frac{\delta}{[n]_q}\right)_{\bar{D}} \leq \omega(f;\delta)_{\bar{D}}.
\end{aligned}
$$

Then, taking supremum over $|z-w| \leq \delta$, we conclude that

$$\omega(P_{n,q}^*(f);\delta)_{\bar{D}} \leq \omega(f;\delta)_{\bar{D}},$$

whence the result. □

Now we obtain an estimation with the help of $\omega(f;\delta)_{\bar{D}}$ for the operators $P_{n,q}^*$ defined by (6.1).

Theorem 12. *For each fixed $n \in \mathbb{N}$, $q \in (0,1)$ and $f \in C\left(\bar{D}\right)$, we have*

$$\left\| P_{n,q}^*(f) - f \right\|_{\bar{D}} \leq 3\omega\left(f; \frac{1}{\sqrt{[n]_q}}\right)_{\bar{D}}$$

for some (finite) positive constant M, where $\|\cdot\|_{\bar{D}}$ denotes the usual sup-norm on $\bar{D}$ as stated before.

Proof. Let $z \in \bar{D}$ and $f \in C\left(\bar{D}\right)$ be fixed. We first observe that

$$\left|P_{n,q}^*(f;z) - f(z)\right| \leq \frac{1}{[n]_q!} \int_0^{1/(1-q)} \left| f\left(\frac{ze^{it}}{[n]_q}\right) - f(z) \right| t^n E_q(-qt) d_q t$$

$$\leq \frac{1}{[n]_q!} \int_0^{1/(1-q)} \omega\left(f; |z| \left|\frac{e^{it}}{[n]_q} - 1\right|\right)_{\bar{D}} t^n E_q(-qt)\, d_q t.$$

Since, for every $z \in D$ and $t \geq 0$,

$$|z| \left|\frac{e^{it}}{[n]_q} - 1\right| \leq \sqrt{\frac{2}{[n]_q^2} - \frac{2\cos t}{[n]_q}} \leq \frac{2|\sin(t/2)|}{\sqrt{[n]_q}} \leq \frac{2}{\sqrt{[n]_q}} \tag{6.5}$$

we may write that

$$\left|P^*_{n,q}(f;z) - f(z)\right| \leq \frac{1}{[n]_q!} \omega\left(f; \frac{2}{\sqrt{[n]_q}}\right)_{\bar{D}} \int_0^{1/(1-q)} t^n E_q(-qt)\, d_q t$$

$$= \omega\left(f; \frac{2}{\sqrt{[n]_q}}\right)_{\bar{D}}$$

Then, by (6.2) we get

$$\left|P^*_{n,q}(f;z) - f(z)\right| \leq 3\omega\left(f; \frac{1}{\sqrt{[n]_q}}\right)_{\bar{D}}$$

Taking supremum over $z \in \bar{D}$ on the last inequality, the proof is completed. □

Using the Lipschitz class functions, we also obtain the next estimation result for the operators $P^*_{n,q}$.

Theorem 13. *For each fixed $n \in \mathbb{N}$, $q \in (0,1)$ and $f \in Lip^*_M(\bar{D};\alpha)$, we have*

$$\left\| P^*_{n,q}(f) - f \right\|_{\bar{D}} \leq \frac{2^\alpha M}{[n]_q^{\alpha/2}}.$$

Proof. Let $z \in \bar{D}$ and $f \in Lip^*_M(\bar{D};\alpha)$ be given. Then, one can write that

$$\left|P^*_{n,q}(f;z) - f(z)\right| \leq \frac{1}{[n]_q!} \int_0^{1/(1-q)} \left| f\left(\frac{ze^{it}}{[n]_q}\right) - f(z) \right| t^n E_q(-qt)\, d_q t$$

$$\leq \frac{M}{[n]_q!} \int_0^{1/(1-q)} |z|^\alpha \left|\frac{e^{it}}{[n]_q} - 1\right|^\alpha t^n E_q(-qt)\, d_q t.$$

By (6.5), we get

$$\left|P^*_{n,q}(f;z) - f(z)\right| \leq \frac{2^\alpha M}{[n]_q^{\alpha/2}}.$$

Taking supremum over $z \in \bar{D}$ in the last inequality, the proof is completed. □

Now replacing the fixed number $q \in (0,1)$ with a sequence (q_n) from the interval $(0,1)$ which satisfies the conditions in (4.1), we can also obtain a statistical approximation result for our complex operators defined by (6.1)

Theorem 14. *Let $A := [a_{jn}]$ be a nonnegative regular summability matrix, and let (q_n) be a sequence from $(0,1)$ satisfying (4.1). Then, for every $f \in C(\bar{D})$, we have*

$$st_A - \lim_n \left\| P^*_{n,q_n}(f) - f \right\|_{\bar{D}} = 0.$$

Proof. Let $f \in C(\bar{D})$. Then, for a given $\varepsilon > 0$, we may write from Theorem 11 that

$$U := \left\{ n \in \mathbb{N} : \left\| P^*_{n,q_n}(f) - f \right\|_{\bar{D}} \geq \varepsilon \right\} \subseteq \left\{ n \in \mathbb{N} : \omega\left(f; \frac{1}{\sqrt{[n]_{q_n}}} \right)_{\bar{D}} \geq \frac{\varepsilon}{3} \right\} =: V.$$

Thus, for every $j \in \mathbb{N}$, we get

$$\sum_{n \in U} a_{jn} \leq \sum_{n \in V} a_{jn}. \tag{6.6}$$

On the other hand, by the right continuity of $\omega(f,\cdot)_{\bar{D}}$ at zero, one can get from (4.1) that

$$st_A - \lim_n \omega\left(f; \frac{1}{\sqrt{[n]_{q_n}}} \right)_{\bar{D}} = 0,$$

which gives

$$\lim_j \sum_{n \in V} a_{jn} = 0. \tag{6.7}$$

Now taking limit as $j \to \infty$ in the both sides of (6.6) and also considering (6.7) we obtain that

$$\lim_j \sum_{n \in U} a_{jn} = 0,$$

which means that

$$st_A - \lim_n \left\| P^*_{n,q_n}(f) - f \right\| = 0.$$

The proof is completed. □

If we replace A by the identity matrix in Theorem 14, then we get the following uniform approximation result.

Corollary 6. *Let (q_n) be a sequence from $(0,1)$ satisfying (4.7). Then, for every $f \in A(\bar{D})$, the sequence $\left\{ P^*_{n,q_n}(f) \right\}_{n \in \mathbb{N}}$ is uniformly convergent to f on $\bar{D}$.*

Finally, we give a statistical approximation result on the space $Lip^*_M(\bar{D};\alpha)$.

Theorem 15. *Let $A := [a_{jn}]$ be a nonnegative regular summability matrix, and let (q_n) be a sequence from $(0,1)$ satisfying (4.1). Then, for every $f \in Lip^*_M(\bar{D};\alpha)$, we have*

$$st_A - \lim_n \| P^*_{n,q_n}(f) - f \|_{\bar{D}} = 0.$$

Proof. Let $z \in \bar{D}$ and $f \in Lip^*_M(\bar{D};\alpha)$. By Theorem 13, we get, for every $\varepsilon > 0$, that

$$\left\{ n \in \mathbf{N} : \| P^*_{n,q_n}(f) - f \|_{\bar{D}} \geq \varepsilon \right\} \subseteq \left\{ n \in \mathbf{N} : \frac{1}{[n]_{q_n}} \geq \left(\frac{\varepsilon}{2^\alpha M} \right)^{2/\alpha} \right\}.$$

Thus, we have, for every $j \in \mathbb{N}$,

$$\sum_{n : \| P^*_{n,q_n}(f) - f \|_{\bar{D}} \geq \varepsilon} a_{jn} \leq \sum_{n : \frac{1}{[n]_{q_n}} \geq \left(\frac{\varepsilon}{2^\alpha M} \right)^{2/\alpha}} a_{jn}.$$

Now taking limit as $j \to \infty$ in and also using (4.7), we immediately obtain that

$$\lim_j \sum_{n \in Z_1} a_{jn} = 0,$$

which gives

$$st_A - \lim_n \| P^*_{n,q_n}(f) - f \|_{\bar{D}} = 0.$$

Thus, the proof is completed. □

The next result is a natural consequence of Theorem 15.

Corollary 7. *Let (q_n) be a sequence from $(0,1)$ satisfying (4.7). Then, for every $f \in Lip^*_M(\bar{D};\alpha)$, the sequence $\{P^*_{n,q_n}(f)\}_{n \in \mathbb{N}}$ is uniformly convergent to f on $\bar{D}$.*

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