Miskolc Mathematical Notes
Vol. 2 (2001), No 1, pp. 31-38

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2001.33

Monotone iterations for differential problems

Tadeusz Jankowski

Mathematical Notes, Miskolc, Vol. 2., No. 1., (2001), pp. 31–38

MONOTONE ITERATIONS FOR DIFFERENTIAL PROBLEMS

Tadeusz Jankowski
Department of Differential Equations, Technical University of Gdansk
11/12 Narutowicz Str., 80–952 Gdansk, Poland
tjank@mif.pg.gda.pl

[Received March 20, 2001]

Abstract. The purpose of this survey paper is to formulate existence results for parameterized non-linear boundary value problems employing the method of upper and lower solutions and the method of quasilinearization.

Mathematical Subject Classification: 34B15
Keywords: Non-linear parametrized boundary value problem, upper and lower solutions, quasilinearization

1. Introduction

Let us consider the differential equation of first order with a parameter of the form

$$\left\{ \begin{array}{l} x'(t) = f(t, x(t), \lambda), \quad t \in J = [0, b] \\ \quad x(0) = k_0 \in R \qquad G(x(b), \lambda) = 0 \end{array} \right. \tag{1.1}$$

where $f \in C(J \times R \times R, R)$, and $G \in C(R \times R, R)$. By a solution of (1.1) we mean a pair $(x, \lambda) \in C^1(J, R) \times R$ for which problem (1.1) is satisfied. Problems with a parameter have been considered for many years. Some of them appeared as mathematical models of physical systems (see, for example [16]).

An important area of research in the qualitative theory of differential equations is the study of existence of solutions. Existence/uniqueness theorems for problem (1.1) can be formulated under the assumption that f and G satisfy the Lipschitz condition with respect to the last two variables with suitable Lipschitz constants or Lipschitz functions (see [1, 2, 8, 15]). In the above mentioned papers the method of successive approximations and comparison technique are used to obtain sufficient conditions on existence/uniqueness solutions for problems with a parameter. An interesting technique for proving existence results is the method based on upper and lower solutions (for details, see [9]). The purpose of this paper is to formulate existence results for problem (1.1) employing the method of upper and lower solutions. As we see one-sided Lipschitz conditions are imposed on f and G (Theorems 1 and 2). If we apply the method of quasilinearization (for details, see [10]), then we can construct monotone sequences which converge quadratically to the unique solution of problem (1.1). We show that if we replace f and G by the sum of two corresponding functions, then monotone iterations converge quadratically to the unique solution of problem (3.1) (see Theorem 3). Theorem 4 generalizes this result. Theorem 5 concerns a more general case when some Lipschiz functions are added to the right–hand sides of problem (3.1). In this case the convergence is only semi–quadratic. This is a survey paper connected with the above technique for problems of type (1.1).

2. Extremal solutions of problem (1.1)

A pair $(v,\alpha)\in C^1(J,R)\times R$ is said to be a lower solution of (1.1) if

$$\begin{cases} v'(t)\le f(t,v(t),\alpha), & t\in J\\ v(0)=k_0 & 0\le G(v(b),\alpha), \end{cases}$$

and an upper solution of (1.1) if the above inequalities are reversed.

The next two theorems give constructive sufficient conditions when problem (1.1) has minimal and maximal solutions. Note that one–sided Lipschitz conditions are imposed on f and G.

Theorem 1 (see [4]). Assume that $f\in C(J\times R\times R,R)$, $G\in C(R\times R,R)$ and

1^o $(y_0,\lambda_0),(z_0,\gamma_0)\in C^1(J,R)\times R$ are lower and upper solutions of problem (1.1) such that $y_0(t)\le z_0(t)$, $t\in J$ and $\lambda_0\le\gamma_0$,

2^o f is nondecreasing with respect to the last variable,

2^o $f(t,\bar u,\lambda)-f(t,u,\lambda)\ge -M(\bar u-u)=$ for $y_0\le u\le\bar u\le z_0$ with $M\ge 0$,

4^o $G(\bar u,\lambda)-G(u,\lambda)\ge Q(\bar u-u)$ for $y_0\le u\le\bar u\le z_0$, $\lambda_0\le\lambda\le\gamma_0$ with $Q\ge 0$,

5^o $G(u,\bar\lambda)-G(u,\lambda)\ge= -N(\bar\lambda-\lambda)$ for $\lambda_0\le\lambda\le\bar\lambda\le\gamma_0$, $y_0\le u\le z_0$ with $N>0$.

Then there exist monotone sequences $\{y_n,\lambda_n\}$, $\{z_n,\gamma_n\}$ such that $y_n(t)\to y(t)$, $z_n(t)\to z(t)$, $t\in J$ and $\lambda_n\to\lambda$, $\gamma_n\to\gamma$ as $n\to\infty$ and this convergence is uniform and monotone on J. Moreover, (y,λ) and (z,γ) are minimal and maximal solutions of problem (1.1), respectively.

Remark 1. We observe that the special cases when f is monotone nondecreasing with respect to the second variable and G is monotone nondecreasing with respect to the first and second variables are covered by Theorem 1. To see this, it is enough to put $M=0$ in condition 3^o, and $Q=0$ in condition 4^o. If G is monotone nondecreasing with respect to the second variable, then there exists $N>0$ such that for $\bar\lambda\ge\lambda$ we have

$$G(u,\bar\lambda)-G(u,\lambda)\ge 0\ge -N(\bar\lambda-\lambda)\ ,$$

which proves that condition 5^o holds also.

Example (see [4]). Let $J=[0,1]$,

$$\begin{cases} x'(t)=x^2(t)+\frac{1}{3}[\sin^2 x(t)+t^2+g(\lambda)]\equiv f(t,x(t),\lambda), & t\in J,\ x(0)=0\\ \quad 0=x(1.1)+\frac{1}{10}\sin[x(1.1)+\lambda]-\lambda\equiv G(x(1.1),\lambda) \end{cases}$$

where

$$g(\lambda) = \begin{cases} 0 & \text{if} \quad \lambda \le 0, \\ \lambda^2 & \text{if} \quad 0 < \lambda < 1, \\ 1 & \text{if} \quad 1 \le \lambda. \end{cases}$$

Since

$$\frac{1}{3}t^2 \le f(t,x,\lambda) \le 1 + x^2,$$

then

$$y_0(t) = \frac{1}{9}t^3, \quad z_0(t) = \tan t, \quad t \in J.$$

Similarly,

$$\frac{1}{9} - \frac{1}{10} - \lambda \le x(1.1) - \frac{1}{10} - \lambda \le G(x(1.1),\lambda) \le x(1.1) + \frac{1}{10} - \lambda \le \tan 1 + \frac{1}{10} - \lambda,$$

then

$$\lambda_0 = \frac{1}{90} \approx 0.01, \quad \gamma_0 = \tan 1 + \frac{1}{10} \approx 1.66.$$

Note that $(y_0,\lambda_0),\ (z_0,\gamma_0)$ are lower and upper solutions of our problem, respectively, and $y_0(t) \le z_0(t),\ t \in J, \quad \lambda_0 < \gamma_0$. Condition 1^o of Theorem 1 holds since g is nondecreasing in λ. Let $\bar{u} \ge u$ and $\bar{\lambda} \ge \lambda$. Using the mean value theorem we see that

$$f(t,\bar{u},\lambda) - f(t,u,\lambda) = (\bar{u})^2 - u^2 + \frac{1}{3}[\sin^2\bar{u} - \sin^2 u] \ge \frac{1}{3}[\sin^2\bar{u} - \sin^2 u] \ge -\frac{1}{3}(\bar{u} - u),$$

$$G(\bar{u},\lambda) - G(u,\lambda) = \bar{u} - u + \frac{1}{10}[\sin(\bar{u}+\lambda) - \sin(u+\lambda)] \ge \frac{9}{10}(\bar{u} - u),$$

$$G(u,\bar{\lambda}) - G(u,\lambda) = \frac{1}{10}[\sin(u+\bar{\lambda}) - \sin(u+\lambda)] - (\bar{\lambda} - \lambda) \ge -\frac{11}{9}(\bar{\lambda} - \lambda).$$

All assumptions of Theorem 1 hold, its assertion is satisfied for this problem.

Theorem 2 (see [4]). Let $f \in C(J \times R \times R, R),\ G \in C(R \times R, R)$ and }

$1^o \quad (y_0,\lambda_0),(z_0,\gamma_0) \in C^1(J,R) \times R$ are lower and upper solutions of problem (1.1) such that $y_0(t) \le z_0(t),\ \ t \in J$ and $\lambda_0 \le \gamma_0$,

$2^o \ \ f(t,u,\bar{\lambda}) - f(t,u,\lambda) \ge= Q(\bar{\lambda} - \lambda)$ for $\lambda_0 \le \lambda \le \bar{\lambda} \le \gamma_0,\ \ y_0 \le u \le z_0$ with $Q \ge 0$,

$3^o \ \ f(t,\bar{u},\lambda) - f(t,u,\lambda) \ge -M(\bar{u} - u) =$ for $y_0 \le u \le \bar{u} \le z_0$ with $M \ge 0$,

$4^o \ \ G$ is nondecreasing with respect to the first variable,

$5^o \ \ G(u,\bar{\lambda}) - G(u,\lambda) \ge= -N(\bar{\lambda} - \lambda)$ for $\lambda_0 \le \lambda \le \bar{\lambda} \le \gamma_0,\ \ y_0 \le u \le z_0$, with $N > 0$.

Then there exist monotone sequences $\{y_n,\lambda_n\},\ \{z_n,\gamma_n\}$ such that $y_n(t) \to y(t)$, $z_n(t) \to z(t),\ \ t \in J$ and $\lambda_n \to \lambda,\ \ \gamma_n \to \gamma$ as $n \to \infty$ and this convergence is uniform and monotone on J. Moreover, (y,λ) and (z,γ) are minimal and maximal solutions of problem (1.1), respectively.

Remark 2. Our results may be extended to a finite system of differential equations of type (1.1). Although this extension follows similar ideas, such a case requires special care since we need to split variables as it is evident in the technique of monotone iterations.

3. Quasilinearization method

Let $y_0, z_0 \in C^1(J, R)$ and $\lambda_0, \gamma_0 \in R$ such that $y_0(t) \le z_0(t)$ on J and $\lambda_0 \le \gamma_0$. Define the closed sets:

$$\Omega = \{(t, y, \lambda) : t \in J, \ \ y_0(t) \le y \le z_0(t), \ \ \lambda_0 \le \lambda \le \gamma_0\},$$
$$\bar{\Omega} = \{(y, \lambda) : y_0(b) \le y \le z_0(b), \ \ \lambda_0 \le \lambda \le \gamma_0\}.$$

Now, instead of (1.1), we shall consider the following problem

$$\begin{cases} x'(t) = f(t, x(t), \lambda) + g(t, x(t), \lambda), \ \ t \in J, \quad x(0) = k_0 \in R, \\ \qquad G(x(b), \lambda) + H(x(b), \lambda) = 0, \end{cases} \tag{3.1}$$

where $f, g \in C(J \times R \times R, R), \ G, H \in C(R \times R, R)$.

The method of quasilinearization offers monotone sequences of approximate solutions that converge quadratically to the unique solution. This problem is considered in the next theorems using less restrictive assumptions.

Theorem 3 (see [5]). Let $f, g, f_y, g_y, f_\lambda, g_\lambda \in C(\Omega, R)$, $G, H, G_y, H_y, G_\lambda, H_\lambda \in C(\bar{\Omega}, R)$. Assume that:

1^o $(y_0, \lambda_0), \ (z_0, \gamma_0) \in C^1(J, R) \times R$ are lower and upper solutions of problem (3.1), respectively, and such that $y_0(t) \le z_0(t), \ t \in J$ and $\lambda_0 \le \gamma_0$,

2^o $-G_\lambda(u, v) \ge K_1, \quad -H_\lambda(u, v) \ge K_2, K = K_1 + K_2 > 0, \bar{L}_1 \le G_y(u, v) \le L_1, \ \ \bar{L}_2 \le H_y(u, v) \le L_2, \bar{L} = \bar{L}_1 + \bar{L}_2 \ge 0, \bar{M}_1 \le f_\lambda(t, \bar{u}, \bar{v}) \le M_1, \ \ \bar{M}_2 \le g_\lambda(t, \bar{u}, \bar{v}) \le M_2, \bar{M} = \bar{M}_1 + \bar{M}_2 \ge 0$

for $(u, v) \in \bar{\Omega}, \ \ (t, \bar{u}, \bar{v}) \in \Omega$, and put $L = L_1 + L_2, \quad M = M_1 + M_2$,

3^o $|f_y(t, u, v)| \le N_1, \ \ |g_y(t, u, v)| \le N_2 =$for $(t, u, v) \in \Omega$, and put $N = N_1 + N_2$,

4^o $S(b) < 1$, where

$$S(t) = \begin{cases} \frac{ML}{K} t & \text{if} \quad N = 0, \\ \frac{ML}{KN}[\exp(Nt) - 1] & \text{if} \quad N > 0, \end{cases}$$

5^o $f_{yy}, f_{y\lambda}, f_{\lambda y}, f_{\lambda\lambda}, g_{yy}, g_{y\lambda}, g_{\lambda y}, g_{\lambda\lambda}$ exist, are continuous and satisfy the relations:

$$f_{yy}(t, u, v) \ge 0, \ \ f_{y\lambda}(t, u, v) \ge 0, \ \ f_{\lambda\lambda}(t, u, v) \ge 0 \text{ for } (t, u, v) \in \Omega,$$

$$g_{yy}(t, u, v) \le 0, \ \ g_{y\lambda}(t, u, v) \le 0, \ \ g_{\lambda\lambda}(t, u, v) \le 0 \text{ for } (t, u, v) \in \Omega,$$

6^o $G_{yy}, G_{y\lambda}, G_{\lambda y}, G_{\lambda\lambda}, H_{yy}, H_{\lambda y}, H_{y\lambda}, H_{\lambda\lambda}$ exist, are continuous and satisfy the relations:

$$G_{yy}(u, v) \ge 0, \ \ G_{y\lambda}(u, v) \ge 0, \ \ G_{\lambda\lambda}(u, v) \ge 0 \ \text{ for } \ (u, v) \in \bar{\Omega},$$

$$H_{yy}(u,v) \quad \leq \quad 0, \;\; H_{y\lambda}(u,v) \leq 0, \;\; H_{\lambda\lambda}(u,v) \leq 0 \;\text{ for }\; (u,v) \in \bar{\Omega}.$$

Then there exist monotone sequences $\{y_n\}, \{z_n\}, \{\lambda_n\}, \{\gamma_n\}$ which converge uniformly to the unique solution (x, λ) of problem (3.1) and the convergence is quadratic, i.e.,

$$\begin{aligned}
\max_{t\in J} |x(t) - y_{n+1}(t)| &\leq a_1 \max_{t\in J} |x(t) - y_n(t)|^2 + a_2 \max_{t\in J} |x(t) - z_n(t)|^2 \\
&\quad + a_3|\lambda - \lambda_n|^2 + a_4|\lambda - \gamma_n|^2, \\
|\lambda - \lambda_{n+1}| &\leq b_1 \max_{t\in J} |x(t) - y_n(t)|^2 + b_2 \max_{t\in J} |x(t) - z_n(t)|^2 \\
&\quad + b_3|\lambda - \lambda_n|^2 + b_4|\lambda - \gamma_n|^2, \\
\max_{t\in J} |x(t) - z_{n+1}(t)| &\leq \bar{a}_1 \max_{t\in J} |x(t) - y_n(t)|^2 + \bar{a}_2 \max_{t\in J} |x(t) - z_n(t)|^2 \\
&\quad + \bar{a}_3|\lambda - \lambda_n|^2 + \bar{a}_4|\lambda - \gamma_n|^2, \\
|\lambda - \gamma_{n+1}| &\leq \bar{b}_1 \max_{t\in J} |x(t) - y_n(t)|^2 + \bar{b}_2 \max_{t\in J} |x(t) - z_n(t)|^2 \\
&\quad + \bar{b}_3|\lambda - \lambda_n|^2 + \bar{b}_4|\lambda - \gamma_n|^2
\end{aligned}$$

for some nonnegative constants $a_i, b_i, \bar{a}_i, \bar{b}_i, \; i = 1, 2, 3, 4$.

4. Generalization of the quasilinearization method

We will use the following notation $f \in C^{0,2,2}(\Omega, R)$ which means that $f, f_y, f_\lambda, f_{yy}$, $f_{y\lambda}, f_{\lambda y}, f_{\lambda\lambda} \in C(\Omega, R)$. The next theorem generalizes the result of Theorem 3.

Theorem 4 (see [6]). Let $f, g \in C^{0,2,2}(\Omega, R), \;\; G, H \in C^2(\bar{\Omega}, R)$. Assume that:

1^o $(y_0, \lambda_0), \; (z_0, \gamma_0) \in C^1(J, R) \times R$ are lower and upper solutions of problem (3.1), respectively, and such that $y_0(t) \leq z_0(t), \; t \in J$ and $\lambda_0 \leq \gamma_0$,

2^o $-G_\lambda(u,v) \geq K_1, \quad -H_\lambda(u,v) \geq K_2$ and $K = K_1 + K_2 > 0$,

$\bar{L}_1 \leq G_y(u,v) \leq L_1, \;\; \bar{L}_2 \leq H_y(u,v) \leq L_2, \bar{L} = \bar{L}_1 + \bar{L}_2$,

$\bar{M}_1 \leq f_\lambda(t, \bar{u}, \bar{v}) \leq M_1, \quad \bar{M}_2 \leq g_\lambda(t, \bar{u}, \bar{v}) \leq M_2, \bar{M} = \bar{M}_1 + \bar{M}_2$ for $(u,v) \in \bar{\Omega}, \;\; (t, \bar{u}, \bar{v}) \in \Omega$, and put $L = L_1 + L_2, \quad M = M_1 + M_2$,

3^o $|f_y(t,u,v)| \leq N_1, \;\; |g_y(t,u,v)| \leq N_2$ for $(t,u,v) \in \Omega$, and put $N = N_1 + N_2$,

4^o $S(b) < 1$, where S is defined as in condition 4^o of Theorem 3,

5^o $\Psi, \Phi \in C^{0,2,2}(\Omega, R)$ and satisfy the relations for $(t,u,v) \in \Omega$ with $F = f + \Phi, \; =$ $T = g + \Psi$

$$\Phi_{yy}(t,u,v) \geq 0, \;\; \Phi_{y\lambda}(t,u,v) \geq 0, \;\; \Phi_{\lambda\lambda}(t,u,v) \geq 0,$$

$$F_{yy}(t,u,v) \geq 0, \;\; F_{y\lambda}(t,u,v) \geq 0, \;\; F_{\lambda\lambda}(t,u,v) \geq 0,$$

$$\Psi_{yy}(t,u,v) \leq 0, \;\; \Psi_{y\lambda}(t,u,v) \leq 0, \;\; \Psi_{\lambda\lambda}(t,u,v) \leq 0,$$

$$T_{yy}(t,u,v) \le 0, \;\; T_{y\lambda}(t,u,v) \le 0, \;\; T_{\lambda\lambda}(t,u,v) \le 0$$

6^o $\Delta, \Gamma \in C^2(\bar{\Omega}, R)$ and satisfy the relations for $(u,v) \in \bar{\Omega}$ with $P = G + \Delta, \; = Q = H + \Gamma$

$$\Delta_{yy}(u,v) \ge 0, \;\; \Delta_{y\lambda}(u,v) \ge 0, \;\; \Delta_{\lambda\lambda}(u,v) \ge 0,$$

$$P_{yy}(u,v) \ge 0, \;\; P_{y\lambda}(u,v) \ge 0, \;\; P_{\lambda\lambda}(u,v) \ge 0,$$

$$\Gamma_{yy}(u,v) \le 0, \;\; \Gamma_{y\lambda}(u,v) \le 0, \;\; \Gamma_{\lambda\lambda}(u,v) \le 0,$$

$$Q_{yy}(u,v) \le 0, \;\; Q_{y\lambda}(u,v) \le 0, \;\; Q_{\lambda\lambda}(u,v) \le 0,$$

7^o $\bar{\eta} \le \Gamma_y(u,v) \le \eta, \; = \bar{\zeta} \le \Delta_y(u,v) \le \zeta, \; \bar{L} + \bar{\zeta} - \zeta + \bar{\eta} - \eta \ge 0, \; \bar{\vartheta} \le \Phi_\lambda(t,\bar{u},\bar{v}) \le \vartheta$, $\bar{\mu} \le \Psi_\lambda(t,\bar{u},\bar{v}) \le \mu, \; \bar{M} + \bar{\vartheta} - \vartheta + \bar{\mu} - \mu \ge 0$ for $(u,v) \in \bar{\Omega}, \; (t,\bar{u},\bar{v}) \in \Omega$.

Then there exist monotone sequences $\{y_n\}, \{z_n\}, \{\lambda_n\}, \{\gamma_n\}$ which converge uniformly to the unique solution of problem (3.1) and the convergence is quadratic.

Remark 3. Note that Theorem 4 contains, as a special case, the result of Theorem 3 if we put $\Phi(t,u,v) = \Psi(t,u,v) = 0$ for $(t,u,v) \in \Omega$, and $\Delta(\bar{u},\bar{v}) = \Gamma(\bar{u},\bar{v}) = 0$ for $(\bar{u},\bar{v}) \in \bar{\Omega}$. Finally, we study the situation when problem (3.1) is replaced by the following one

$$\begin{cases} x'(t) = \bar{f}(t,x(t),\lambda), \; t \in J, \;\; x(0) = k_0, \\ \qquad 0 = \bar{G}(x(b),\lambda) \end{cases} \tag{4.1}$$

with $\bar{f} = f + \tilde{f} + g + \tilde{g}, \; = \bar{G} = G + \tilde{G} + H + \tilde{H}$. We get weak quadratic convergence only if we assume that $f, g \in= C^{0,2,2}(\Omega, R), \; G, H \in C^2(\bar{\Omega}, R), \; \tilde{f}, \tilde{g} \in C^{0,1,1}(\Omega, R), = \tilde{G}, \tilde{H} \in C^1(\bar{\Omega}, R)$. This general case is considered in the next theorem.

Theorem 5 (see [7]). Let $f, g \in C^{0,2,2}(\Omega, R), \;\; G, H \in C^2(\bar{\Omega}, R), \;\; \tilde{f}, \tilde{g} \in C(\Omega, R), \;\; = \tilde{G}, \tilde{H} \in C(\bar{\Omega}, R)$. Assume that:

1^o $(y_0, \lambda_0), \; (z_0, \gamma_0) \in C^1(J, R) \times R$ are lower and upper solutions of problem (4.1), respectively, and such that $y_0(t) \le z_0(t), \; t \in J$ and $\lambda_0 \le \gamma_0$,

2^o conditions 5^o and 6^o of Theorem 4 hold,

3^o $-G_\lambda(u,v) \ge K_1, \quad -H_\lambda(u,v) \ge K_2, \; \bar{L}_1 \le G_y(u,v) \le L_1, \;\; \bar{L}_2 \le H_y(u,v) \le L_2$, $\bar{M}_1 \le f_\lambda(t,\bar{u},\bar{v}) \le M_1, \quad \bar{M}_2 \le g_\lambda(t,\bar{u},\bar{v}) \le M_2$ for $(u,v) \in \bar{\Omega}, \quad (t,\bar{u},\bar{v}) \in \Omega$; $|f_y(t,u,v)| \le N_1, \;\; |g_y(t,u,v)| \le N_2$ for $(t,u,v) \in \Omega$,

5^o $\tilde{F}_y, \tilde{F}_\lambda, \tilde{\Phi}_y, = \tilde{\Phi}_\lambda, \tilde{P}_y, \tilde{P}_\lambda, \tilde{\Delta}_y, \tilde{\Delta}_\lambda$ exist, are nondecreasing in the last two variables, while $\tilde{T}_y, \tilde{T}_\lambda, \tilde{\Psi}_y, \tilde{\Psi}_\lambda, \tilde{Q}_y, \tilde{Q}_\lambda, \tilde{\Gamma}_y, \tilde{\Gamma}_\lambda$ exist and are nonincreasing in the last two variables with $\tilde{F} = \tilde{f} + \tilde{\Phi}, \; \tilde{T} = \tilde{g} + \tilde{\Psi}, \; \tilde{P} = \tilde{G} + \tilde{\Delta}, \; = \tilde{Q} = \tilde{H} + \tilde{\Gamma}$,

6^o $\bar{M}_3 \le \tilde{f}_\lambda(t,\bar{u},\bar{v}) \le M_3, \; \bar{M}_4 \le \tilde{g}_\lambda(t,\bar{u},\bar{v}) \le M_4$,

$K_3 \le -\tilde{G}_\lambda(u,v) \le \bar{K}_3, \; K_4 \le -\tilde{H}_\lambda(u,v) \le \bar{K}_4, \;\; \bar{L}_3 \le \tilde{G}_y(u,v) \le L_3, \;\; \bar{L}_4 \le \tilde{H}_y(u,v) \le L_4, \;\; |\tilde{f}_y(t,\bar{u},\bar{v})| \le N_3, \; |\tilde{g}_y(t,\bar{u},\bar{v})| \le N_4, \; \bar{\eta}_1 \le \tilde{\Gamma}_y(u,v) \le \eta_1, \; \bar{\zeta}_1 \le \tilde{\Delta}_y(u,v) \le \zeta_1, \;\; \bar{\vartheta}_1 \le \tilde{\Phi}_\lambda(t,\bar{u},\bar{v}) \le \vartheta_1, \;\; \bar{\mu}_1 \le \tilde{\Psi}_\lambda(t,\bar{u},\bar{v}) \le \mu_1$ for $(u,v) \in \Omega, \; (t,\bar{u},\bar{v}) \in \bar{\Omega}$

7^o put $L = L_1+L_2+L_3+L_4$, $K = K_1+K_2+K_3+K_4 > 0$, $M = M_1+M_2+M_3+M_4$, $N = N_1 + N_2 + N_3 + N_4$, $\bar{L} = \bar{L}_1 + \bar{L}_2 + \bar{L}_3 + \bar{L}_4 + \bar{\zeta}_1 - \zeta_1 + \bar{\eta}_1 - \eta_1$, $\bar{M} = \bar{M}_1 + \bar{M}_2 + \bar{M}_3 + \bar{M}_4 + \bar{\vartheta}_1 - \vartheta_1 + \bar{\mu}_1 - \mu_1$,

8^o condition 7^o of Theorem 4 holds,

9^o $S(b) < 1$, where S is defined as in condition 4^o of Theorem 3.

Then there exist monotone sequences $\{y_n\}, \{z_n\}, \{\lambda_n\}, \{\gamma_n\}$ which converge uniformly and monotonically to the unique solution of problem (4.1) and the convergence is semi–quadratic, i.e.,

$$\max_{t\in J} |x(t) - y_{n+1}(t)| \le a_1 \max_{t\in J} |x(t) - y_n(t)|^2 + a_2 \max_{t\in J} |x(t) - z_n(t)|^2 + a_3|\lambda - \lambda_n|^2$$

$$+a_4|\lambda - \gamma_n|^2 + a_5 \max_{t\in J} |x(t) - y_n(t)| + a_6|\lambda - \lambda_n|,$$

$$|\lambda - \lambda_{n+1}| \le b_1 \max_{t\in J} |x(t) - y_n(t)|^2 + b_2 \max_{t\in J} |x(t) - z_n(t)|^2 + b_3|\lambda - \lambda_n|^2$$

$$+b_4|\lambda - \gamma_n|^2 + b_5 \max_{t\in J} |x(t) - y_n(t)| + b_6|\lambda - \lambda_n|,$$

$$\max_{t\in J} |x(t) - z_{n+1}(t)| \le \bar{a}_1 \max_{t\in J} |x(t) - y_n(t)|^2 + \bar{a}_2 \max_{t\in J} |x(t) - z_n(t)|^2 + \bar{a}_3|\lambda - \lambda_n|^2$$

$$+\bar{a}_4|\lambda - \gamma_n|^2 + \bar{a}_5 \max_{t\in J} |x(t) - z_n(t)| + \bar{a}_6|\lambda - \gamma_n|$$

$$|\lambda - \gamma_{n+1}| \le \bar{b}_1 \max_{t\in J} |x(t) - y_n(t)|^2 + \bar{b}_2 \max_{t\in J} |x(t) - z_n(t)|^2 + \bar{b}_3|\lambda - \lambda_n|^2$$

$$+\bar{b}_4|\lambda - \gamma_n|^2 + \bar{b}_5 \max_{t\in J} |x(t) - z_n(t)| + \bar{b}_6|\lambda - \gamma_n|,$$

for some nonnegative constants $a_i, b_i, \bar{a}_i, \bar{b}_i$, $i = 1, 2, 3, 4, 5, 6$.

REFERENCES

[1] Goma, A.: *The method of successive approximations in a two–point boundary value problem with a parameter*, Ukr. Mat. Zh., **29**, (1977), 800–807. (in Russian)

[2] Jankowski, T.: *Existence, uniqueness and approximate solutions of problems with a parameter*, Zesz. Nauk. Politech. Gdansk., Mat., **16**, (1993), 3–167.

[3] Jankowski, T. and Lakshmikantham, V.: *Monotone iterations for differential equations with a parameter*, J. Appl. Math. Stoch. Anal., **10**, (1997), 273–278.

[4] Jankowski, T.: *Monotone iterations for first order differential equations with a parameter*, Acta Math. Hungar., **84**, (1999), 65–80.

[5] Jankowski, T. and McRae, F. A.: *An extension of the method of quasilinearization for differential problems with a parameter*, Nonlinear Studies, **6**, (1999), 21–44.

[6] Jankowski, T.: *Generalizations of the method of quasilinearization for differential problems with a parameter*, Dynamic Systems and Applications, **8**, (1999), 53–72.

[7] Jankowski, T.: *Generalization of the method of quasilinearization*, International Journal of Non–linear Differential Equations, **5**, (1999), 122–147.

[8] Kurpiel, N. S. and Marusiak, A. G.: *A multipoint boundary value problem for differential equations with parameters*, Ukr. Mat. Zh., **32**, (1980), 223–226. (in Russian)

[9] Ladde, G. S., Lakshmikantham, V. and Vatsala, A. S.: *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman, 1985.

[10] Lakshmikantham, V. and Vatsala, A. S.: *Generalized Quasilinearization for Nonlinear Problems*, Kluwer, Dordrecht, 1998.

[11] Lakshmikantham, V. and Malek, S.: *Generalized quasilinearization*, Nonlinear World, **1**, (1994), 59–63.

[12] Lakshmikantham, V., Leela, S. and Sivasundaram, S.: *Extensions of the method of quasilinearization*, J. Otimization Theory Appl., **87**, (1995), 379–401.

[13] Lakshmikantham, V., Leela, S. and McRae, F.A.: *Improved generalized quasilinearization method*, Nonlinear Analysis **24**, (1995), 1627–1637.

[14] Lakshmikantham, V. and Shahzad, N.: *Further generalization of generalized quasilinearization method*, J. Appl. Math. Stoch. Anal., **7**, (1994), 545–552.

[15] Pomentale, T.: *A constructive theorem of existence and uniqueness for the problem* $y' = f(x, y, \lambda),\ y(a) = \alpha,\ y(b) = \beta$, Z. Angew. Math. Mech., **56**, (1976), 387–388.

[16] Zawischa, K.: *Über die Differentialgleichung* $y' = kf(x, y)$ *deren Lösungskurve durch zwei gegebene Punkte hindurchgehen soll*, Monatsh. Math. Phys., **37**, (1930), 103–124.