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# On Riemannian manifolds admitting $W_2$-curvature tensor

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# ON RIEMANNIAN MANIFOLDS ADMITTING $W_2$-CURVATURE TENSOR

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*Abstract.* The object of this paper is to study the properties of flat spacetimes under some conditions regarding the $W_2$-curvature tensor. In the first section, several results are obtained on the geometrical symmetries of this curvature tensor. It is shown that in a spacetime with $W_2$-curvature tensor filled with a perfect fluid, the energy momentum tensor satisfying the Einstein's equations with a cosmological constant is a quadratic conformal Killing tensor. It is also proved that a necessary and sufficient condition for the energy momentum tensor to be a quadratic Killing tensor is that the scalar curvature of this space must be constant. In a radiative perfect fluid, it is shown that the sectional curvature is constant.

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## 1. INTRODUCTION

In 1970, Pokhariyal and Mishra, [9] introduced a new curvature tensor in an n-dimensional manifold $(M,g)$ denoted by $W_2$ and defined by

$$W_2(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}(g(X,Z)S(Y,U) - g(Y,Z)S(X,U)). \tag{1.1}$$

Some authors have studied the $W_2$-curvature tensor on some special manifolds before, [3, 5, 8, 16].

A non-flat n-dimensional Riemannian manifold $(M,g)$ is called generalized recurrent Riemannian manifold [4] if its curvature tensor satisfies the condition

$$\nabla R = A \otimes R + B \otimes G \tag{1.2}$$

where $A$ and $B$ non-zero 1-forms. $\otimes$ is the tensor product, $\nabla$ denotes the Levi-Civita connection, and $G$ is a tensor type (0,4) given by

$$G(X,Y,Z,U) = g(X,U)g(Y,Z) - g(X,Z)g(Y,U) \tag{1.3}$$

for all $X,Y,Z,U \in \chi(M)$, where $\chi(M)$ is the Lie algebra of smooth vector fields on $M$. This manifold is denoted by $(GK_n)$. These manifolds have been studied by some authors before, [1, 2, 10, 11], etc.

In this paper, the properties of a $W_2$-flat spacetime are studied. Some theorems on the $W_2$ curvature tensor are proved.

## 2. Some properties of $W_2$-curvature tensor

**Definition 1.** Let $(M,g)$ be a manifold with Levi-Civita connection $\nabla$. A quadratic Killing tensor is a generalization of a Killing vector and is defined as a second order symmetric tensor $A$ satisfying the condition [13, 15]

$$(\nabla_X A)(Y,Z) + (\nabla_Y A)(Z,X) + (\nabla_Z A)(X,Y) = 0 \tag{2.1}$$

**Definition 2.** Let $(M,g)$ be a manifold with Levi-Civita connection $\nabla$. A quadratic conformal Killing tensor is an analogous generalization of a conformal Killing vector and is defined as a second order symmetric tensor $A$ satisfying the condition [13, 15]

$$\begin{aligned}&(\nabla_X A)(Y,Z) + (\nabla_Y A)(Z,X) + (\nabla_Z A)(X,Y) = \\ &k(X)g(Y,Z) + k(Y)g(Z,X) + k(Z)g(X,Y).\end{aligned} \tag{2.2}$$

Now, we have the following theorems:

**Theorem 1.** *If the Ricci tensor of $M$ admitting $W_2$-curvature tensor is a quadratic conformal Killing tensor then $\tilde{W}_2(Y,Z)$ (which is type of (0,2)) is also quadratic conformal Killing tensor.*

*Proof.* By (1.1) we have

$$\tilde{W}_2(Y,Z) = \frac{n}{n-1}\left(S(Y,Z) - \frac{r}{n}g(Y,Z)\right) \tag{2.3}$$

where $S(Y,Z)$ and $r$ are the Ricci tensor and the scalar curvature of $M$, respectively. If we take the covariant derivative of (2.3), we find

$$(\nabla_X \tilde{W}_2)(Y,Z) = \frac{n}{n-1}(\nabla_X S)(Y,Z) - \frac{1}{n-1}(\nabla_X r)g(Y,Z) \tag{2.4}$$

Permutating the indices $X,Y,Z$ cyclically in (2.4) and adding the three equations, we get

$$\begin{aligned}&(\nabla_X \tilde{W}_2)(Y,Z) + (\nabla_Y \tilde{W}_2)(Z,X) + (\nabla_Z \tilde{W}_2)(X,Y) \\ &= \frac{n}{n-1}((\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y)) \\ &- \frac{1}{n-1}((\nabla_X r)g(Y,Z) + (\nabla_Y r)g(Z,X) + (\nabla_Z r)g(X,Y)).\end{aligned} \tag{2.5}$$

If we assume that the Ricci tensor of $M$ is a quadratic conformal Killing tensor, from (2.2) and (2.5), we obtain that

$$\begin{aligned}&(\nabla_X \tilde{W}_2)(Y,Z)+(\nabla_Y \tilde{W}_2)(Z,X)+(\nabla_Z \tilde{W}_2)(X,Y)\\&=(\frac{n}{n-1}a(X)-\frac{1}{n-1}(\nabla_X r))g(Y,Z)\\&+(\frac{n}{n-1}a(Y)-\frac{1}{n-1}(\nabla_Y r))g(Z,X)\\&+(\frac{n}{n-1}a(Z)-\frac{1}{n-1}(\nabla_Z r))g(X,Y).\end{aligned} \tag{2.6}$$

By taking $\frac{n}{n-1}a(X)-\frac{1}{n-1}(\nabla_X r)=\alpha(X)$, it can be seen that

$$\begin{aligned}&(\nabla_X \tilde{W}_2)(Y,Z)+(\nabla_Y \tilde{W}_2)(Z,X)+(\nabla_Z \tilde{W}_2)(X,Y)\\&=\alpha(X)g(Y,Z)+\alpha(Y)g(Z,X)+\alpha(Z)g(X,Y).\end{aligned}$$

This relation completes the proof. □

**Theorem 2.** *Let the Ricci tensor of $M$ admitting a $W_2$-curvature tensor be a quadratic Killing tensor. A necessary and sufficient condition for $\tilde{W}_2(X,Y)$ to be a quadratic Killing tensor is that the scalar curvature of $M$ be constant.*

*Proof.* Let us consider that the Ricci tensor of $M$ is a quadratic Killing tensor. From equations (2.1) and (2.5), we have

$$\begin{aligned}&(\nabla_X \tilde{W}_2)(Y,Z)+(\nabla_Y \tilde{W}_2)(Z,X)+(\nabla_Z \tilde{W}_2)(X,Y)\\&-\frac{1}{n-1}((\nabla_X r)g(Y,Z)+(\nabla_Y r)g(Z,X)+(\nabla_Z r)g(X,Y)).\end{aligned} \tag{2.7}$$

If $\tilde{W}_2(X,Y)$ is a quadratic Killing tensor then we get

$$(\nabla_X r)g(Y,Z)+(\nabla_Y r)g(Z,X)+(\nabla_Z r)g(X,Y)=0 \tag{2.8}$$

Walker's Lemma,[14] states the following. If $a(X,Y)$ and $b(X)$ are numbers that satisfy $a(X,Y)=a(Y,X)$ and

$$a(X,Y)b(Z)+a(Y,Z)b(X)+a(Z,X)b(Y)=0 \tag{2.9}$$

for all $X,Y,Z$, then either all the $a(X,Y)$ are zero or all the $b(X)$ are zero. Hence, by the above Lemma, we get from (2.8) and (2.9) that either $g(X,Y)=0$ or $(\nabla_X r)=0$. As $g(X,Y)\neq 0$, we get $(\nabla_X r)=0$, i.e., the scalar curvature of $M$ is constant. Conversely, if the scalar curvature is constant, then, from (2.7), $\tilde{W}_2(X,Y)$ is a quadratic Killing tensor. Thus, the proof is completed. □

The geometrical symmetries of a Riemannian manifold are expressed through the equation

$$L_\xi A-2\Omega A=0 \tag{2.10}$$

where $A$ represents a geometrical/physical quantity $L_\xi$, denotes the Lie derivative with respect to the vector field $\xi$ and $\Omega$ is a scalar, [7].

One of the most simple and widely used example is the metric inheritance symmetry for which $A = g$ in (2.10); in this case, $\xi$ is the Killing vector field if $\Omega$ is zero,i.e.,

$$(L_\xi g)(X,Y) = 2\Omega g(X,Y) \tag{2.11}$$

A Riemannian manifold $M$ is said to admit a symmetry called a curvature collineation *(CC)* provided there exists a vector field $\xi$ such that

$$(L_\xi R)(X,Y)Z = 0 \tag{2.12}$$

where $R(X,Y)Z$ is the Riemannian curvature tensor, [6].

Now, we shall investigate the role of such symmetry inheritance for the $W_2$-curvature tensor of a Riemannian manifold.

**Theorem 3.** *If a Riemannian manifold $M$ admitting $W_2$-curvature tensor with a Killing vector $\xi$ is a* (CC) *then the Lie derivative of the $W_2$-curvature tensor is zero.*

*Proof.* If $\xi$ is a Killing vector of $M$ then we have

$$(L_\xi g)(X,Y) = 0 \tag{2.13}$$

Since $M$ admits a *(CC)* then we have also from (2.12)

$$(L_\xi S)(X,Y) = 0 \tag{2.14}$$

By taking the Lie derivative of (1.1) and using (2.12)-(2.14), we obtain

$$(L_\xi W)(X,Y)Z = 0$$

The proof is completed. □

**Theorem 4.** *If a Riemannian manifold $M$ admitting $W_2$-curvature tensor with a conformal Killing vector $\xi$ has a symmetry inheritance then the $W_2$-curvature tensor has also symmetry inheritance property.*

*Proof.* Let us assume that $\xi$ of $M$ is a conformal Killing vector and $M$ has a symmetry inheritance. By taking the Lie derivative of (1.1) and using the equations (2.10) and (2.11), we find

$$\begin{aligned}(L_\xi W)(X,Y,Z,U) &= 2\Omega R(X,Y,Z,U)\\ &+ \frac{2\Omega}{n-1}(g(X,Z)S(Y,U) - g(Y,Z)S(X,U))\\ &= 2\Omega W(X,Y,Z,U).\end{aligned}$$

The proof is completed. □

## 3. $W_2$-FLAT SPACETIMES

**Theorem.** [12] A $W_2$-flat space is an Einstein space, i.e,

$$S(X,Y) = \frac{r}{n} g(X,Y) \tag{3.1}$$

We denote the $W_2$ flat spacetime as $(W_2FS)_n$ and we consider that our space is a perfect fluid. In local coordinates, a perfect fluid is a spacetime $(M,g)$ satisfying the Einstein equations

$$S_{ij} - \frac{r}{2} g_{ij} + \lambda g_{ij} = kT_{ij} \tag{3.2}$$

where $S_{ij}$ and $r$ denote the Ricci tensor and the scalar curvature, respectively. $\lambda$ is the cosmological constant and $T_{ij}$ is the energy-momentum tensor. We can state the following theorem:

**Theorem 5.** *For a $(W_2FS)_4$, the energy-momentum tensor satisfying the Einstein's equations with a cosmological constant is locally symmetric.*

*Proof.* In a $(W_2FS)_4$, by (3.1), we have

$$S_{ij} = \frac{r}{4} g_{ij} \tag{3.3}$$

and then we get from (3.2) and (3.3)

$$T_{ij} = \frac{1}{k}(\lambda - \frac{r}{4}) g_{ij} \tag{3.4}$$

By taking the covariant derivative of (3.4), we get

$$\nabla_k T_{ij} = -\frac{1}{4k}(\nabla_k r) g_{ij}. \tag{3.5}$$

Since $r$ is constant in $(W_2FS)_4$,

$$\nabla_k r = 0 \tag{3.6}$$

then (3.5) reduces to

$$\nabla_k T_{ij} = 0. \tag{3.7}$$

This completes the proof. □

**Theorem 6.** *For a $(W_2S)_4$, whose the Ricci tensor is quadratic Killing, a necessary and sufficient condition the energy-momentum tensor satisfying the Einstein's equations with a cosmological constant be a quadratic Killing tensor is that the scalar curvature of this space must be constant.*

*Proof.* If $T_{ij}$ and $R_{ij}$ are quadratic Killing tensors, (3.5) reduces to

$$(\nabla_k r) g_{ij} + (\nabla_i r) g_{jk} + (\nabla_j r) g_{ki} = 0 \tag{3.8}$$

By Walker's Lemma,[14], from (2.9) we get $\nabla_k r = 0$. Thus we can say that the scalar curvature of this space is constant. Conversely, if $r$ is constant, from (3.5), it can be obtained that

$$\nabla_k T_{ij} + \nabla_i T_{jk} + \nabla_j T_{ki} = 0$$

Thus, the proof is completed. □

Now, we consider that a radiative perfect fluid in $(W_2S)_4$. Thus, we have

$$T_{ij} = \sigma a_i a_j, \qquad a_i a^i = -1 \tag{3.9}$$

In this case, by taking the covariant derivative of (3.9), we find

$$\nabla_k T_{ij} = (\nabla_k \sigma) a_i a_j + \sigma((\nabla_k a_i) a_j + (\nabla_k a_j) a_i) \tag{3.10}$$

Multiplying (3.9) by $g^{ij}$, we get

$$T = -\sigma \tag{3.11}$$

By taking the covariant derivative of (3.11), it can be found that

$$\nabla_k T = -\nabla_k \sigma \tag{3.12}$$

Now, we can state the following theorem:

**Theorem 7.** *For a radiative perfect fluid in $(W_2S)_4$, if the energy-momentum tensor satisfying the Einstein's equations with a cosmological constant is generalized recurrent then the integral curves of the vector field $a_i$ are geodesics.*

*Proof.* For a radiative perfect fluid in $(W_2S)_4$, if $T_{ij}$ is generalized recurrent then from (1.2)

$$\nabla_k T_{ij} = \lambda_k T_{ij} + \beta_k g_{ij} \tag{3.13}$$

and

$$\nabla_k T = \lambda_k T + 4\beta_k \tag{3.14}$$

Moreover, by (3.12) and (3.14), we get

$$\nabla_k \sigma = \lambda_k \sigma - 4\beta_k \tag{3.15}$$

Putting equations (3.13) and (3.15) in (3.10), we have the following relation

$$\lambda_k T_{ij} + \beta_k g_{ij} = (\lambda_k \sigma - 4\beta_k) a_i a_j + \sigma((\nabla_k a_i) a_j + (\nabla_k a_j) a_i) \tag{3.16}$$

By (3.9), (3.16) takes the form

$$\lambda_k \sigma a_i a_j + \beta_k g_{ij} = \lambda_k \sigma a_i a_j - 4\beta_k a_i a_j + \sigma((\nabla_k a_i) a_j + (\nabla_k a_j) a_i) \tag{3.17}$$

Multiplying (3.17) by $g^{ij}$ and using (3.9), we obtain

$$2\sigma a^i (\nabla_k a_i) = 0 \tag{3.18}$$

Since, in a radiative perfect fluid, $\sigma \neq 0$ then from (3.18), we finally get

$$a^i (\nabla_k a_i) = 0 \tag{3.19}$$

From (3.19), we can say that the integral curves of the vector field $a_i$ are geodesics. In this case, the proof is completed. □

Now, by using the above theorem, we can state the following:

**Theorem 8.** *For a radiative perfect fluid in* $(W_2FS)_4$*, the sectional curvature is constant.*

*Proof.* For a radiative perfect fluid in $(W_2FS)_4$, the scalar curvature of this space is constant. Thus, we can say that for a radiative perfect fluid in $(W_2FS)_4$, the scalar curvature of this space is constant.

The sectional curvature of a Riemannian manifold $M$ is in the form

$$K(\pi) = \frac{R_{hijk}X^hY^iX^jY^k}{(g_{ij}g_{hk} - g_{ik}g_{hj})X^hY^iX^jY^k}. \tag{3.20}$$

By (1.1), if $M$ is a $W_2$ flat manifold then we have

$$R_{ijkh} = \frac{1}{n-1}(g_{jk}S_{ih} - g_{ik}S_{jh}). \tag{3.21}$$

Thus, multiplying (3.21) by $g^{ih}$ and using (3.21) again, we get

$$R_{ijkh} = \frac{r}{n(n-1)}(g_{jk}g_{ih} - g_{ik}g_{jh}). \tag{3.22}$$

Finally, from (3.20) and (3.22), it can be seen that

$$K(\pi) = \frac{r}{n(n-1)}. \tag{3.23}$$

□

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