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A GENERALIZATION OF ČEBYŠEV TYPE INEQUALITIES FOR FIRST DIFFERENTIABLE MAPPINGS

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Abstract. In this paper, we improve and further generalize some Čebyšev type inequalities involving functions whose derivatives belong to L_p spaces via certain integral identities.

1. INTRODUCTION

In 1882, P. L. Čebyšev proved the following important integral inequality [13],[9, p.207]:

$$T(f,g) \leq \frac{1}{12}(b-a)^2 \left\| f' \right\|_\infty \left\| g' \right\|_\infty, \tag{1.1}$$

where $f,g : [a,b] \to \mathbb{R}$ are absolutely continuous functions whose derivatives $f',g' \in L_\infty[a,b]$ and

$$T(f,g) = \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right), \tag{1.2}$$

which is called the Čebyšev functional, provided the integrals in (1.2) exist.

During the past few years many researchers have given considerable attention to the inequality of (1.1). Various generalizations, extensions and variants of this inequality have appeared in the literature, see ([1–13]) and the references cited therein.

Recently, Pachpatte [8] established a new Čebyšev type inequality involving functions whose derivatives belong to L_p spaces:

Theorem 1. *Let $f,g : [a,b] \to \mathbb{R}$ are absolutely continuous functions whose derivatives $f',g' \in L_p[a,b]$, $p > 1$. Then, we have inequalities*

$$T(f,g) \leq \frac{1}{(b-a)^3} \left\| f' \right\|_p \left\| g' \right\|_p \int_a^b (B(x))^{\frac{2}{q}} dx,$$

and

$$T(f,g) \le \frac{1}{2(b-a)^2} \int_a^b \left[g(x) \left\| f' \right\|_p + f(x) \left\| g' \right\|_p \right] (B(x))^{\frac{1}{q}} \, dx, \qquad (1.3)$$

where

$$B(x) = \frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1},$$

for $x \in [a,b]$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Motivated by the results of Pachpatte, in the present paper we establish some new Čebyšev type inequalities for $\frac{1}{p} + \frac{1}{q} = 1$. The analysis used in the proofs is elementary and our results provide new estimates for these types of inequalities.

2. MAIN RESULTS

Let $[a,b] \subset \mathbb{R}$, $a < b$; and as usual for any function $h \in L_p[a,b]$, $p > 1$, we define $\|h\|_p = \left(\int_a^b |h(t)|^p \, dt \right)^{\frac{1}{p}}$. We use the following notation to simplify the details of presentation. For suitable functions $f, g : [a,b] \to \mathbb{R}$ and $\lambda \in [0,1]$, we set

$$\begin{aligned}
T_\lambda(f,g) &\\
&= \frac{1}{b-a} \left(1 - \frac{\lambda}{2} \right) \int_a^b f(x)g(x)dx \\
&\quad - \frac{\lambda}{4(b-a)^2} \left\{ \int_a^b [(x-b)f(b) - (x-a)f(a)]g(x)dx \right. \\
&\quad \left. + \int_a^b [(x-b)g(b) - (x-a)g(a)]f(x)dx \right\} \\
&\quad - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right).
\end{aligned}$$

Now, let us state our main results.

Theorem 2. *Let* $f, g : [a,b] \to \mathbb{R}$ *be absolutely continuous functions whose derivatives* $f', g' \in L_p[a,b]$, $p > 1$. *Then, the following inequality holds:*

$$|T_\lambda(f,g)| \le \frac{1}{2(b-a)^2} \int_a^b \left[g(x) \left\| f' \right\|_p + f(x) \left\| g' \right\|_p \right] (Z(x))^{\frac{1}{q}} \, dx, \qquad (2.1)$$

where

$$Z(x) = \frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \left(\left(\frac{\lambda}{2} \right)^{q+1} + \left(1 - \frac{\lambda}{2} \right)^{q+1} \right),$$

for $x \in [a,b]$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\lambda \in [0,1]$.

Proof. From the hypotheses, we can write the following identities,

$$\frac{1}{b-a}\int_a^b p(x,t)f'(t)dt$$

$$= \left(1-\frac{\lambda}{2}\right)f(x) - \lambda\frac{(x-b)\,f(b)-(x-a)\,f(a)}{2(b-a)} - \frac{1}{b-a}\int_a^b f(t)dt, \quad (2.2)$$

and

$$\frac{1}{b-a}\int_a^b p(x,t)g'(t)dt$$

$$= \left(1-\frac{\lambda}{2}\right)g(x) - \lambda\frac{(x-b)\,g(b)-(x-a)\,g(a)}{2(b-a)} - \frac{1}{b-a}\int_a^b g(t)dt, \quad (2.3)$$

for $x \in [a,b]$, $\lambda \in [0,1]$, where

$$p(x,t) = \begin{cases} t - \left(a+\lambda\dfrac{x-a}{2}\right), & a \le t \le x \\[2mm] t - \left(b-\lambda\dfrac{b-x}{2}\right), & x < t \le b \end{cases}.$$

A simple proof of equalities of (2.2) and (2.3) can be done by integrating in the left hand side and using the identity of $p(x,t)$. The details are left for the interested reader.

Multiplying both sides of (2.2) and (2.3) by $g(x)$ and $f(x)$ respectively and adding the resulting identities, we get

$$\frac{1}{b-a}g(x)\int_a^b p(x,t)f'(t)dt + \frac{1}{b-a}f(x)\int_a^b p(x,t)g'(t)dt \quad (2.4)$$

$$= 2\left(1-\frac{\lambda}{2}\right)f(x)g(x) - \lambda\frac{(x-b)\,f(b)-(x-a)\,f(a)}{2(b-a)}g(x)$$

$$- \lambda\frac{(x-b)\,g(b)-(x-a)\,g(a)}{2(b-a)}f(x)$$

$$- \frac{1}{b-a}g(x)\int_a^b f(t)dt - \frac{1}{b-a}f(x)\int_a^b g(t)dt.$$

Multiplying both sides of (2.4) by $\frac{1}{b-a}$, we get

$$\frac{2}{b-a}\left(1-\frac{\lambda}{2}\right)f(x)g(x) \quad (2.5)$$

$$- \lambda\frac{(x-b)\,f(b)-(x-a)\,f(a)}{2(b-a)^2}g(x) - \lambda\frac{(x-b)\,g(b)-(x-a)\,g(a)}{2(b-a)^2}f(x)$$

$$- \frac{1}{(b-a)^2}g(x)\int_a^b f(t)dt - \frac{1}{(b-a)^2}f(x)\int_a^b g(t)dt$$

$$= \frac{1}{(b-a)^2} g(x) \int_a^b p(x,t) f'(t) dt + \frac{1}{(b-a)^2} f(x) \int_a^b p(x,t) g'(t) dt.$$

Integrating both sides of (2.5) with respect to x over $[a,b]$, we get

$$\frac{2}{b-a} \left(1 - \frac{\lambda}{2}\right) \int_a^b f(x) g(x) dx \tag{2.6}$$

$$- \frac{\lambda}{2(b-a)^2} \left\{ \int_a^b [(x-b) f(b) - (x-a) f(a)] g(x) dx \right.$$

$$+ \int_a^b [(x-b) g(b) - (x-a) g(a)] f(x) dx \Bigg\}$$

$$-2 \left(\frac{1}{b-a} \int_a^b g(x) dx\right) \left(\frac{1}{b-a} \int_a^b f(x) dx\right)$$

$$= \frac{1}{(b-a)^2} \int_a^b g(x) \left(\int_a^b p(x,t) f'(t) dt\right) dx$$

$$+ \frac{1}{(b-a)^2} \int_a^b f(x) \left(\int_a^b p(x,t) g'(t) dt\right) dx.$$

After the rewriting of the equality of (2.5), we obtain

$$T_\lambda(f,g) = \frac{1}{2(b-a)^2} \int_a^b \left[g(x) \int_a^b p(x,t) f'(t) dt + f(x) \int_a^b p(x,t) g'(t) dt \right] dx.$$
$$\tag{2.7}$$

From (2.7) and using the properties of modulus, Hölder's integral inequality, we have

$$|T_\lambda(f,g)| \tag{2.8}$$

$$\leq \frac{1}{2(b-a)^2} \int_a^b \left[|g(x)| \int_a^b |p(x,t)| |f'(t)| dt + |f(x)| \int_a^b |p(x,t)| |g'(t)| dt \right] dx$$

$$\leq \frac{1}{2(b-a)^2} \int_a^b \left[|g(x)| \left(\int_a^b |p(x,t)|^q dt\right)^{\frac{1}{q}} \left(\int_a^b |f'(t)|^p dt\right)^{\frac{1}{p}} \right.$$

$$\left. + |f(x)| \left(\int_a^b |p(x,t)|^q dt\right)^{\frac{1}{q}} \left(\int_a^b |g'(t)|^p dt\right)^{\frac{1}{p}} \right] dx$$

$$= \frac{1}{2(b-a)^2} \int_a^b \left[g(x) \|f'\|_p + f(x) \|g'\|_p \right] \left(\int_a^b |p(x,t)|^q dt\right)^{\frac{1}{q}} dx.$$

A simple calculation shows that

$$\int_a^b |p(x,t)|^q \, dt \tag{2.9}$$

$$= \int_a^x \left| t - \left(a + \lambda \frac{x-a}{2} \right) \right|^q dt + \int_x^b \left| t - \left(b - \lambda \frac{b-x}{2} \right) \right|^q dt$$

$$= (x-a)^q \int_a^x \left| \frac{t-a}{x-a} - \frac{\lambda}{2} \right|^q dt + (b-x)^q \int_x^b \left| \frac{t-b}{b-x} - \frac{\lambda}{2} \right|^q dt$$

$$= \frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \left(\left(\frac{\lambda}{2} \right)^{q+1} + \left(1 - \frac{\lambda}{2} \right)^{q+1} \right)$$

$$= Z(x).$$

Using (2.8) and (2.9), we get the required inequality in (2.1). The proof is complete. $\square$

Remark 1. If we choose $\lambda = 0$ in (2.1), then we get (1.3).

Corollary 1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function whose derivative $f' \in L_p[a,b]$, $p > 1$. Then, the following inequality holds:*

$$\left| \left(1 - \frac{\lambda}{2} \right) f(x) - \lambda \frac{(x-b) f(b) - (x-a) f(a)}{2(b-a)} - \frac{1}{b-a} \int_a^b f(t)dt \right|$$

$$\leq \frac{1}{b-a} \| f' \|_p \, (Z(x))^{\frac{1}{q}} . \tag{2.10}$$

Proof. From (2.2) using the properties of modulus and applying Hölder's integral inequalty, we have

$$\left| \left(1 - \frac{\lambda}{2} \right) f(x) - \lambda \frac{(x-b) f(b) - (x-a) f(a)}{2(b-a)} - \frac{1}{b-a} \int_a^b f(t)dt \right|$$

$$\leq \frac{1}{b-a} \left(\int_a^b |f'(t)|^p \, dt \right)^{\frac{1}{p}} \left(\int_a^b |p(x,t)|^q \, dt \right)^{\frac{1}{q}}$$

$$= \frac{1}{b-a} \| f' \|_p \, (Z(x))^{\frac{1}{q}} . \qquad \square$$

Remark 2. i) If we choose $\lambda = 1$ in (2.10), then we get

$$\left| \frac{1}{2} f(x) - \frac{(x-b) f(b) - (x-a) f(a)}{2(b-a)} - \frac{1}{b-a} \int_a^b f(t)dt \right|$$

$$\leq \frac{1}{2(b-a)} \left(\frac{(x-a)^{q+1} + (b-x)^{q+1}}{(q+1)} \right)^{\frac{1}{q}} \|f'\|_p.$$

ii) For $\lambda = 0$ in (2.10), we have,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{1}{(b-a)} \left(\frac{(x-a)^{q+1} + (b-x)^{q+1}}{(q+1)} \right)^{\frac{1}{q}} \|f'\|_p.$$

Theorem 3. *Let* $f, g : [a,b] \to \mathbb{R}$ *be absolutely continuous functions whose derivatives* $f', g' \in L[a,b]$. *If* $|f'|^q$ *and* $|g'|^q$ *is convex on* $[a,b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then, the following inequality holds:*

$$|T(f,g)|$$
$$\leq \frac{1}{2} \int_a^b \left\{ \left[|f(x)| \left(\frac{|g'(a)|^q + |g'(b)|^q}{2} \right)^{\frac{1}{q}} \right. \right.$$
$$\left. \left. + |g(x)| \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \right] M^{\frac{1}{p}}(x) \right\} dx \qquad (2.11)$$

where

$$M(x) = \frac{(b-x)^{p+1} + (x-a)^{p+1}}{(b-a)^{p+1}(p+1)}.$$

Proof. From the hypotheses, we can write the following identities,

$$f(x) = \frac{1}{b-a} \int_a^b f(x)dx + (b-a) \int_0^1 P(x,t) f'(ta + (1-t)b)dt, \qquad (2.12)$$

and

$$g(x) = \frac{1}{b-a} \int_a^b g(x)dx + (b-a) \int_0^1 P(x,t) g'(ta + (1-t)b)dt, \qquad (2.13)$$

for $x \in [a,b]$, where

$$P(x,t) = \begin{cases} t, & 0 \leq t \leq \frac{b-x}{b-a} \\ t-1, & \frac{b-x}{b-a} < t \leq 1 \end{cases}.$$

A simple proof of equalities of (2.12) and (2.13) can be obtained by integrating in the right hand side and using the identity of $p(x,t)$. The details are left for the interested reader.

Multiplying both sides of (2.12) and (2.13) by $g(x)$ and $f(x)$ respectively and adding the resulting identities, we get

$$2f(x)g(x) \qquad (2.14)$$

$$= \frac{g(x)}{b-a} \int_a^b f(x)dx + (b-a)g(x) \int_0^1 P(x,t) f'(ta+(1-t)b)dt$$
$$+ \frac{f(x)}{b-a} \int_a^b g(x)dx + (b-a)f(x) \int_0^1 P(x,t) g'(ta+(1-t)b)dt.$$

Integrating both sides of (2.14) with respect to x over $[a,b]$ and dividing both sides of the resulting identity $(b-a)$, we get

$$\frac{2}{b-a} \int_a^b f(x)g(x)dx \qquad (2.15)$$
$$= 2\left(\frac{1}{b-a} \int_a^b g(x)dx\right)\left(\frac{1}{b-a} \int_a^b f(x)dx\right)$$
$$+ \int_a^b g(x)\left[\int_0^1 P(x,t) f'(ta+(1-t)b)dt\right]dx$$
$$+ \int_a^b f(x)\left[\int_0^1 P(x,t) g'(ta+(1-t)b)dt\right]dx.$$

Rewriting the identity (2.15), we get

$$T(f,g) = \frac{1}{2} \int_a^b g(x)\left[\int_0^1 P(x,t) f'(ta+(1-t)b)dt\right]dx \qquad (2.16)$$
$$+ \frac{1}{2} \int_a^b f(x)\left[\int_0^1 P(x,t) g'(ta+(1-t)b)dt\right]dx.$$

From (2.16), using the properties of modulus and rearranging the terms, we get

$$|T(f,g)|$$
$$\leq \frac{1}{2} \int_a^b |f(x)|\left[\int_0^1 |P(x,t)|\,|g'(ta+(1-t)b)|\,dt\right]dx$$
$$+ \frac{1}{2} \int_a^b |g(x)|\left[\int_0^1 |P(x,t)|\,|f'(ta+(1-t)b)|\,dt\right]dx.$$

Applying Hölder's integral inequality and the convexity of $|f'|^q$ and $|g'|^q$, we obtain

$$|T(f,g)| \leq \frac{1}{2} \int_a^b \left[|f(x)|\left(\int_0^1 |P(x,t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |g'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right.$$
$$\left. + |g(x)|\left(\int_0^1 |P(x,t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right]dx$$

$$= \quad \frac{1}{2} \int_a^b \left\{ \left[|f(x)| \left(\frac{|g'(a)|^q + |g'(b)|^q}{2} \right)^{\frac{1}{q}} \right. \right. \tag{2.17}$$

$$\left. \left. + |g(x)| \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \right] \left(\int_0^1 |P(x,t)|^p \, dt \right)^{\frac{1}{p}} \right\} dx.$$

A simple computation gives

$$\int_0^1 |P(x,t)|^p \, dt \quad = \quad \int_0^{\frac{b-x}{b-a}} t^p \, dt + \int_{\frac{b-x}{b-a}}^1 (t-1)^p \, dt \tag{2.18}$$

$$= \quad \frac{(b-x)^{p+1} + (x-a)^{p+1}}{(b-a)^{p+1} (p+1)} = M(x).$$

Using (2.18) in (2.18), we get the required inequality in (2.11). $\square$

Corollary 2. *If in theorem 3 we require that* $|f'(x)| \le K$, $K > 0$ *and* $|g'(x)| \le N$, $N > 0$, *then the following inequality holds*

$$|T(f,g)| \le \frac{1}{2} \left(\int_a^b [|f(x)| N + |g(x)| K] M^{\frac{1}{p}}(x) dx \right),$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

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