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The matrix version for the multivariable Humbert polynomials

Rabia Aktaş, Bayram Çekim, and Recep Şahin



THE MATRIX VERSION FOR THE MULTIVARIABLE HUMBERT POLYNOMIALS

RABİA AKTAŞ, BAYRAM ÇEKİM, AND RECEP ŞAHİN

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Abstract. In this paper, the matrix extension of the multivariable Humbert polynomials is introduced. Various families of linear, multilinear and multilateral generating matrix functions of these matrix polynomials are presented. Miscellaneous applications are also discussed.

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1. INTRODUCTION

It is well-known that special matrix functions appear in the study of many areas. Generalization of the property of orthogonality [11, 12], Rodrigues formula [6, 10], a second-order Sturm-Liouville differential equation [10], a three-term matrix recurrence [6, 7], relation between different orthogonal matrix polynomials [21] are theoretical examples. Statistics, group representation theory [17], scattering theory [15], differential equations [18, 19], Fourier series expansions [9], interpolation and quadrature [22, 23], splines [8], and medical imaging [5] are areas of application of orthogonal matrix polynomials.

Throughout this paper, for a matrix $A \in \mathbb{C}^{N \times N}$, its spectrum is denoted by $\sigma(A)$. The two-norm of A , which will be denoted by $\|A\|$, is defined by

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

where, for a vector $y \in \mathbb{C}^N$, $\|y\|_2 = (y^T y)^{1/2}$ is the Euclidean norm of y . I and θ will denote the identity matrix and the null matrix in $\mathbb{C}^{N \times N}$, respectively. We say that a matrix A in $\mathbb{C}^{N \times N}$ is a positive stable if $\Re(\lambda) > 0$ for all $\lambda \in \sigma(A)$, where $\sigma(A)$ is the set of the eigenvalues of A . If A_0, A_1, \dots, A_n are elements of $\mathbb{C}^{N \times N}$ and $A_n \neq \theta$, then we call

$$P(x) = A_n x^n + A_{n-1} x^{n-1} + \dots + A_1 x + A_0$$

a matrix polynomial of degree n in x . From [20], one can see

$$(P)_n = P(P + I)(P + 2I)\dots(P + (n - 1)I); n \geq 1; (P)_0 = I. \quad (1.1)$$

For any matrix A in $\mathbb{C}^{N \times N}$, the authors exploited the following relation due to [20]

$$(1 - x)^{-A} = \sum_{n=0}^{\infty} \frac{(A)_n}{n!} x^n, |x| < 1. \quad (1.2)$$

Gould [16] presented a systematic study of an interesting generalization of the Humbert and the Gegenbauer polynomials and several other polynomial systems, that is called generalized Humbert polynomials and defined by

$$(C - mx + yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, C) t^n \quad (1.3)$$

where m is a positive integer and the other parameters are unrestricted (see also [26, p. 77, 86]).

Aktas *et al.* [1] present a systematic investigation of a multivariable extension of the Humbert polynomials generated by

$$\prod_{i=1}^r \{(C_i - m_i x_i t + y_i t^{m_i})^{-\alpha_i}\} = \sum_{n=0}^{\infty} P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) t^n \quad (1.4)$$

$$(|m_i x_i t - y_i t^{m_i}| < |C_i| ; i = 1, 2, \dots, r)$$

where $\mathbf{x} = (x_1, \dots, x_r)$, $\mathbf{y} = (y_1, \dots, y_r)$, $\mathbf{C} = (C_1, \dots, C_r)$, $\mathbf{m} = (m_1, \dots, m_r)$, $m_i = 1, 2, \dots$ ($i = 1, 2, \dots, r$) and the other parameters are unrestricted.

The main objective of this paper is to construct a matrix version of the multivariable Humbert polynomials given by (1.4) and the derivation of various families of multilinear and mixed multilateral generating matrix functions for these matrix polynomials. We present some special cases of our results and also obtain several recurrence relations for these matrix polynomials.

2. MATRIX EXTENSION OF THE MULTIVARIABLE HUMBERT POLYNOMIALS

The main object of this section is to present a systematic investigation of the matrix extension of the multivariable Humbert polynomials generated by

$$\prod_{i=1}^r \{(C_i - m_i x_i t + y_i t^{m_i})^{-A_i}\} = \sum_{n=0}^{\infty} P_n^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) t^n \quad (2.1)$$

$$(|m_i x_i t - y_i t^{m_i}| < |C_i| ; i = 1, 2, \dots, r)$$

where $A_i \in \mathbb{C}^{N \times N}$, $\mathbf{x} = (x_1, \dots, x_r)$, $\mathbf{y} = (y_1, \dots, y_r)$, $\mathbf{C} = (C_1, \dots, C_r)$, $\mathbf{m} = (m_1, \dots, m_r)$, $m_i = 1, 2, \dots$ ($i = 1, 2, \dots, r$) and the other parameters are unrestricted.

(2.1) yields the following explicit representation:

$$\begin{aligned}
 & P_n^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\
 = & \sum_{m_1 k_1 + \dots + m_r k_r + n_1 + \dots + n_r = n} \\
 & \frac{(A_1)_{n_1 + k_1} C_1^{-A_1 - (n_1 + k_1)I} \dots (A_r)_{n_r + k_r} C_r^{-A_r - (n_r + k_r)I}}{n_1! \dots n_r! k_1! \dots k_r!} \\
 & \times m_1^{n_1} \dots m_r^{n_r} (-1)^{k_1 + \dots + k_r} x_1^{n_1} \dots x_r^{n_r} y_1^{k_1} \dots y_r^{k_r} \\
 = & \sum_{m_1 k_1 + \dots + m_r k_r + n_1 + \dots + n_r = n} \prod_{p=1}^r \\
 & \left\{ \frac{(A_p)_{n_p + k_p} C_p^{-A_p - (n_p + k_p)I}}{n_p! k_p!} m_p^{n_p} (-1)^{k_p} x_p^{n_p} y_p^{k_p} \right\} \tag{2.2}
 \end{aligned}$$

where, as usual, $(A)_n$ denotes the Pochhammer symbol given by (1.1).

We notice that the case $r = 1$ in (2.1) reduces to the matrix version of the generalized Humbert polynomials introduced by Gould [16]. In this case, it is generated by

$$(C - mxt + yt^m)^{-A} = \sum_{n=0}^{\infty} P_n^{(A)}(m, x, y, C) t^n \tag{2.3}$$

where $|mxt - yt^m| < |C|$, $A \in \mathbb{C}^{N \times N}$, m is a positive integer and the other parameters are unrestricted. For the special cases of (2.3), including Gegenbauer matrix polynomials, we refer [19].

It is clear that the case

$$C_i = 1, m_i = 1, y_i = 0, i = 1, 2, \dots, r$$

of the polynomials of (2.1) reduces to matrix version of the Chan-Chyan-Srivastava multivariable polynomials, which is generated by [14]

$$\prod_{i=1}^r \left\{ (1 - x_i t)^{-A_i} \right\} = \sum_{n=0}^{\infty} g_n^{(A_1, \dots, A_r)}(x_1, \dots, x_r) t^n \tag{2.4}$$

$$\left(A_i \in \mathbb{C}^{N \times N} (i = 1, 2, \dots, r) ; |t| < \min \left\{ |x_1|^{-1}, \dots, |x_r|^{-1} \right\} \right).$$

Since $A_i = \alpha_i \in \mathbb{C}$ for $N = 1$ in (2.4), we obtained the generating function of the Chan-Chyan-Srivastava multivariable polynomials [3].

On the other hand, if we choose $C_i = 1, m_i = i, x_i = 0, y_i = -x_i, i = 1, 2, \dots, r$ in (2.1), we get a matrix version of the multivariable Lagrange-Hermite

polynomials, which is generated by [14]

$$\prod_{i=1}^r \left\{ (1 - x_i t^i)^{-A_i} \right\} = \sum_{n=0}^{\infty} h_n^{(A_1, \dots, A_r)}(x_1, \dots, x_r) t^n$$

$$\left(A_i \in \mathbb{C}^{N \times N} \ (i = 1, 2, \dots, r) ; |t| < \min \left\{ |x_1|^{-1}, |x_2|^{-1/2}, \dots, |x_r|^{-1/r} \right\} \right). \quad (2.5)$$

Since $A_i = \alpha_i \in \mathbb{C}$ for $N = 1$ in (2.5), we have the multivariable Lagrange-Hermite polynomials presented by Altın and Erkuş [2]. Furthermore, we should remark that the case $r = 2$ of the polynomials corresponds to the familiar (two-variable) Lagrange-Hermite polynomials considered by Dattoli *et al.* [4].

Moreover, the special case

$$C_i = 1, \ x_i = 0, \ y_i = -x_i, \ i = 1, 2, \dots, r$$

gives the matrix version of the Erkus-Srivastava multivariable polynomials generated by [14]

$$\prod_{i=1}^r \left\{ (1 - x_i t^{m_i})^{-A_i} \right\} = \sum_{n=0}^{\infty} u_n^{(A_1, \dots, A_r)}(x_1, \dots, x_r) t^n,$$

$$A_i \in \mathbb{C}^{N \times N} \ (i = 1, 2, \dots, r),$$

$$|t| < \min \left\{ |x_1|^{-1/m_1}, |x_2|^{-1/m_2}, \dots, |x_r|^{-1/m_r} \right\}. \quad (2.6)$$

Since $A_i = \alpha_i \in \mathbb{C}$ for $N = 1$ in (2.6), we have the Erkus-Srivastava multivariable polynomials generated by [13].

3. AN APPLICATION OF SRIVASTAVA'S THEOREM ON MIXED GENERATING FUNCTIONS

Srivastava [25] (see also the subsequent treatise on the subject by Srivastava and Manocha [26, p. 378, Theorem 12]) obtained a family of mixed generating functions for certain general multivariable and multiparameter sequences of functions. Our generating function (2.1) fits easily into the general setting of Srivastava's theorem. Thus, by applying this general result to the generating function (2.1), we obtain the

following family of mixed generating functions for matrix version of the multivariable Humbert polynomials given by (2.1)

$$\sum_{n=0}^{\infty} P_n^{(A_1+\lambda_1 nI, \dots, A_r+\lambda_r nI)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) t^n = \frac{\prod_{i=1}^r \{(C_i - m_i x_i \xi + y_i \xi^{m_i})^{-A_i}\}}{1 - \xi \left\{ \sum_{i=1}^r \lambda_i m_i (C_i + y_i \xi^{m_i})^{-1} \left[-y_i \xi^{m_i-1} + x_i \frac{(C_i - m_i y_i \xi^{m_i} + y_i \xi^{m_i})}{(C_i - m_i x_i \xi + y_i \xi^{m_i})} \right] \right\}}, \tag{3.1}$$

$$\left(\xi = \xi(t) := t \prod_{i=1}^r \{(C_i - m_i x_i \xi + y_i \xi^{m_i})^{-\lambda_i}\}; \lambda_i \in \mathbb{C}, \right. \\ \left. C_i \neq 0 \text{ and } A_i \in \mathbb{C}^{N \times N} \ (i = 1, \dots, r) \right)$$

where all of the matrices commute with each other. In a special case, it is easily seen that (3.1) would at once reduce to the generating function (2.1) when $\lambda_i = 0 \ (i = 1, \dots, r)$. For the special case of $N = 1$, (3.1) gives mixed generating function for the multivariable Humbert polynomials given by [1]. Furthermore, the special case of $N = 1$ and $r = 1$ of (3.1) reduces to the mixed generating function for the generalized Humbert polynomials in [24].

4. BILINEAR AND BILATERAL GENERATING MATRIX FUNCTIONS

In this section, we derive several families of bilinear and bilateral generating matrix functions for matrix version of the multivariable Humbert polynomials which are generated by (2.1) and given explicitly by (2.2).

We begin by stating the following theorem.

Theorem 1. *Corresponding to an identically non-vanishing function $\Omega_\mu(\mathbf{z})$ of s complex variables $z_1, \dots, z_s \ (s \in \mathbb{N})$ and of complex order μ , let*

$$\Lambda_{\mu, \nu}(\mathbf{z}; w) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(\mathbf{z}) w^k \tag{4.1}$$

where $(a_k \neq 0, \ \mu, \nu \in \mathbb{C})$; $\mathbf{z} = (z_1, \dots, z_s)$ and

$$\Theta_{n, p, \mu, \nu}(\mathbf{x}, \mathbf{y}; \zeta) := \sum_{k=0}^{[n/p]} a_k P_{n-pk}^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \Omega_{\mu+\nu k}(\mathbf{z}) \zeta^k \tag{4.2}$$

where $n, p \in \mathbb{N}$; $A_i \in \mathbb{C}^{N \times N}$; $\mathbf{x} = (x_1, \dots, x_r)$; $\mathbf{y} = (y_1, \dots, y_r)$; $\mathbf{C} = (C_1, \dots, C_r)$; $\mathbf{m} = (m_1, \dots, m_r)$, $m_i = 1, 2, \dots$ ($i = 1, 2, \dots, r$). Then we have

$$\sum_{n=0}^{\infty} \Theta_{n,p,\mu,v} \left(\mathbf{x}, \mathbf{y}; \mathbf{z}; \frac{\eta}{t^p} \right) t^n = \prod_{i=1}^r \left\{ (C_i - m_i x_i t + y_i t^{m_i})^{-A_i} \right\} \Lambda_{\mu,v}(\mathbf{z}; \eta) \quad (4.3)$$

provided that each member of (4.3) exists.

Proof. Let T denote the left-hand side of the equality (4.3) of Theorem 1. Then, upon substituting the polynomials

$$\Theta_{n,p,\mu,v} \left(\mathbf{x}, \mathbf{y}; \mathbf{z}; \frac{\eta}{t^p} \right)$$

from definition (4.2) into the left-hand side of (4.3), we find

$$T = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/p \rfloor} a_k P_{n-pk}^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \Omega_{\mu+vk}(\mathbf{z}) \eta^k t^{n-pk}. \quad (4.4)$$

Replacing n by $n + pk$, we can write

$$\begin{aligned} T &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k P_n^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \Omega_{\mu+vk}(\mathbf{z}) \eta^k t^n \\ &= \sum_{n=0}^{\infty} P_n^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) t^n \sum_{k=0}^{\infty} a_k \Omega_{\mu+vk}(\mathbf{z}) \eta^k \\ &= \prod_{i=1}^r \left\{ (C_i - m_i x_i t + y_i t^{m_i})^{-A_i} \right\} \Lambda_{\mu,v}(\mathbf{z}; \eta), \end{aligned}$$

which completes the proof. \square

In a similar manner, we can give the next result.

Theorem 2. For a non-vanishing function $\Omega_{\mu}(\mathbf{z})$ of s complex variables z_1, \dots, z_s ($s \in \mathbb{N}$) and for $p \in \mathbb{N}$, $\mu, v \in \mathbb{C}$, $\mathbf{z} = (z_1, \dots, z_s)$, $A := (A_1, \dots, A_r)$, $B := (B_1, \dots, B_r)$, $A_i, B_i \in \mathbb{C}^{N \times N}$ for $i = 1, 2, \dots, r$, let

$$\mathcal{E}_{\mu,v,\mathbf{C},\mathbf{m}}^{n,p}(\mathbf{x}, \mathbf{y}; \mathbf{z}; w) := \sum_{k=0}^{\lfloor n/p \rfloor} a_k P_{n-pk}^{(A_1+B_1, \dots, A_r+B_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \Omega_{\mu+vk}(\mathbf{z}) w^k \quad (4.5)$$

where $a_k \neq 0$; $n, k \in \mathbb{N}_0$; $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Then we have

$$\begin{aligned} &\sum_{k=0}^n \sum_{l=0}^{\lfloor k/p \rfloor} a_l P_{n-k}^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) P_{k-pl}^{(B_1, \dots, B_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \Omega_{\mu+vl}(\mathbf{z}) w^l \\ &= \mathcal{E}_{\mu,v,\mathbf{C},\mathbf{m}}^{n,p}(\mathbf{x}, \mathbf{y}; \mathbf{z}; w) \end{aligned} \quad (4.6)$$

provided that each member of (4.6) exists where the matrices commute with each other.

5. SPECIAL CASES AND SOME FURTHER PROPERTIES

It is possible to give many applications of the theorems obtained in the previous sections with the help of appropriate choices of the multivariable functions $\Omega_{\mu+\nu k}(\mathbf{z})$, $\mathbf{z} = (z_1, \dots, z_s)$, $k \in \mathbb{N}_0$, $s \in \mathbb{N}$. For example, if we set

$$s = r \text{ and } \Omega_{\mu+\nu k}(\mathbf{z}) = h_{\mu+\nu k}^{(B_1, \dots, B_r)}(\mathbf{z})$$

in Theorem 1, where the matrix version of the multivariable Lagrange-Hermite polynomials

$$h_n^{(B_1, \dots, B_r)}(\mathbf{x})$$

are generated by (2.5), then we obtain the following result which provides a class of bilateral generating matrix functions for the matrix version of the multivariable Lagrange-Hermite polynomials and for the matrix version of the multivariable Humbert polynomials given explicitly by (2.2).

Corollary 1. If $\Lambda_{\mu, \nu}(\mathbf{z}; w) := \sum_{k=0}^{\infty} a_k h_{\mu+\nu k}^{(B_1, \dots, B_r)}(\mathbf{z}) w^k$, $a_k \neq 0$, $\mu, \nu \in \mathbb{N}_0$, $\mathbf{z} = (z_1, \dots, z_r)$ and

$$\Theta_{n, p, \mu, \nu}(\mathbf{x}, \mathbf{y}; \mathbf{z}; \zeta) := \sum_{k=0}^{[n/p]} a_k P_{n-pk}^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) h_{\mu+\nu k}^{(B_1, \dots, B_r)}(\mathbf{z}) \zeta^k$$

where $n \in \mathbb{N}_0$; $p \in \mathbb{N}$; $A_i, B_i \in \mathbb{C}^{N \times N}$; $\mathbf{x} = (x_1, \dots, x_r)$; $\mathbf{y} = (y_1, \dots, y_r)$; $\mathbf{C} = (C_1, \dots, C_r)$; $\mathbf{m} = (m_1, \dots, m_r)$, $m_i = 1, 2, \dots$ ($i = 1, 2, \dots, r$), then

$$\sum_{n=0}^{\infty} \Theta_{n, p, \mu, \nu}(\mathbf{x}, \mathbf{y}; \mathbf{z}; \frac{\eta}{t^p}) t^n = \prod_{i=1}^r \left\{ (C_i - m_i x_i t + y_i t^{m_i})^{-A_i} \right\} \Lambda_{\mu, \nu}(\mathbf{z}; \eta) \quad (5.1)$$

provided that each member of (5.1) exists.

Remark 1. Using the generating relation (2.5) for the matrix version of the multivariable Lagrange-Hermite polynomials and setting $a_k = 1$, $\mu = 0$, $\nu = 1$, we obtain

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} P_{n-pk}^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) h_k^{(B_1, \dots, B_r)}(\mathbf{z}) \eta^k t^{n-pk} = \left(\prod_{i=1}^r (C_i - m_i x_i t + y_i t^{m_i})^{-A_i} \right) \left(\prod_{i=1}^r (1 - z_i \eta^i)^{-B_i} \right),$$

where

$$|\eta| < \min \left\{ |z_1|^{-1}, |z_2|^{-1/2}, \dots, |z_r|^{-1/r} \right\},$$

$$|m_i x_i t - y_i t^{m_i}| < |C_i| \quad ; \quad i = 1, 2, \dots, r.$$

Also, if we choose $s = 2r$ and $\Omega_{\mu+\nu k}(\mathbf{z}) = P_{\mu+\nu k}^{(E_1, \dots, E_r)}(\mathbf{m}, \mathbf{t}, \omega, \mathbf{C})$, $\mu, \nu \in \mathbb{N}_0$, $\mathbf{t} = (t_1, \dots, t_r)$, $\omega = (\omega_1, \dots, \omega_r)$ in Theorem 2, we obtain the following class of bilinear generating matrix functions for the matrix version of the multivariable Humbert polynomials given explicitly by (2.2).

Corollary 2. *If*

$$\begin{aligned} & \Lambda_{\mu, \nu, \mathbf{C}, \mathbf{m}}^{n, p}(\mathbf{x}, \mathbf{y}; \mathbf{t}, \omega; w) \\ & := \sum_{k=0}^{[n/p]} a_k P_{n-pk}^{(A_1+B_1, \dots, A_r+B_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) P_{\mu+\nu k}^{(E_1, \dots, E_r)}(\mathbf{m}, \mathbf{t}, \omega, \mathbf{C}) w^k \\ & (a_k \neq 0; p \in \mathbb{N}; n, k, \mu, \nu \in \mathbb{N}_0) \end{aligned}$$

where $A_i, B_i, E_i \in \mathbb{C}^{N \times N}$ for $i = 1, 2, \dots, r$, then

$$\begin{aligned} & \sum_{k=0}^n \sum_{l=0}^{[k/p]} a_l P_{n-k}^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\ & P_{k-pl}^{(B_1, \dots, B_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) P_{\mu+\nu l}^{(E_1, \dots, E_r)}(\mathbf{m}, \mathbf{t}, \omega, \mathbf{C}) w^l \\ & = \Lambda_{\mu, \nu, \mathbf{C}, \mathbf{m}}^{n, p}(\mathbf{x}, \mathbf{y}; \mathbf{t}, \omega; w) \end{aligned} \quad (5.2)$$

provided that each member of (5.2) exists where $A_i B_j = B_j A_i$ for $i, j = 1, 2, \dots, r$.

For example, if we set

$$s = 1 \quad \text{and} \quad \Omega_{\mu+\nu k}(y) = L_{\mu+\nu k}^{(E, \lambda)}(y)$$

in Theorem 1, where the n th Laguerre matrix polynomials $L_n^{(E, \lambda)}(x)$ are defined by [18]

$$L_n^{(E, \lambda)}(x) = \sum_{k=0}^n \frac{(-1)^k \lambda^k}{k! (n-k)!} (E+I)_n [(E+I)_k]^{-1} x^k,$$

where E is a matrix in $\mathbb{C}^{N \times N}$, $E+nI$ is invertible for every integer $n \geq 0$ and λ is a complex number with $\Re(\lambda) > 0$ and they are generated by

$$\sum_{n=0}^{\infty} L_n^{(E, \lambda)}(x) t^n = (1-t)^{-(E+I)} \exp\left(\frac{-\lambda x t}{1-t}\right), \quad (5.3)$$

$$|t| < 1, \quad 0 < x < \infty,$$

then we obtain the following result which provides a class of bilateral generating matrix functions for the matrix version of the multivariable Humbert and Laguerre matrix polynomials.

Corollary 3. If $\Lambda_{\mu,\nu}(z; w) := \sum_{k=0}^{\infty} a_k L_{\mu+\nu k}^{(E,\lambda)}(z) w^k$ where $(a_k \neq 0, \mu, \nu \in \mathbb{N}_0)$; and

$$\Theta_{n,p,\mu,\nu}(\mathbf{x}, \mathbf{y}; z; \zeta) := \sum_{k=0}^{[n/p]} a_k P_{n-pk}^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) L_{\mu+\nu k}^{(E,\lambda)}(z) \zeta^k$$

where $n, p \in \mathbb{N}$. Then we have

$$\sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu}(\mathbf{x}; \mathbf{y}; z; \frac{\eta}{t^p}) t^n = \prod_{i=1}^r \left\{ (C_i - m_i x_i t + y_i t^{m_i})^{-A_i} \right\} \Lambda_{\mu,\nu}(z; \eta) \quad (5.4)$$

provided that each member of (5.4) exists.

Remark 2. Using the generating relation (5.3) for the Laguerre matrix polynomials and taking $a_k = 1, \mu = 0, \nu = 1$, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} P_{n-pk}^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) L_k^{(E,\lambda)}(z) \eta^k t^{n-pk} \\ &= \prod_{i=1}^r \left\{ (C_i - m_i x_i t + y_i t^{m_i})^{-A_i} \right\} \\ & \quad \times (1 - \eta)^{-(E+I)} \exp\left(\frac{-\lambda z \eta}{1 - \eta}\right), \end{aligned} \quad (5.5)$$

where $|\eta| < 1, 0 < z < \infty$.

Remark 3. For $r = 1$ in (5.5), we have a bilateral generating matrix function of the Humbert (2.3) and Laguerre matrix polynomials:

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} P_{n-pk}^{(A)}(m, x, y, C) L_k^{(E,\lambda)}(z) \eta^k t^{n-pk} \\ &= (C - mxt + yt^m)^{-A} (1 - \eta)^{-(E+I)} \exp\left(\frac{-\lambda z \eta}{1 - \eta}\right). \end{aligned}$$

Remark 4. For $r = 1$ and $s = 2$ in Theorem 1, setting $\Omega_{\mu+\nu k}(z) = P_n^{(B)}(m, x, y, C) (B \in \mathbb{C}^{N \times N})$ and taking $a_k = 1, \mu = 0, \nu = 1$, we have bilinear generating matrix function for the Humbert matrix polynomials:

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} P_{n-pk}^{(A)}(m, x, y, C) P_k^{(B)}(m, x, y, C) \eta^k t^{n-pk} \\ &= (C - mxt + yt^m)^{-A} (C - mx\eta + y\eta^m)^{-B}. \end{aligned}$$

Furthermore, for every suitable choice of the coefficients a_k ($k \in \mathbb{N}_0$), if the multivariable function $\Omega_{\mu+\nu k}(\mathbf{y})$, $\mathbf{y}=(y_1, \dots, y_s)$, ($s \in \mathbb{N}$), is expressed as an appropriate product of several simpler functions, then the assertions of Theorems 1 and 2 can be applied in order to derive various families of multilinear and multilateral generating matrix functions for the matrix version of the multivariable Humbert polynomials given explicitly by (2.2).

We now discuss some further properties of matrix version of the multivariable Humbert polynomials given by (2.2). First of all, the generating matrix relation (2.1) yields the following addition formula for these multivariable polynomials:

$$\begin{aligned} & P_n^{(A_1+B_1, \dots, A_r+B_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\ &= \sum_{k=0}^n P_{n-k}^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) P_k^{(B_1, \dots, B_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \end{aligned}$$

where $A_i, B_i \in \mathbb{C}^{N \times N}$, $A_i B_j = B_j A_i$ for $i, j = 1, 2, \dots, r$.

On the other hand, the multivariable Humbert matrix polynomials satisfy the following equation:

$$\sum_{j=1}^r \left(x_j \frac{\partial}{\partial x_j} + m_j y_j \frac{\partial}{\partial y_j} \right) P_n^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) = n P_n^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}). \quad (5.6)$$

If we differentiate each member of the generating function (2.1) with respect to x_j and y_j ($j = 1, 2, \dots, r$), we obtain the following (differential) recurrence relations for the matrix version of the multivariable Humbert polynomials:

$$\begin{aligned} & \frac{\partial}{\partial x_j} P_n^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\ &= \sum_{k=0}^{n-1} \sum_{l=0}^{[k/m_j]} \frac{(-1)^l (k+l-lm_j)! A_j (m_j)^{k-lm_j+1}}{(k-lm_j)! l! C_j^{k-l(m_j-1)+1}} x_j^{k-lm_j} y_j^l \\ & \quad \times P_{n-k-1}^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \end{aligned} \quad (5.7)$$

for $n \geq 1$, and

$$\begin{aligned} & \frac{\partial}{\partial y_j} P_n^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\ &= - \sum_{k=0}^{n-m_j} \sum_{l=0}^{[k/m_j]} \frac{(-1)^l (k+l-lm_j)! A_j (m_j)^{k-lm_j}}{(k-lm_j)! l! C_j^{k-l(m_j-1)+1}} x_j^{k-lm_j} y_j^l \\ & \quad \times P_{n-k-m_j}^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \end{aligned} \quad (5.8)$$

where $n \geq m_j$ and m_j ($j = 1, 2, \dots, r$) is a positive integer and all matrices are commutative. By applying (5.6), (5.7) and (5.8), the following recurrence relation for the matrix polynomials (given explicitly by (2.2)) can be easily derived:

$$\begin{aligned} & \sum_{j=1}^r \sum_{k=0}^{n-1} \sum_{l=0}^{\lfloor k/m_j \rfloor} \frac{(-1)^l (k+l-lm_j)! A_j(m_j)^{k-lm_j+1}}{(k-lm_j)! l! C_j^{k-l(m_j-1)+1}} \\ & x_j^{k-lm_j+1} y_j^l P_{n-k-1}^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\ & - \sum_{j=1}^r \sum_{k=0}^{n-m_j} \sum_{l=0}^{\lfloor k/m_j \rfloor} \frac{(-1)^l (k+l-lm_j)! A_j(m_j)^{k-lm_j+1}}{(k-lm_j)! l! C_j^{k-l(m_j-1)+1}} \\ & x_j^{k-lm_j} y_j^{l+1} P_{n-k-m_j}^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\ & = n P_n^{(A_1, \dots, A_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}), \text{ for } n \geq m_j \end{aligned}$$

where m_j ($j = 1, 2, \dots, r$) is a positive integer and all matrices commute.

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Authors' addresses

Rabia Aktaş

Ankara University, Faculty of Science, Department of Mathematics, Tandoğan TR-06100, Ankara, Turkey

E-mail address: raktas@science.ankara.edu.tr

Bayram Çekim

Gazi University, Faculty of Science, Department of Mathematics, Teknik Okullar TR-06500, Ankara, Turkey

E-mail address: bayramcekim@gazi.edu.tr

Recep Şahin

Ankara University, Faculty of Science, Department of Mathematics, Tandoğan TR-06100, Ankara, Turkey

E-mail address: sahin@science.ankara.edu.tr