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COMMENTS ON "NEW HYPERGEOMETRIC IDENTITIES ARISING FROM GAUSS'S SECOND SUMMATION THEOREM"

MEDHAT K. RAKHA, ARJUN K. RATHIE, PURNIMA CHOPRA, AND RICHARD B. PARIS

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Abstract. In 1997, Exton [J. Comput. Appl. Math. **88** (1997) 269–274] obtained a general transformation involving hypergeometric functions by elementary manipulation of series. A number of hypergeometric identities that had not been previously recorded in the literature were then deduced by application of Gauss' second summation theorem and other known hypergeometric summation theorems. However, some of the results stated by Exton contain errors. It is the purpose of this note to present the corrected forms of these hypergeometric identities.

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1. INTRODUCTION

The generalized hypergeometric function with p numeratorial and q denominatorial parameters is defined by the series [2, p. 41]

$$ {}_pF_q\left[\begin{matrix} a_1,a_2,\ldots,a_p \\ b_1,b_2,\ldots,b_q \end{matrix};x\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\ldots(a_p)_n}{(b_1)_n(b_2)_n\ldots(b_q)_n}\frac{x^n}{n!}, \tag{1.1}$$

where $(a)_n=\Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol (or ascending factorial). When $q=p$ this series converges for $|x|<\infty$, but when $q=p-1$ convergence occurs when $|x|<1$. However, when only one of the parameters a_j is a negative integer or zero, then the series (1.1) terminates and so always converges since it becomes a polynomial in x of degree $-a_j$.

In [1], Exton obtained a number of hypergeometric identities which had not been previously recorded in the literature. Exton based his investigation on the following general transformation which he obtained by elementary manipulation of series [1, Eq. (1.8)]

$$\sum_{n=0}^{\infty}\frac{(c_1)_n\ldots(c_p)_n}{(d_1)_n\ldots(d_q)_n}\frac{(\frac{1}{2}a)_n(-2x)^n}{n!}\,{}_{p+1}F_q\left[\begin{matrix} a+2n,c_1+n,\ldots,c_p+n \\ d_1+n,\ldots,d_q+n \end{matrix};x\right] \tag{1.2}$$

$$= {}_{2p+1}F_{2q}\left[\begin{matrix} \frac{1}{2}a, & \frac{1}{2}c_1,\ldots,\frac{1}{2}c_p, & \frac{1}{2}(1+c_1),\ldots,\frac{1}{2}(1+c_p) \\ & \frac{1}{2}d_1,\ldots,\frac{1}{2}d_q, & \frac{1}{2}(1+d_1),\ldots,\frac{1}{2}(1+d_q) \end{matrix};-4^{p-q}x^2\right].$$

The domain of validity in the x-plane depends on the particular values of p, q and the various parameters. For example, if $p=q$ then (1.2) holds in general for $|x|<1$. When one of the numeratorial parameters c_j equals a negative integer, the resulting expansion (1.2) involves *finite* sums and convergence at $x=\pm 1$ is assured. Then, upon letting $x=\pm 1$ in (1.2) followed by application of Gauss' second summation theorem and other well-known summation formulas listed in [2, Appendix III], Exton deduced several interesting hypergeometric identities.

2. Exton's results in corrected form

We have discovered that some of these results are incorrect. Following the same method used by Exton, notably the equation (1.2), we find that his corrected results are as stated below. In the following N denotes a positive integer.

Exton's result (2.5) should read

$$\frac{(b-a)_N}{(b)_N}\,{}_3F_2\left[\begin{matrix} -N,\frac{1}{2}a,1+a-b \\ \frac{1}{2}(1+a-b-N),\frac{1}{2}(2+a-b-N)\end{matrix};\frac{1}{2}\right]$$
$$= {}_3F_2\left[\begin{matrix} -\frac{1}{2}N,\frac{1}{2}-\frac{1}{2}N,\frac{1}{2}a \\ \frac{1}{2}b,\frac{1}{2}+\frac{1}{2}b\end{matrix};-1\right];$$

Exton's result (3.1) should read

$$\frac{(1+a)_N}{(1+\frac{1}{2}a)_N}\,{}_2F_1\left[\begin{matrix} -N,\frac{1}{2}a \\ \frac{1}{2}+\frac{1}{2}a\end{matrix};\frac{1}{2}\right]$$
$$= {}_3F_2\left[\begin{matrix} -\frac{1}{2}N,\frac{1}{2}-\frac{1}{2}N,\frac{1}{2}a \\ \frac{1}{2}(1+a+N),\frac{1}{2}(2+a+N)\end{matrix};-1\right];$$

Exton's result (3.2) should read

$$\frac{(1+a)_N(1+\frac{1}{2}a-b)_N}{(1+\frac{1}{2}a)_N(1+a-b)_N}\,{}_3F_2\left[\begin{matrix} -N,\ \frac{1}{2}a,\ b \\ \frac{1}{2}+\frac{1}{2}a,\ b-\frac{1}{2}a-N\end{matrix};\frac{1}{2}\right]$$
$$= {}_5F_4\left[\begin{matrix} -\frac{1}{2}N,\ \frac{1}{2}-\frac{1}{2}N,\ \frac{1}{2}a,\ \frac{1}{2}b,\ \frac{1}{2}+\frac{1}{2}b \\ \frac{1}{2}(1+a-b),\ \frac{1}{2}(2+a-b),\ \frac{1}{2}(1+a+N),\ \frac{1}{2}(2+a+N)\end{matrix};-1\right];$$

Exton's result (3.3) should read

$$\frac{(1+a)_N}{(\frac{1}{2}+\frac{1}{2}a)_N}\,{}_1F_0\left[\begin{matrix} -N \\ -\end{matrix};\frac{1}{2}\right] = \frac{2^{-N}(1+a)_N}{(\frac{1}{2}+\frac{1}{2}a)_N}$$
$$= {}_4F_3\left[\begin{matrix} -\frac{1}{2}N,\ \frac{1}{2}-\frac{1}{2}N,\ \frac{1}{2}a,\ 1+\frac{1}{4}a \\ \frac{1}{4}a,\ \frac{1}{2}(1+a+N),\ \frac{1}{2}(2+a+N)\end{matrix};-1\right].$$

We also note that Exton's results for the equations (2.9), (3.4) and (3.6) are correct, but his (3.7) is not a new result since the right-hand side is the same as that in (3.6) and so yields a simple identity between two ${}_2F_1(\frac{1}{2})$ functions. In addition, we cannot derive the results in (3.5), (3.8) and (3.9), nor could we verify them numerically. They should therefore be taken to be incorrect.

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Authors' addresses

Medhat K. Rakha
Department of Mathematics, College of Science, Sultan Qaboos University, P.O. Box 36, Al-Khoud 123, Muscat, Oman
E-mail address: medhat@squ.edu.om

Arjun K. Rathie
Department of Mathematics, Vedant College of Engineering Technology, Tulsi, Bundi, Rajasthan State, India
E-mail address: akrathie@rediffmail.com

Purnima Chopra
Department of Mathematics, Marudhar Engineering College, Raisar, Bikaner, Rajasthan State, India
E-mail address: purnimachopra@rediffmail.com

Richard B. Paris
School of Computing, University of Abertay Dundee, Dundee DD1 1HG, UK
E-mail address: r.paris@abertay.ac.uk