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THE VORONOVSKAJA TYPE THEOREM FOR A GENERAL CLASS OF SZÁSZ-MIRAKJAN OPERATORS

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Abstract. The paper is devoted to defining a new general class of linear and positive operators depending on a certain function φ. This new class of linear and positive operators generalizes the Szász-Mirakjan operators. For this new class of operators we establish a convergence theorem and the evaluation of the rate of convergence, in terms of the modulus of continuity.

1. INTRODUCTION

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In this section we recall some results from [12], which we shall use in the present paper.

Let I, J be real intervals and $I \cap J \neq \varnothing$. For any $n, k \in \mathbb{N}_0$, $n \neq 0$ consider the functions $\varphi_{n,k} : J \to \mathbb{R}$, with the property that $\varphi_{n,k}(x) \geq 0$, for any $x \in J$ and the linear positive functionals $A_{n,k} : E(I) \to \mathbb{R}$.

For any $n \in \mathbb{N}$ define the operator $L_n : E(I) \to F(J)$, by

$$(L_n f)(x) = \sum_{k=0}^{\infty} \varphi_{n,k}(x) A_{n,k}(f), \qquad (1.1)$$

where $E(I)$ is a linear space of real-valued functions defined on I, for which the operators (1.1) are convergent and $F(J)$ is a subset of the set of real-valued functions defined on J.

Remark 1. [12] The operators $(L_n)_{n \in \mathbb{N}}$ are linear and positive on $E(I \cap J)$.

For $n, i \in \mathbb{N}_0$, $n \neq 0$, let the function ψ_x be defined by $\psi_x(t) = t - x$, $(x, t) \in [0, +\infty[\times [0, +\infty[$ and define $T_{n,i}^*$ by

$$(T_{n,i}^* L_n)(x) = n^i (L_n \psi_x^i)(x) = n^i \sum_{k=0}^{\infty} \varphi_{n,k}(x) A_{n,k}(\psi_x^i), \ x \in I \cap J. \qquad (1.2)$$

In what follows $s \in \mathbb{N}_0$ is even and we suppose that the following two conditions hold:

- there exists the smallest α_s, $\alpha_{s+2} \in [0, +\infty[$, so that

$$\lim_{n \to \infty} \frac{(T_{n,j}^* L_n)(x)}{n^{\alpha_j}} = B_j(x) \in \mathbb{R}, \tag{1.3}$$

for any $x \in I \cap J$ and $j \in \{s, s+2\}$,

$$\alpha_{s+2} < \alpha_s + 2 \tag{1.4}$$

- $I \cap J$ is an interval.

Theorem 1 ([12]). *Let $f \in E(I)$ be a function. If $x \in I \cap J$ and f is s times differentiable in a neighborhood of x, $f^{(s)}$ is continuous in x, then*

$$\lim_{n \to \infty} n^{s-\alpha_s} \left((L_n f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{n^i i!} (T_{n,i}^* L_n)(x) \right) = 0. \tag{1.5}$$

Assume that f is s times differentiable on I and there exists an interval $K \subset I \cap J$ such that, there exists $n(s) \in \mathbb{N}$ and the constants $k_j \in \mathbb{R}$ depending on K, so that for $n \geq n(s)$ and $x \in K$, the following

$$\frac{(T_{n,j}^* L_n)(x)}{n^{\alpha_j}} \leq k_j \tag{1.6}$$

holds, for $j \in \{s, s+2\}$.
Then, the convergence expressed by (1.5) is uniform on K and

$$n^{s-\alpha_s} \left| (L_n f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{n^i i!} (T_{n,i}^* L_n)(x) \right| \tag{1.7}$$

$$\leq \frac{1}{s!} (k_s + k_{s+2}) \omega_1 \left(f^{(s)}; \frac{1}{\sqrt{n^{2+\alpha_s-\alpha_{s+2}}}} \right),$$

for any $x \in K$, $n \geq n(s)$, where $\omega_1(f;\delta)$ denotes the modulus of continuity [1], (the first order modulus of smoothness) of the function f.

In [11], C. Mortici defined the sequence of operators

$$\varphi S_n : C^2([0, +\infty[) \to C^\infty([0, +\infty[),$$

given by

$$(\varphi S_n f)(x) = \frac{1}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k f\left(\frac{k}{n}\right), \tag{1.8}$$

for any $x \in [0, +\infty[$ and $n \in \mathbb{N}$, where $\varphi : \mathbb{R} \to]0, +\infty[$ is an analytic function. These are called the φ-Szász-Mirakjan operators, because in the case when $\varphi(y) = e^y$, they reduce to the classical Mirakjan-Favard-Szász operators [5, 10] and [15].

Remark 2. Similar generalization of this type are the operators defined and studied by Jakimovski and Leviatan [7] or the operators defined by Baskakov in 1957 (see, e.g., the book [2], subsection 5.3.11, p. 344, where they are attributed to Mastroianni). Another recently similar generalization of this type are the operators defined and studied by authors in [3, 4, 8, 9] and [13].

Remark 3. The classical Mirakjan-Favard-Szász operators $S_n : C_2([0, +\infty[) \to C([0, +\infty[)$ are defined by

$$(S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \tag{1.9}$$

where

$$C_2([0, +\infty[) = \left\{ f \in C([0, +\infty[) : \lim_{x \to \infty} \frac{f(x)}{1+x^2} \text{ exists and is finite} \right\}.$$

In the following, we shall use the classical definition of Mirakjan-Favard-Szász operators, i.e., $f \in C_2([0, +\infty[)$.

The purpose of this paper is to introduce a new linear positive operators of Szász-Mirakjan type, in order to use it in the theory of uniform approximation of functions, which generalize some new, respectively older results, cited at the adequate moment. We shall prove uniform convergence, Voronovskaja type formulas and the order of approximation, for these new linear positive operators.

2. AUXILIARY RESULTS

We consider an analytic function $\varphi : [0, +\infty[\to]0, +\infty[$ and it follows

$$\varphi(x) = \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} x^k. \tag{2.1}$$

By differentiation of the relation (2.1), we get

$$\varphi^{(1)}(x) = \sum_{k=1}^{\infty} \frac{\varphi^{(k)}(0)}{(k-1)!} x^{k-1} \tag{2.2}$$

and

$$\varphi^{(2)}(x) = \sum_{k=2}^{\infty} \frac{\varphi^{(k)}(0)}{(k-2)!} x^{k-2}. \tag{2.3}$$

We consider the function $\alpha : [0, +\infty[\to \mathbb{R}$ defined by

$$\alpha(x) = \frac{\varphi^{(1)}(x)}{\varphi(x)} - 1, \tag{2.4}$$

for any $x \in [0, +\infty[$.

Remark 4. From (2.4) it follows that α is also an analytic function.

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence, so that for any $n \in \mathbb{N}$, $a_n > 0$ and we assume that the function φ has the properties

$$\lim_{x \to \infty} \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} = 1 \tag{2.5}$$

and

$$\lim_{x \to \infty} \left(\frac{a_n}{n}\right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} = 1, \tag{2.6}$$

where $x \in [0, +\infty[$.

Lemma 1. *The following identity*

$$\varphi(x) = c \cdot e^{x + \int\limits_0^x \alpha(t)dt} \tag{2.7}$$

holds, for any $x \in [0, +\infty[$, where $c > 0$.

Proof. From (2.4), we get

$$\varphi^{(1)}(x) - (1 + \alpha(x))\varphi(x) = 0,$$

for any $x \in [0, +\infty[$. This relation is equivalent to

$$\left(\varphi(x) \cdot e^{-x - \int\limits_0^x \alpha(t)dt}\right)^{(1)} = 0,$$

for any $x \in [0, +\infty[$. Because $\varphi : [0, +\infty[\to]0, +\infty[$ exists $c > 0$, so that

$$\varphi(x) \cdot e^{-x - \int\limits_0^x \alpha(t)dt} = c,$$

for any $x \in [0, +\infty[$.
The relation (2.7) follows immediately from the above equality. □

3. THE φ-SZÁSZ-MIRAKJAN TYPE OPERATORS

We define the sequence of operators $\varphi S_n : C_2([0, +\infty[) \to C([0, +\infty[)$, given by

$$(\varphi S_n f)(x) = \frac{1}{\varphi(a_n x)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (a_n x)^k f\left(\frac{k}{n}\right), \tag{3.1}$$

for any $x \in [0, +\infty[$ and $n \in \mathbb{N}$. The operators (3.1) are called the φ-Szász-Mirakjan type operators, because in the case when $a_n = n$, for any $n \in \mathbb{N}$, the φ-Szász-Mirakjan operators [11] are obtained. In the case when $a_n = n$, for any $n \in \mathbb{N}$ and $\varphi(x) = e^x$, for any $x \in [0, +\infty[$, the operators (3.1) become the classical Mirakjan-Favard-Szász operators.

Lemma 2. *Let* $e_j(x) = x^j$, $j \in \{0, 1, 2\}$ *be the test functions. The φ-Szász-Mirakjan type operators satisfy*

  i) $(\varphi S_n e_0)(x) = 1$,

  ii) $(\varphi S_n e_1)(x) = \dfrac{a_n}{n} \dfrac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} x,$

  iii) $(\varphi S_n e_2)(x) = \left(\dfrac{a_n}{n}\right)^2 \dfrac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} x^2 + \dfrac{1}{n} \dfrac{a_n}{n} \dfrac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} x,$
    for any $x \in [0, +\infty[$ and $n \in \mathbb{N}$.

Proof. We get

$$(\varphi S_n e_0)(x) = \frac{1}{\varphi(a_n x)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (a_n x)^k = 1,$$

taking (2.1) into account.
Next, we get

$$(\varphi S_n e_1)(x) = \frac{1}{\varphi(a_n x)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (a_n x)^k \frac{k}{n}$$

$$= \frac{a_n}{n} \frac{x}{\varphi(a_n x)} \sum_{k=1}^{\infty} \frac{\varphi^{(k)}(0)}{(k-1)!} (a_n x)^{k-1} = \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} x,$$

taking (2.2) into account. In a similar way, one obtains

$$(\varphi S_n e_2)(x) = \frac{1}{\varphi(a_n x)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (a_n x)^k \left(\frac{k}{n}\right)^2$$

$$= \frac{1}{n^2} \frac{1}{\varphi(a_n x)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (a_n x)^k (k(k-1) + k)$$

$$= \left(\frac{a_n}{n}\right)^2 \frac{x^2}{\varphi(a_n x)} \sum_{k=2}^{\infty} \frac{\varphi^{(k)}(0)}{(k-2)!} (a_n x)^{k-2} + \frac{a_n}{n^2} \frac{x}{\varphi(a_n x)} \sum_{k=1}^{\infty} \frac{\varphi^{(k)}(0)}{(k-1)!} (a_n x)^{k-1}$$

$$= \left(\frac{a_n}{n}\right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} x^2 + \frac{1}{n} \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} x,$$

taking (2.2) and (2.3) into account. $\qquad\square$

Lemma 3. *Let $e_j(x) = x^j$, $j \in \{3, 4\}$ be the test functions. The φ-Szász-Mirakjan type operators satisfy*

$$(\varphi S_n e_3)(x) = \left(\frac{a_n}{n}\right)^3 \frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)} x^3 + \frac{3}{n} \left(\frac{a_n}{n}\right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} x^2 \qquad (3.2)$$

$$+ \frac{1}{n^2} \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} x$$

and

$$(\varphi S_n e_4)(x) = \left(\frac{a_n}{n}\right)^4 \frac{\varphi^{(4)}(a_n x)}{\varphi(a_n x)} x^4 + \frac{6}{n} \left(\frac{a_n}{n}\right)^3 \frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)} x^3 \qquad (3.3)$$

$$+ \frac{7}{n^2} \left(\frac{a_n}{n}\right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} x^2 + \frac{1}{n^3} \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} x,$$

for any $x \in [0, +\infty[$ and $n \in \mathbb{N}$.

Proof. Because the function φ is analytic by differentiation, it follows

$$\varphi^{(3)}(x) = \sum_{k=3}^{\infty} \frac{\varphi^{(k)}(0)}{(k-3)!} x^{k-3}, \quad \varphi^{(4)}(x) = \sum_{k=4}^{\infty} \frac{\varphi^{(k)}(0)}{(k-4)!} x^{k-4}.$$

Using the above relations and (2.1), (2.2), then from (3.1), it follows

$$(\varphi S_n e_3)(x) = \frac{1}{\varphi(a_n x)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (a_n x)^k \left(\frac{k}{n}\right)^3$$

$$= \frac{1}{n^3} \frac{1}{\varphi(a_n x)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (a_n x)^k \cdot u_1(k)$$

$$= \left(\frac{a_n}{n}\right)^3 \frac{x^3}{\varphi(a_n x)} \sum_{k=3}^{\infty} \frac{\varphi^{(k)}(0)}{(k-3)!} (a_n x)^{k-3}$$

$$+ \frac{3}{n} \left(\frac{a_n}{n}\right)^2 \frac{x^2}{\varphi(a_n x)} \sum_{k=2}^{\infty} \frac{\varphi^{(k)}(0)}{(k-2)!} (a_n x)^{k-2} + \frac{1}{n^2} \frac{a_n}{n} \frac{x}{\varphi(a_n x)} \sum_{k=1}^{\infty} \frac{\varphi^{(k)}(0)}{(k-1)!} (a_n x)^{k-1}$$

$$= \left(\frac{a_n}{n}\right)^3 \frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)} x^3 + \frac{3}{n} \left(\frac{a_n}{n}\right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} x^2 + \frac{1}{n^2} \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} x,$$

where $u_1(k) = k(k-1)(k-2) + 3k(k-1) + k$.

$$(\varphi S_n e_4)(x) = \frac{1}{\varphi(a_n x)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (a_n x)^k \left(\frac{k}{n}\right)^4$$

$$= \frac{1}{n^4} \frac{1}{\varphi(a_n x)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (a_n x)^k \cdot u_2(k)$$

$$= \left(\frac{a_n}{n}\right)^4 \frac{x^4}{\varphi(a_n x)} \sum_{k=4}^{\infty} \frac{\varphi^{(k)}(0)}{(k-4)!}(a_n x)^{k-4} + \frac{6}{n}\left(\frac{a_n}{n}\right)^3 \frac{x^3}{\varphi(a_n x)} \sum_{k=3}^{\infty} \frac{\varphi^{(k)}(0)}{(k-3)!}(a_n x)^{k-3}$$

$$+ \frac{7}{n^2}\left(\frac{a_n}{n}\right)^2 \frac{x^2}{\varphi(a_n x)} \sum_{k=2}^{\infty} \frac{\varphi^{(k)}(0)}{(k-2)!}(a_n x)^{k-2} + \frac{1}{n^3}\frac{a_n}{n}\frac{x}{\varphi(a_n x)} \sum_{k=1}^{\infty} \frac{\varphi^{(k)}(0)}{(k-1)!}(a_n x)^{k-1}$$

$$= \left(\frac{a_n}{n}\right)^4 \frac{\varphi^{(4)}(a_n x)}{\varphi(a_n x)}x^4 + \frac{6}{n}\left(\frac{a_n}{n}\right)^3 \frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)}x^3 + \frac{7}{n^2}\left(\frac{a_n}{n}\right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)}x^2$$

$$+ \frac{1}{n^3}\frac{a_n}{n}\frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)}x,$$

where $u_2(k) = k(k-1)(k-2)(k-3) + 6k(k-1)(k-2) + 7k(k-1) + k$. □

Lemma 4. *For any $x \in [0, +\infty[$ and $n \in \mathbb{N}$, the following*

$$(T_{n,0}^* \varphi S_n)(x) = 1, \tag{3.4}$$

$$(T_{n,1}^* \varphi S_n)(x) = n\left(\frac{a_n}{n}\frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} - 1\right)x, \tag{3.5}$$

$$(T_{n,2}^* \varphi S_n)(x) = n^2 \left(\left(\left(\frac{a_n}{n}\right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} - 2\frac{a_n}{n}\frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} + 1\right)x^2 \tag{3.6}$$

$$+ \frac{1}{n}\frac{a_n}{n}\frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)}x\right),$$

$$(T_{n,4}^* \varphi S_n)(x) = n^4 \left(\left(\left(\frac{a_n}{n}\right)^4 \frac{\varphi^{(4)}(a_n x)}{\varphi(a_n x)} - 4\left(\frac{a_n}{n}\right)^3 \frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)}\right. \tag{3.7}$$

$$+ 6\left(\frac{a_n}{n}\right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} - 4\frac{a_n}{n}\frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} + 1\right)x^4$$

$$+ \frac{6}{n}\left(\left(\frac{a_n}{n}\right)^3 \frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)} - 2\left(\frac{a_n}{n}\right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} + \frac{a_n}{n}\frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)}\right)x^3$$

$$+ \frac{1}{n^2}\left(7\left(\frac{a_n}{n}\right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} - 4\frac{a_n}{n}\frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)}\right)x^2 + \frac{1}{n^3}\frac{a_n}{n}\frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)}x\right)$$

hold.

Proof. Taking into account relations (1.2), (3.1), Lemma 2 and Lemma 3, we get

$$(T_{n,0}^* \varphi S_n)(x) = (\varphi S_n e_0)(x) = 1,$$

$$(T_{n,1}^* \varphi S_n)(x) = n(\varphi S_n \psi_x)(x) = n\left((\varphi S_n e_1)(x) - x(\varphi S_n e_0)(x)\right)$$

$$= n\left(\frac{a_n}{n}\frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} - 1\right)x,$$

$$(T_{n,2}^* \varphi S_n)(x) = n^2(\varphi S_n \psi_x^2)(x) = n^2\left((\varphi S_n e_2)(x) - 2x(\varphi S_n e_1)(x) + x^2(\varphi S_n e_0)(x)\right)$$

$$= n^2\left(\left(\left(\frac{a_n}{n}\right)^2\frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} - 2\frac{a_n}{n}\frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} + 1\right)x^2 + \frac{1}{n}\frac{a_n}{n}\frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)}x\right),$$

$$(T_{n,4}^* \varphi S_n)(x) = n^4(\varphi S_n \psi_x^4)(x) = n^4\left((\varphi S_n e_4)(x) - 4x(\varphi S_n e_3)(x)\right.$$

$$\left. +6x^2(\varphi S_n e_2)(x) - 4x^3(\varphi S_n e_1)(x) + x^4(\varphi S_n e_0)(x)\right)$$

$$= n^4\left(\left(\left(\frac{a_n}{n}\right)^4\frac{\varphi^{(4)}(a_n x)}{\varphi(a_n x)} - 4\left(\frac{a_n}{n}\right)^3\frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)}\right.\right.$$

$$\left.+6\left(\frac{a_n}{n}\right)^2\frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} - 4\frac{a_n}{n}\frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} + 1\right)x^4$$

$$+\frac{6}{n}\left(\left(\frac{a_n}{n}\right)^3\frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)} - 2\left(\frac{a_n}{n}\right)^2\frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} + \frac{a_n}{n}\frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)}\right)x^3$$

$$+\frac{1}{n^2}\left(7\left(\frac{a_n}{n}\right)^2\frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} - 4\frac{a_n}{n}\frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)}\right)x^2 + \frac{1}{n^3}\frac{a_n}{n}\frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)}x\right).$$

$\square$

Remark 5. Taking (2.5) and (2.6) into account, for $x \in [0, +\infty[$ we have

$$\lim_{n\to\infty}\left(\left(\frac{a_n}{n}\right)^2\frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} - 2\frac{a_n}{n}\frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} + 1\right) = 0.$$

In the following, we assume that there exist γ, δ, so that $0 < \gamma \le 1$, $\delta \le 2$, $\gamma < \delta$, and the function φ verifies the conditions

$$\lim_{n\to\infty} n^\gamma\left(\left(\frac{a_n}{n}\right)^2\frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} - 2\frac{a_n}{n}\frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} + 1\right) = \beta_2(x), \qquad (3.8)$$

$$\lim_{n\to\infty} n^{\delta-1}\left(\left(\frac{a_n}{n}\right)^3\frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)} - 2\left(\frac{a_n}{n}\right)^2\frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} + \frac{a_n}{n}\frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)}\right) = \beta_3(x)$$

$$(3.9)$$

and

$$\lim_{n\to\infty} n^\delta\left(\left(\frac{a_n}{n}\right)^4\frac{\varphi^{(4)}(a_n x)}{\varphi(a_n x)} - 4\left(\frac{a_n}{n}\right)^3\frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)} + 6\left(\frac{a_n}{n}\right)^2\frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)}\right. \qquad (3.10)$$

$$-4\frac{a_n}{n}\frac{\varphi^{(1)}(a_nx)}{\varphi(a_nx)}+1\Bigg)=\beta_4(x),$$

for any $x\in[0,+\infty[$, where β_2,β_3,β_4 are functions, $\beta_2,\beta_3,\beta_4:[0,+\infty[\to\mathbb{R}$.

Lemma 5. *For any $x\in[0,+\infty[$, the following identities*

$$\lim_{n\to\infty}(T^*_{n,0}\varphi S_n)(x)=1, \tag{3.11}$$

$$\lim_{n\to\infty}\frac{(T^*_{n,2}\varphi S_n)(x)}{n^{2-\gamma}}=\beta_2(x)\cdot x^2+\epsilon(\gamma)\cdot x, \tag{3.12}$$

$$\lim_{n\to\infty}\frac{(T^*_{n,4}\varphi S_n)(x)}{n^{4-\delta}}=\beta_4(x)\cdot x^4+6\beta_3(x)\cdot x^3+3\eta(\delta)\cdot x^2 \tag{3.13}$$

hold and there exists $n_0\in\mathbb{N}$, so that

$$(T^*_{n,0}\varphi S_n)(x)=1=k_0, \tag{3.14}$$

$$\frac{(T^*_{n,2}\varphi S_n)(x)}{n^{2-\gamma}}\le m_2(K)\cdot b^2+b+1=k_2, \tag{3.15}$$

$$\frac{(T^*_{n,4}\varphi S_n)(x)}{n^{4-\delta}}\le m_4(K)\cdot b^4+6m_3(K)\cdot b^3+3b^2+1=k_4, \tag{3.16}$$

for any $x\in K=[0,b]$, $b>0$, any $n\in\mathbb{N}$, $n\ge n_0$ and $m_2(K):=\sup\limits_{x\in K}|\beta_2(x)|$,

$m_3(K):=\sup\limits_{x\in K}|\beta_3(x)|$, $m_4(K):=\sup\limits_{x\in K}|\beta_4(x)|$, where $\epsilon(\gamma)=\begin{cases}1,\ \gamma=1\\0,\ 0<\gamma<1\end{cases}$

and $\eta(\delta)=\begin{cases}1,\ \delta=2\\0,\ \delta<2.\end{cases}$

Proof. The identities (3.11)-(3.13) follow from Lemma 4, while (3.14)-(3.16) follow from (3.11)-(3.13) by taking the definition of the limit into account. $\square$

Let us assume that $I=J=[0,+\infty[$, $E(I)=C_2([0,+\infty[)$, $F(J)=C([0,+\infty[)$, the functions $\varphi_{n,k}:[0,+\infty[\to\mathbb{R}$ let be defined by $\varphi_{n,k}(x)=\dfrac{1}{\varphi(a_nx)}\dfrac{\varphi^{(k)}(0)}{k!}(a_nx)^k$, for any $x\in[0,+\infty[$, any $n,k\in\mathbb{N}_0$, $n\ne0$ and the functionals $A_{n,k}:C_2([0,+\infty[)\to\mathbb{R}$ let be defined by $A_{n,k}(f)=f\left(\dfrac{k}{n}\right)$, for any $n,k\in\mathbb{N}_0$, $n\ne0$. In this way we get the φ-Szász-Mirakjan type operators.

Theorem 2. *Let the function $f \in E(I)$ be given. If $x \in [0, +\infty[$, f is s times differentiable in a neighborhood of x, then:*

$$\lim_{n \to \infty} (\varphi S_n f)(x) = f(x), \tag{3.17}$$

if $s = 0$ and

$$\lim_{n \to \infty} n^{\gamma} \left((\varphi S_n f)(x) - f(x) - \left(\frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} - 1 \right) x f^{(1)}(x) \right) \tag{3.18}$$

$$= \frac{1}{2} \left(\beta_2(x) \cdot x^2 + \epsilon(\gamma) \cdot x \right) f^{(2)}(x),$$

if $s = 2$.
If f is s times differentiable on $[0, +\infty[$, then the convergence from (3.17) and (3.18) is uniform on any compact interval $K = [0, b] \subset [0, +\infty[$.
Moreover, we get

$$|(\varphi S_n f)(x) - f(x)| \le (1 + k_2) \omega_1 \left(f; \frac{1}{\sqrt{n^{\gamma}}} \right), \tag{3.19}$$

for any $f \in C_2([0, +\infty[)$, any $x \in K$ and $n \in \mathbb{N}$, $n \ge n_0$.

Proof. It follows from Theorem 1, with $\alpha_0 = 0$, $\alpha_2 = 2 - \gamma$ and $\alpha_4 = 4 - \delta$, Lemma 4 and Lemma 5. $\square$

Remark 6. If $a_n = n$, for any $n \in \mathbb{N}$ and $\varphi(x) = e^x$, for any $x \in [0, +\infty[$ we get the well-known results for the classical Szász-Mirakjan operators. In this case $\gamma = 1$, $\delta = 2$, $\beta_2(x) = \beta_3(x) = \beta_4(x) = 0$, for any $x \in [0, +\infty[$ and $k_2 = b$, $k_4 = 3b^2 + b$, (see [12]).

4. OTHER FORM FOR THE SZÁSZ-MIRAKJAN OPERATORS

In this section, we assume that $c = 1$ in the relation (2.7). In [6] or [14], we can find some known results about differentiation of composed functions.

Theorem 3 ([14]). *Let I, J be two intervals of real numbers and let $f : I \to \mathbb{R}$, $g : J \to \mathbb{R}$ be two real-valued functions, such that $f(I) \subset J$. If f is k times differentiable on I and g is k times differentiable on J, $k \in \mathbb{N}$, then the function $g \circ f : I \to \mathbb{R}$ is also k times differentiable on I and the following identity*

$$(g \circ f)^{(k)}(x) = \sum_{i=1}^{k} \left(g^{(i)} \circ f \right)(x) \sum_{\substack{a_1 + a_2 + \dots + a_k = i \\ a_1 + 2a_2 + \dots + ka_k = k}} \frac{k!}{a_1! a_2! \dots a_k!} \tag{4.1}$$

$$\cdot \left(\frac{f^{(1)}(x)}{1!} \right)^{a_1} \left(\frac{f^{(2)}(x)}{2!} \right)^{a_2} \cdot \dots \cdot \left(\frac{f^{(k)}(x)}{k!} \right)^{a_k}$$

holds, for any $x \in I$.

Theorem 4. *If $k \in \mathbb{N}$, then the following identity*

$$\varphi^{(k)}(0) = \sum_{i=1}^{k} \sum_{\substack{a_1+a_2+\ldots+a_k=i \\ a_1+2a_2+\ldots+ka_k=k}} \frac{k!}{a_1!a_2!\ldots a_k!} \left(\frac{1+\alpha(0)}{1!}\right)^{a_1} \qquad (4.2)$$

$$\cdot \left(\frac{\alpha^{(1)}(0)}{2!}\right)^{a_2} \cdot \ldots \cdot \left(\frac{\alpha^{(k-1)}(0)}{k!}\right)^{a_k}$$

holds.

Proof. One applies Theorem 3, with $g(x) = e^x$, for $x \in \mathbb{R}$ and $f(x) = x + \int\limits_0^x \alpha(t)dt$, for $x \in [0,+\infty[$. $\qquad\square$

Remark 7. The relation (3.1) represents the form of φ-Szász-Mirakjan type operators, by using the function φ. By replacing the form of $\varphi^{(k)}(0)$, from the relation (4.2) in (3.1), we get the form of the φ-Szász-Mirakjan type operators, where appears the function α.

Now we can give some applications.

Example 1. If $a_n = n$, $n \in \mathbb{N}$ and $\alpha(x) = 0$, for any $x \in [0,+\infty[$, from (4.2) follows that $\varphi^{(k)}(0) = 1$, for any $k \in \mathbb{N}$. In this case, we get the classical Szász-Mirakjan operators.

Example 2. Suppose that $\alpha(x) = \dfrac{1}{(x+1)^2}$, for any $x \in [0,+\infty[$. Then $\alpha^{(p)}(x) = \dfrac{(-1)^p(p+1)!}{(x+1)^{p+2}}$, $\varphi(x) = e^{x+1-\frac{1}{x+1}}$, for any $x \in [0,+\infty[$, any $p \in \mathbb{N}_0$ and we get

$$\varphi^{(k)}(0) = \sum_{i=1}^{k} \sum_{\substack{a_1+a_2+\ldots+a_k=i \\ a_1+2a_2+\ldots+ka_k=k}} \frac{k!}{a_1!a_2!\ldots a_k!} 2^{a_1}(-1)^{1a_2+2a_3+\ldots+(k-1)a_k}.$$

Taking into account $a_1 + 2a_2 + \ldots + ka_k = k$ and $a_1 + a_2 + \ldots + a_k = i$, we get

$$\varphi^{(k)}(0) = \sum_{i=1}^{k} \sum_{\substack{a_1+a_2+\ldots+a_k=i \\ a_1+2a_2+\ldots+ka_k=k}} \frac{k!}{a_1!a_2!\ldots a_k!}(-1)^{k-i}2^{a_1},$$

from where, it follows

$$\varphi^{(k)}(0) = (-1)^k \sum_{i=1}^{k} \sum_{\substack{a_1+a_2+\ldots+a_k=i \\ a_1+2a_2+\ldots+ka_k=k}} (-1)^i \frac{k!}{a_1!a_2!\ldots a_k!}2^{a_1}. \qquad (4.3)$$

If we consider $a_n = n + \frac{1}{n}$, for any $n \in \mathbb{N}$, the operators defined by (3.1) become

$$(\varphi S_n f)(x) = e^{\frac{1}{(n+\frac{1}{n})x+1} - (n+\frac{1}{n})x - 1} \sum_{k=0}^{\infty} \frac{(-1)^k \sum_{i=1}^{k} \sum_{\substack{a_1+a_2+...+a_k=i \\ a_1+2a_2+...+ka_k=k}} (-1)^i \frac{k!}{a_1!a_2!...a_k!} 2^{a_1}}{k!}$$

$$\cdot \left(\left(n + \frac{1}{n} \right) x \right)^k f \left(\frac{k}{n} \right),$$

for any $f \in C_2([0, +\infty[)$, $x \in [0, +\infty[$ and $n \in \mathbb{N}$.

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