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# Sharp function estimates and boundedness for Toeplitz type operator related to general singular integral operator

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# SHARP FUNCTION ESTIMATES AND BOUNDEDNESS FOR TOEPLITZ TYPE OPERATOR RELATED TO GENERAL SINGULAR INTEGRAL OPERATOR

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*Abstract.* In this paper, we establish the sharp maximal function estimates for the Toeplitz type operator related to the singular integral operator with general kernel. As an application, we obtain the boundedness of the operator on Lebesgue, Morrey and Triebel-Lizorkin spaces.

2000 *Mathematics Subject Classification:* 42B20; 42B25

*Keywords:* Toeplitz type operator, singular integral operator, sharp maximal function, Morrey space, Triebel-Lizorkin space, $BMO$, Lipschitz function

## 1. INTRODUCTION

As the development of singular integral operators (see [7, 19, 20]), their commutators have been well studied. In [4, 17, 18] the authors prove that the commutators generated by the singular integral operators and $BMO$ functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [2]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [3, 8, 14] the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [1], some singular integral operators with general kernel are introduced, and the boundedness for the operators and their commutators generated by $BMO$ and Lipschitz functions are obtained (see [1, 10]). In [9, 11], some Toeplitz type operators related to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by $BMO$ and Lipschitz functions are obtained. In this paper, we will study the Toeplitz type operators generated by the singular integral operators with general kernel and the Lipschitz and $BMO$ functions.

## 2. PRELIMINARIES

First, let us introduce some notations. Throughout this paper, $Q$ will denote a cube of $R^n$ with sides parallel to the axes. For any locally integrable function $f$, the sharp maximal function of $f$ is defined by

$$M^{\#}(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-f_Q|dy,$$

where, and in what follows, $f_Q=|Q|^{-1}\int_Q f(x)dx$. It is well-known that (see [7, 19])

$$M^{\#}(f)(x)\approx\sup_{Q\ni x}\inf_{c\in C}\frac{1}{|Q|}\int_Q|f(y)-c|dy.$$

We say that $f$ belongs to $BMO(R^n)$ if $M^{\#}(f)$ belongs to $L^{\infty}(R^n)$ and define $||f||_{BMO}=||M^{\#}(f)||_{L^{\infty}}$. It has been known that (see [19])

$$||f-f_{2^kQ}||_{BMO}\le Ck||f||_{BMO}.$$

Let

$$M(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)|dy.$$

For $\eta>0$, let $M_\eta(f)(x)=M(|f|^\eta)^{1/\eta}(x)$.

For $0<\eta<n$ and $1\le r<\infty$, set

$$M_{\eta,r}(f)(x)=\sup_{Q\ni x}\left(\frac{1}{|Q|^{1-r\eta/n}}\int_Q|f(y)|^r dy\right)^{1/r}.$$

The $A_p$ weight is defined by (see [7]), for $1<p<\infty$,

$$A_p=\left\{w\in L^1_{loc}(R^n):\sup_Q\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1}<\infty\right\}$$

and

$$A_1=\{w\in L^p_{loc}(R^n):M(w)(x)\le Cw(x),a.e.\}.$$

For $\beta>0$ and $p>1$, let $\dot{F}_p^{\beta,\infty}(R^n)$ be the homogeneous Triebel-Lizorkin space (see [14]).

For $\beta>0$, the Lipschitz space $Lip_\beta(R^n)$ is the space of functions $f$ such that

$$||f||_{Lip_\beta}=\sup_{\substack{x,y\in R^n\\ x\neq y}}\frac{|f(x)-f(y)|}{|x-y|^\beta}<\infty.$$

**Definition 1.** Let $\varphi$ be a positive, increasing function on $R^+$ and there exists a constant $D>0$ such that

$$\varphi(2t)\le D\varphi(t)\ \text{ for }\ t\ge 0.$$

Let $f$ be a locally integrable function on $R^n$. Set, for $1\le p<\infty$,

$$||f||_{L^{p,\varphi}}=\sup_{x\in R^n,\ d>0}\left(\frac{1}{\varphi(d)}\int_{Q(x,d)}|f(y)|^p dy\right)^{1/p},$$

where $Q(x,d) = \{y \in R^n : |x-y| < d\}$. The generalized Morrey space is defined by

$$L^{p,\varphi}(R^n) = \{f \in L^1_{loc}(R^n) : ||f||_{L^{p,\varphi}} < \infty\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n) = L^{p,\delta}(R^n)$, which is the classical Morrey space (see [15, 16]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n) = L^p(R^n)$, which is the Lebesgue space (see [13]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [5, 6, 12, 13]).

In this paper, we will study some singular integral operators as following (see [1]).

**Definition 2.** Let $T : S \to S'$ be a linear operator such that $T$ is bounded on $L^2(R^n)$ and there exists a locally integrable function $K(x,y)$ on $R^n \times R^n \setminus \{(x,y) \in R^n \times R^n : x = y\}$ such that

$$T(f)(x) = \int_{R^n} K(x,y) f(y) dy$$

for every bounded and compactly supported function $f$, where $K$ satisfies: there is a sequence of positive constant numbers $\{C_j\}$ such that for any $j \geq 1$,

$$\int_{2|y-z|<|x-y|} (|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)|) dx \leq C,$$

and

$$\left(\int_{2^j|z-y|\leq|x-y|<2^{j+1}|z-y|} (|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)|)^q dy\right)^{1/q} \leq C_j (2^j|z-y|)^{-n/q'},$$

where $1 < q' < 2$ and $1/q + 1/q' = 1$.

Moreover, let $b$ be a locally integrable function on $R^n$. The Toeplitz type operator related to $T$ is defined by

$$T_b = \sum_{k=1}^{m} T^{k,1} M_b T^{k,2},$$

where $T^{k,1}$ are $T$ or $\pm I$ (the identity operator), $T^{k,2}$ are the bounded linear operators on $L^p(R^n)$ for $1 < p < \infty$, $k = 1, ..., m$, $M_b(f) = bf$.

Note that the classical Calderón-Zygmund singular integral operator satisfies Definition 2 with $C_j = 2^{-j\delta}$ (see [7, 19]). And note that the commutator $[b,T](f) = bT(f) - T(bf)$ is a particular operator of the Toeplitz type operator $T_b$. The Toeplitz type operator $T_b$ are the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [17, 18]). The main purpose of this paper is to prove the sharp maximal inequalities for the Toeplitz type operator $T_b$. As the application, we obtain the the $L^p$-norm inequality, Morrey and Triebel-Lizorkin spaces boundedness for the Toeplitz type operators $T_b$.

## 3. THEOREMS

We shall prove the following theorems.

**Theorem 1.** *Let $T$ be the singular integral operator as in Definition 2, the sequence $\{C_j\} \in l^1$, $0 < \beta < 1$, $q' \le s < \infty$ and $b \in Lip_\beta(R^n)$. If $g \in L^u(R^n)(1 < u < \infty)$ and $T_1(g) = 0$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$M^\#(T_b(f))(\tilde{x}) \le C||b||_{Lip_\beta} \sum_{k=1}^m M_{\beta,s}(T^{k,2}(f))(\tilde{x}).$$

**Theorem 2.** *Let $T$ be the singular integral operator as in Definition 2, the sequence $\{2^{j\beta}C_j\} \in l^1$, $0 < \beta < 1$, $q' \le s < \infty$ and $b \in Lip_\beta(R^n)$. If $g \in L^u(R^n)(1 < u < \infty)$ and $T_1(g) = 0$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$\sup_{Q \ni \tilde{x}} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T_b(f)(x) - C_0| \, dx \le C||b||_{Lip_\beta} \sum_{k=1}^m M_s(T^{k,2}(f))(\tilde{x}).$$

**Theorem 3.** *Let $T$ be the singular integral operator as in Definition 2, the sequence $\{jC_j\} \in l^1$, $q' \le s < \infty$ and $b \in BMO(R^n)$. If $g \in L^u(R^n)(1 < u < \infty)$ and $T_1(g) = 0$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$M^\#(T_b(f))(\tilde{x}) \le C||b||_{BMO} \sum_{k=1}^m M_s(T^{k,2}(f))(\tilde{x}).$$

**Theorem 4.** *Let $T$ be the singular integral operator as in Definition 2, the sequence $\{C_j\} \in l^1$, $0 < \beta < \min(1, n/q')$, $q' < p < n/\beta$, $1/r = 1/p - \beta/n$ and $b \in Lip_\beta(R^n)$. If $g \in L^u(R^n)(1 < u < \infty)$ and $T_1(g) = 0$, then $T_b$ is bounded from $L^p(R^n)$ to $L^r(R^n)$.*

**Theorem 5.** *Let $T$ be the singular integral operator as in Definition 2, the sequence $\{C_j\} \in l^1$, $0 < \beta < \min(1, n/q')$, $q' < p < n/\beta$, $1/r = 1/p - \beta/n$, $0 < D < 2^n$ and $b \in Lip_\beta(R^n)$. If $g \in L^u(R^n)(1 < u < \infty)$ and $T_1(g) = 0$, then $T_b$ is bounded from $L^{p,\varphi}(R^n)$ to $L^{r,\varphi}(R^n)$.*

**Theorem 6.** *Let $T$ be the singular integral operator as in Definition 2, the sequence $\{2^{j\beta}C_j\} \in l^1$, $0 < \beta < \min(1, n/q')$, $q' < p < n/\beta$ and $b \in Lip_\beta(R^n)$. If $g \in L^u(R^n)(1 < u < \infty)$ and $T_1(g) = 0$, then $T_b$ is bounded from $L^p(R^n)$ to $\dot{F}_p^{\beta,\infty}(R^n)$.*

**Theorem 7.** *Let $T$ be the singular integral operator as in Definition 2, the sequence $\{jC_j\} \in l^1$, $q' < p < n/\beta$ and $b \in BMO(R^n)$. If $g \in L^u(R^n)(1 < u < \infty)$ and $T_1(g) = 0$, then $T_b$ is bounded on $L^p(R^n)$.*

**Theorem 8.** *Let $T$ be the singular integral operator as in Definition 2, the sequence $\{jC_j\} \in l^1$, $q' < p < n/\beta$, $0 < D < 2^n$ and $b \in BMO(R^n)$. If $g \in L^u(R^n)(1 < u < \infty)$ and $T_1(g) = 0$, then $T_b$ is bounded on $L^{p,\varphi}(R^n)$.*

## 4. Proofs of the Theorems

To prove the theorems, we need the following lemmas.

**Lemma 1.** *(see [1]) Let $T$ be the singular integral operator as in Definition 2. Then $T$ is bounded on $L^p(R^n)$ for $1 < p < \infty$.*

**Lemma 2.** *(see [14]) For $0 < \beta < 1$ and $1 < p < \infty$, we have*

$$||f||_{\dot{F}_p^{\beta,\infty}} \approx \left|\left|\sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| dx\right|\right|_{L^p}$$
$$\approx \left|\left|\sup_{Q \ni \cdot} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c| dx\right|\right|_{L^p}.$$

**Lemma 3.** *(see [7]) Let $0 < p < \infty$ and $w \in \cup_{1 \le r < \infty} A_r$. Then, for any smooth function $f$ for which the left-hand side is finite,*

$$\int_{R^n} M(f)(x)^p w(x) dx \le C \int_{R^n} M^{\#}(f)(x)^p w(x) dx.$$

**Lemma 4.** *(see [2]). Suppose that $0 < \eta < n$, $1 \le s < p < n/\eta$ and $1/r = 1/p - \eta/n$. Then*

$$||M_{\eta,s}(f)||_{L^r} \le C||f||_{L^p}.$$

**Lemma 5.** *Let $1 < p < \infty$, $0 < D < 2^n$. Then, for any smooth function $f$ for which the left-hand side is finite,*

$$||M(f)||_{L^{p,\varphi}} \le C||M^{\#}(f)||_{L^{p,\varphi}}.$$

*Proof.* For any cube $Q = Q(x_0, d)$ in $R^n$, we know $M(\chi_Q) \in A_1$ for any cube $Q = Q(x, d)$ by [7]. Noticing that $M(\chi_Q) \le 1$ and $M(\chi_Q)(x) \le \frac{d^n}{(|x-x_0|-d)^n}$ if $x \in Q^c$, by Lemma 3, we have, for $f \in L^{p,\varphi}(R^n)$,

$$\int_Q M(f)(x)^p dx = \int_{R^n} M(f)(x)^p \chi_Q(x) dx$$
$$\le \int_{R^n} M(f)(x)^p M(\chi_Q)(x) dx \le C \int_{R^n} M^{\#}(f)(x)^p M(\chi_Q)(x) dx$$

$$= C\left(\int_Q M^\#(f)(x)^p M(\chi_Q)(x)dx + \sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ} M^\#(f)(x)^p M(\chi_Q)(x)dx\right)$$
$$\le C\left(\int_Q M^\#(f)(x)^p dx + \sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ} M^\#(f)(x)^p \frac{|Q|}{|2^{k+1}Q|}dx\right)$$
$$\le C\left(\int_Q M^\#(f)(x)^p dx + \sum_{k=0}^{\infty}\int_{2^{k+1}Q} M^\#(f)(x)^p 2^{-kn}dy\right)$$
$$\le C||M^\#(f)||^p_{L^{p,\varphi}}\sum_{k=0}^{\infty}2^{-kn}\varphi(2^{k+1}d)$$
$$\le C||M^\#(f)||^p_{L^{p,\varphi}}\sum_{k=0}^{\infty}(2^{-n}D)^k\varphi(d)$$
$$\le C||M^\#(f)||^p_{L^{p,\varphi}}\varphi(d),$$

thus

$$\left(\frac{1}{\varphi(d)}\int_Q M(f)(x)^p dx\right)^{1/p} \le C\left(\frac{1}{\varphi(d)}\int_Q M^\#(f)(x)^p dx\right)^{1/p}$$

and

$$||M(f)||_{L^{p,\varphi}} \le C||M^\#(f)||_{L^{p,\varphi}}.$$

This finishes the proof. □

**Lemma 6.** *Let* $0 < D < 2^n$, $1 \le s < p < n/\eta$ *and* $1/r = 1/p - \eta/n$. *Then*

$$||M_{\eta,s}(f)||_{L^{r,\varphi}} \le C||f||_{L^{p,\varphi}}.$$

**Lemma 7.** *Let* $T$ *be the bounded linear operators on* $L^q(R^n)$ *for any* $1 < q < \infty$. *Then, for* $1 < p < \infty$, $0 < D < 2^n$,

$$||T(f)||_{L^{p,\varphi}} \le C||f||_{L^{p,\varphi}}.$$

The proofs of the last two Lemmas are similar to that of Lemma 5 by Lemma 4, we omit the details.

*Proof of Theorem 1.* It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant $C_0$, the following inequality holds:

$$\frac{1}{|Q|}\int_Q |T_b(f)(x) - C_0|dx \le C||b||_{Lip_\beta}\sum_{k=1}^{m} M_{\beta,s}(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k = 1,...,m)$. Fix a cube $Q = Q(x_0,d)$ and $\tilde{x} \in Q$. Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$T_b(f)(x) = T_{b-b_Q}(f)(x) = T_{(b-b_Q)\chi_{2Q}}(f)(x) + T_{(b-b_Q)\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x).$$

Then

$$\frac{1}{|Q|}\int_Q |T_b(f)(x)-f_2(x_0)|dx \le \frac{1}{|Q|}\int_Q |f_1(x)|dx + \frac{1}{|Q|}\int_Q |f_2(x)-f_2(x_0)|dx = I_1+I_2.$$

For $I_1$, by Hölder's inequality, we obtain

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|dx \\
&\le \left(\frac{1}{|Q|}\int_{R^n} |T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|^s dx\right)^{1/s} \\
&\le C|Q|^{-1/s}\left(\int_{R^n} |M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|^s dx\right)^{1/s} \\
&\le C|Q|^{-1/s}\left(\int_{2Q} (|b(x)-b_Q||T^{k,2}(f)(x)|)^s dx\right)^{1/s} \\
&\le C|Q|^{-1/s}||b||_{Lip_\beta}|2Q|^{\beta/n}|Q|^{1/s-\beta/n}\left(\frac{1}{|Q|^{1-s\beta/n}}\int_Q |T^{k,2}(f)(x)|^s dx\right)^{1/s} \\
&\le C||b||_{Lip_\beta}M_{\beta,s}(T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
I_1 &\le \sum_{k=1}^m \frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|dx \\
&\le C||b||_{Lip_\beta}\sum_{k=1}^m M_{\beta,s}(T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

For $I_2$, by the boundedness of $T$ and recalling that $s > q'$, we get, for $x \in Q$,

$$\begin{aligned}
&|T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x) - T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x_0)| \\
&\le \int_{(2Q)^c} |b(y)-b_{2Q}||K(x,y)-K(x_0,y)||T^{k,2}(f)(y)|dy \\
&= \sum_{j=1}^\infty \int_{2^j d\le|y-x_0|<2^{j+1}d} |b(y)-b_{2Q}||K(x,y)-K(x_0,y)||T^{k,2}(f)(y)|dy \\
&\le C||b||_{Lip_\beta}\sum_{j=1}^\infty |2^{j+1}Q|^{\beta/n}\left(\int_{2^j d\le|y-x_0|<2^{j+1}d} |K(x,y)-K(x_0,y)|^q dy\right)^{1/q}
\end{aligned}$$

$$\times \left( \int_{2^{j+1}Q} |T^{k,2}(f)(y)|^{q'} dy \right)^{1/q'}$$

$$\leq C||b||_{Lip_\beta} \sum_{j=1}^{\infty} |2^{j+1}Q|^{\beta/n} C_j (2^j d)^{-n/q'} |2^{j+1}Q|^{1/q'-\beta/n}$$

$$\times \left( \frac{1}{|2^{j+1}Q|^{1-s\beta/n}} \int_{2^{j+1}Q} |T^{k,2}(f)(y)|^s dy \right)^{1/s}$$

$$\leq C||b||_{Lip_\beta} M_{\beta,s}(T^{k,2}(f))(\tilde{x}) \sum_{j=1}^{\infty} C_j$$

$$\leq C||b||_{Lip_\beta} M_{\beta,s}(T^{k,2}(f))(\tilde{x}),$$

thus

$$I_2 \leq \frac{1}{|Q|} \int_Q \sum_{k=1}^{m} |T^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x) - T^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x_0)| dx$$

$$\leq C||b||_{Lip_\beta} \sum_{k=1}^{m} M_{\beta,s}(T^{k,2}(f))(\tilde{x}).$$

These complete the proof of Theorem 1. □

*Proof of Theorem 2.* It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant $C_0$, the following inequality holds:

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |T_b(f)(x) - C_0| dx \leq C||b||_{Lip_\beta} \sum_{k=1}^{m} M_s(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T^{k,1}$ are $T (k = 1, ..., m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. For $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$, write

$$T_b(f)(x) = T_{b-b_Q}(f)(x) = T_{(b-b_Q)\chi_{2Q}}(f)(x) + T_{(b-b_Q)\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x)$$

and

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |T_b(f)(x) - f_2(x_0)| dx \leq \frac{1}{|Q|^{1+\beta/n}} \int_Q |f_1(x)| dx + \frac{1}{|Q|^{1+\beta/n}} \int_Q |f_2(x) - f_2(x_0)| dx = I_3 + I_4.$$

By using the same argument as in the proof of Theorem 1, we get

$$I_3 \leq \sum_{k=1}^{m} \frac{C}{|Q|^{\beta/n}} ||b||_{Lip_\beta} |2Q|^{\beta/n} |Q|^{-1/s} \left( \int_{2Q} |T^{k,2}(f)(x)|^s dx \right)^{1/s}$$

$$\leq C||b||_{Lip_\beta} \sum_{k=1}^{m} \left( \frac{1}{|2Q|} \int_{2Q} |T^{k,2}(f)(x)|^s dx \right)^{1/s}$$
$$\leq C||b||_{Lip_\beta} \sum_{k=1}^{m} M_s(T^{k,2}(f))(\tilde{x}),$$

$$I_4 \leq \sum_{k=1}^{m} \frac{1}{|Q|^{1+\beta/n}} \int_Q \sum_{j=1}^{\infty} \int_{2^j d \leq |y-x_0| < 2^{j+1} d} |b(y) - b_{2Q}|$$
$$\times |K(x,y) - K(x_0,y)||T^{k,2}(f)(y)| dy dx$$
$$\leq \sum_{k=1}^{m} \frac{C}{|Q|^{1+\beta/n}} \int_Q \sum_{j=1}^{\infty} ||b||_{Lip_\beta} |2^{j+1}Q|^{\beta/n} \left( \int_{2^{j+1}Q} |T^{k,2}(f)(y)|^{q'} dy \right)^{1/q'}$$
$$\times \left( \int_{2^j d \leq |y-x_0| < 2^{j+1} d} |K(x,y) - K(x_0,y)|^q dy \right)^{1/q} dx$$
$$\leq C||b||_{Lip_\beta} \sum_{k=1}^{m} |Q|^{-\beta/n} \sum_{j=1}^{\infty} |2^{j+1}Q|^{\beta/n} C_j (2^j d)^{-n/q'} |2^{j+1}Q|^{1/q'}$$
$$\times \left( \frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |T^{k,2}(f)(y)|^s dy \right)^{1/s}$$
$$\leq C||b||_{Lip_\beta} \sum_{k=1}^{m} M_s(T^{k,2}(f))(\tilde{x}) \sum_{j=1}^{\infty} 2^{j\beta} C_j$$
$$\leq C||b||_{Lip_\beta} \sum_{k=1}^{m} M_s(T^{k,2}(f))(\tilde{x}).$$

This completes the proof of Theorem 2. □

*Proof of Theorem 3.* It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant $C_0$, the following inequality holds:

$$\frac{1}{|Q|} \int_Q |T_b(f)(x) - C_0| dx \leq C||b||_{BMO} \sum_{k=1}^{m} M_s(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k = 1, ..., m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. For $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$, similar to the proof of Theorem 1, we have

$$T_b(f)(x) = T_{b-b_Q}(f)(x) = T_{(b-b_Q)\chi_{2Q}}(f)(x)$$
$$+ T_{(b-b_Q)\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x)$$

and

$$\begin{aligned}&\frac{1}{|Q|}\int_Q |T_b(f)(x)-f_2(x_0)|\,dx\\&\le \frac{1}{|Q|}\int_Q |f_1(x)|dx+\frac{1}{|Q|}\int_Q |f_2(x)-f_2(x_0)|dx = I_5+I_6.\end{aligned}$$

For $I_5$, choose $1<r<s$, by Hölder's inequality and the boundedness of $T$, we obtain

$$\begin{aligned}&\frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|dx\\&\le \left(\frac{1}{|Q|}\int_{R^n}|T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|^r dx\right)^{1/r}\\&\le C|Q|^{-1/r}\left(\int_{R^n}|M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|^r dx\right)^{1/r}\\&\le C|Q|^{-1/r}\left(\int_{2Q}|T^{k,2}(f)(x)|^s dx\right)^{1/s}\left(\int_{2Q}|b(x)-b_Q|^{sr/(s-r)}dx\right)^{(s-r)/sr}\\&\le C||b||_{BMO}\left(\frac{1}{|Q|}\int_{2Q}|T^{k,2}(f)(x)|^s dx\right)^{1/s}\\&\le C||b||_{BMO}M_s(T^{k,2}(f))(\tilde{x}),\end{aligned}$$

thus

$$\begin{aligned}I_5 &\le \sum_{k=1}^m \frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|dx\\&\le C||b||_{BMO}\sum_{k=1}^m M_s(T^{k,2}(f))(\tilde{x}).\end{aligned}$$

For $I_6$, recalling that $s>q'$, taking $1<p<\infty, 1<r<s$ with $1/p+1/q+1/r=1$, by the boundedness of $T$, we get, for $x\in Q$,

$$\begin{aligned}&|T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x)-T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x_0)|\\&\le \sum_{j=1}^\infty \int_{2^j d\le|y-x_0|<2^{j+1}d}|K(x,y)-K(x_0,y)||b(y)-b_{2Q}||T^{k,2}(f)(y)|dy\\&\le \sum_{j=1}^\infty\left(\int_{2^j d\le|y-x_0|<2^{j+1}d}|K(x,y)-K(x_0,y)|^q dy\right)^{1/q}\\&\quad\times\left(\int_{2^{j+1}Q}|b(y)-b_Q|^p dy\right)^{1/p}\left(\int_{2^{j+1}Q}|T^{k,2}(f)(y)|^r dy\right)^{1/r}\end{aligned}$$

$$\leq C||b||_{BMO}\sum_{j=1}^{\infty}C_j(2^jd)^{-n/q'}j(2^jd)^{n/p}(2^jd)^{n/s} \cdot\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|T^{k,2}(f)(y)|^s dy\right)^{1/s}$$

$$\leq C||b||_{BMO}M_s(T^{k,2}(f))(\tilde{x})\sum_{j=1}^{\infty}jC_j$$

$$\leq C||b||_{BMO}M_s(T^{k,2}(f))(\tilde{x}),$$

thus

$$I_6 \leq \frac{1}{|Q|}\int_Q\sum_{k=1}^{m}|T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x)-T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x_0)|dx$$

$$\leq C||b||_{BMO}\sum_{k=1}^{m}M_s(T^{k,2}(f))(\tilde{x}).$$

This completes the proof of Theorem 3. □

*Proof of Theorem 4.* Choose $q' < s < p$ in Theorem 1, we have, by Lemma 1, 3 and 4,

$$\begin{aligned}||T_b(f)||_{L^r} &\leq \|M(T_b(f))\|_{L^r} \leq C\|M^{\#}(T_b(f))\|_{L^r}\\ &\leq C||b||_{Lip_\beta}\sum_{k=1}^{m}\|M_{\beta,s}(T^{k,2}(f))\|_{L^r} \leq C||b||_{Lip_\beta}\sum_{k=1}^{m}\|T^{k,2}(f)\|_{L^p}\\ &\leq C||b||_{Lip_\beta}\|f\|_{L^p}.\end{aligned}$$

This completes the proof. □

*Proof of Theorem 5.* Choose $q' < s < p$ in Theorem 1, we have, by Lemma 5, 6 and 7,

$$\begin{aligned}||T_b(f)||_{L^{r,\varphi}} &\leq \|M(T_b(f))\|_{L^{r,\varphi}} \leq C\|M^{\#}(T_b(f))\|_{L^{r,\varphi}}\\ &\leq C||b||_{Lip_\beta}\sum_{k=1}^{m}\|M_{\beta,s}(T^{k,2}(f))\|_{L^{r,\varphi}} \leq C||b||_{Lip_\beta}\sum_{k=1}^{m}\|T^{k,2}(f)\|_{L^{p,\varphi}}\\ &\leq C||b||_{Lip_\beta}\|f\|_{L^{p,\varphi}}.\end{aligned}$$

This completes the proof. □

*Proof of Theorem 6.* Choose $q' < s < p$ in Theorem 2, we have, by Lemma 1, 2 and 3,

$$||T_b(f)||_{\dot{F}_p^{\beta,\infty}} \leq C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T_b(f)(x) - C_0| dx \right\|_{L^p}$$
$$\leq C||b||_{Lip_\beta} \sum_{k=1}^m \|M_s(T^{k,2}(f))\|_{L^p} \leq C||b||_{Lip_\beta} \sum_{k=1}^m \|T^{k,2}(f)\|_{L^p}$$
$$\leq C||b||_{Lip_\beta}||f||_{L^p}.$$

This completes the proof. □

*Proof of Theorem 7.* Choose $q' < s < p$ in Theorem 3, we have, by Lemma 1, 3 and 4,

$$||T_b(f)||_{L^p} \leq \|M(T_b(f))\|_{L^p} \leq C\|M^\#(T_b(f))\|_{L^p}$$
$$\leq C||b||_{BMO} \sum_{k=1}^m \|M_s(T^{k,2}(f))\|_{L^p} \leq C||b||_{BMO} \sum_{k=1}^m \|T^{k,2}(f)\|_{L^p}$$
$$\leq C||b||_{BMO}\|f\|_{L^p}.$$

This completes the proof. □

*Proof of Theorem 8.* Choose $q' < s < p$ in Theorem 3, we have, by Lemma 5, 6 and 7,

$$||T_b(f)||_{L^{p,\varphi}} \leq \|M(T_b(f))\|_{L^{p,\varphi}} \leq C\|M^\#(T_b(f))\|_{L^{p,\varphi}}$$
$$\leq C||b||_{BMO} \sum_{k=1}^m \|M_s(T^{k,2}(f))\|_{L^{p,\varphi}} \leq C||b||_{BMO} \sum_{k=1}^m \|T^{k,2}(f)\|_{L^{p,\varphi}}$$
$$\leq C||b||_{BMO}\|f\|_{L^{p,\varphi}}.$$

This completes the proof. □

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