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MAPPING BIJECTIVELY σ-ALGEBRAS ONTO POWER SETS

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Dedicated to the memory of my Father

Abstract. As an application of the so-called "optimal measure" we attempt to seek sets whose power sets are equinumerous with σ-algebras, which seems to be new information about σ-algebras.

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1. Introduction

Some new information about σ-algebras is investigated, consisting of mapping bijectively σ-algebras onto power sets. Such σ-algebras, in fact, form a rather broad class. A special grouping of the so-called optimal measures is used in our investigation (for more about optimal measures cf. [1-4]. We provide constructively a bijective mapping that will serve the purpose. In the proof we first characterize set-inclusion as well as some asymptotic behaviors of sequences of measurable sets. Without loss of generality we shall restrict ourselves to infinite σ-algebras, since the opposite case can be easily done.

Throughout this communication $(\Omega, \mathcal{F})$ will stand for an arbitrary measurable space, with both Ω and $\mathcal{F}$ being infinite sets (where, as usual, the elements of $\mathcal{F}$ are referred to as measurable sets).

By an optimal measure we mean a set function $p^* : \mathcal{F} \to [0, 1]$ which fulfills the following axioms:

P1. $p^*(\emptyset) = 0$ and $p^*(\Omega) = 1$.
P2. $p^*(B \bigcup E) = p^*(B) \bigvee p^*(E)$ for all measurable sets B and E (where $\bigvee$ stands for the maximum).
P3. $p^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} p^*(E_n) = \bigwedge_{n=1}^{\infty} p^*(E_n)$, for every decreasing sequence of measurable sets (E_n), where $\bigwedge$ stands for the minimum.

In [2] we have obtained the following results for all optimal measures p^*.

By (p^*)-atom we mean a measurable set H, $p^*(H) > 0$ such that whenever $B \in \mathcal{F}$, $B \subset H$, then $p^*(B) = p^*(H)$ or $p^*(B) = 0$.

A p^*-atom H is decomposable if there exists a subatom $B \subset H$ such that $p^*(B) = p^*(H) = p^*(H \backslash B)$. If no such subatom exists, we shall say that H is indecomposable.

Fundamental Optimal Measure Theorem. *Let $(\Omega, \mathcal{F})$ be a measurable space and p^* an optimal measure on it. Then there exists a collection $\mathcal{H}(p^*) = \{H_n : n \in J\}$ of disjoint indecomposable p^*-atoms, where J is some countable (i.e. finite or countably infinite) index-set such that for any measurable set B, with $p^*(B) > 0$, we have that*

$$p^*(B) = \max\left\{p^*\left(B \bigcap H_n\right) : n \in J\right\}.$$

Moreover, the only limit point of the set $\{p^*(H_n) : n \in J\}$ is 0 provided that J is a countably infinite set. ($\mathcal{H}(p^*)$ is referred to as a p^*-generating countable system.)

NOTATIONS.

1. $\mathcal{P}$ will denote the set of all optimal measures defined on $(\Omega, \mathcal{F})$.
2. $\mathcal{P}_\infty$ is the set of all optimal measures whose generating systems are countably infinite.
3. For every $A \in \mathcal{F}$, we write $\overline{A}$ for the complement of A.
4. $\mathbb{N}$ stands for the set of counting numbers (or positive integers).
5. $A \subset B$ means set A is a proper subset of set B.
6. $A \subseteq B$ means set A is a subset of set B.
7. $\mathbb{P}(A)$ stands for the power set of set A.

2. Main results

Definition 2.1. We say that an optimal measure $p^* \in \mathcal{P}_\infty$ is of order-one if there is a unique indecomposable p^**-atom* H such that $p^*(H) = 1$. (Any such atom will be referred to as an order-one-atom and the set of all order-one optimal measures will be denoted by $\widetilde{\mathcal{P}^1_\infty}$.)

Example 1. Fix a sequence $(\omega_n) \subset \Omega$ and define $p_0^* \in \mathcal{P}_\infty$ by

$$p_0^*(B) = \max\left\{\frac{1}{n} : \omega_n \in B\right\}.$$

Then $p_0^* \in \widetilde{\mathcal{P}^1_\infty}$.

In fact, via the Structure Theorem, there is an indecomposable p_0^**-atom* H such that $p_0^*(H) = 1$. This is possible if and only if $\omega_1 \in H$. We note that there is no other indecomposable p_0^*-atom H^* with $H^* \cap H = \emptyset$ such that $p_0^*(H^*) = 1$, otherwise necessarily it would ensue that $\omega_1 \in H^*$, which is absurd. Hence $p_0^* \in \widetilde{\mathcal{P}^1_\infty}$.

FURTHER NOTATIONS.

If H is the order-one-atom of some $p^* \in \widetilde{\mathcal{P}^1_\infty}$, we write $p = \left\{q^* \in \widetilde{\mathcal{P}^1_\infty} : q^*(H) = 1\right\}$. We then refer to the elements of the class p as representing members of the class, and call H the unitary atom of the class. (If the unitary atom of a class is the order-one-atom of a representing member, we shall speak of representation.)

We further denote by $\mathcal{P}_\infty^1$ the set of all p classes.

If A is a nonempty measurable set and $p \in \mathcal{P}_\infty^1$, the identity $p(A) = 1$ (resp. the inequality $p(A) < 1$) will simply mean that $p^*(A) = 1$ (resp. $p^*(A) < 1$) for any representing member $p^* \in p$. We shall also write $p(A) = 0$ to mean that $p^*(A) = 0$ whenever $p^* \in p$.

Write ∇ for the set of all unitary atoms on the measurable space $(\Omega, \mathcal{F})$.

Lemma 2.1. *Let $A, B \in \mathcal{F}$ and $p \in \mathcal{P}_\infty^1$ be arbitrary. In order that $p(A \cap B) = 1$ it is necessary and sufficient that $p(A) = 1$ and $p(B) = 1$.*

Proof. As the necessity is obvious, we only have to show the sufficiency. In fact, assume that $p(A) = 1$ and $p(B) = 1$. Let H be the unitary atom of class p, and let p^* denote an arbitrary but fixed representing member in the class. Without loss of generality we may assume that p^* is a representation of p (i.e. H is the order-one atom of p^*). Then $p^*(H) = 1$. Clearly, $p^*(A \cap H) = 1$ and $p^*(B \cap H) = 1$. Hence $p^*\left(\overline{A} \cap H \cap \overline{B}\right) = 0$. It is enough to prove that both identities $p^*\left(A \cap H \cap \overline{B}\right) = 0$ and $p^*\left(\overline{A} \cap H \cap B\right) = 0$ are valid. On the contrary, assume that at least one of these identities fails to hold: $p^*\left(A \cap H \cap \overline{B}\right) = 0$, say. Then $p^*\left(A \cap H \cap \overline{B}\right) = 1$. Now, since $p^*(H \cap B) = 1$, it ensues that either $p^*(A \cap H \cap B) = 1$ or $p^*\left(\overline{A} \cap H \cap B\right) = 1$. Then combining each of these last identities with $p^*\left(A \cap H \cap \overline{B}\right) = 1$, we have that $p^*\left(A \cap H \cap \overline{B}\right) = 1$ and $p^*(A \cap H \cap B) = 1$, or $p^*\left(A \cap H \cap \overline{B}\right) = 1$ and $p^*\left(\overline{A} \cap H \cap B\right) = 1$. This violates that H is an order-one-atom (because the sets $A \cap H \cap B$, $A \cap H \cap \overline{B}$ and $\overline{A} \cap H \cap B$ are pairwise disjoint). **q.e.d.**

Remark 2.0. Let $p \in \mathcal{P}_\infty^1$ be arbitrary. Then the identity $p(\emptyset) = 0$ holds.

Remark 2.1. Let $A \in \mathcal{F}$ and $p \in \mathcal{P}_\infty^1$ be arbitrary. Then the identities $p(A) = 1$ and $p\left(\overline{A}\right) = 1$ cannot hold simultaneously, i.e., for no representing member p^* of class p the identities $p^*(A) = 1$ and $p^*\left(\overline{A}\right) = 1$ hold at the same time.

In fact, assume the contrary. Then *Lemma 2.1* would imply that

$$p(A) = p\left(\overline{A}\right) = 1 = p\left(A \cap \overline{A}\right) = p(\emptyset) = 0$$

which is absurd, indeed. **q.e.d.**

Definition 2.2. For any $A \in \mathcal{F}$ define the set $\Delta(A)$ by

1. $\Delta(A) \subseteq \mathcal{P}_\infty^1$.
2. If $p \in \Delta(A)$, then $p(A) = 1$.

Remark 2.2. Let $A \in \mathcal{F}$. Then $\Delta(A) = \emptyset$ if and only if $A = \emptyset$.

Remark 2.3. If H is a unitary atom (with p its corresponding class), then $\Delta(H) = \{p\}$.

Let $A \in \mathcal{F}$ and denote by $\triangledown_A$ the set of all unitary atoms H such that $p(A) = 1$, where $\Delta(H) = \{p\}$. It is clear that $\triangledown_A \bigcap \triangledown_{\overline{A}} = \emptyset$ and $\triangledown_A \bigcup \triangledown_{\overline{A}} = \triangledown$. From this observation the following lemma is straightforward:

Lemma 2.2. *For every set $A \in \mathcal{F}$, we have that $\Delta(\overline{A}) = \overline{\Delta(A)}$.*

Proposition 2.3. *Let A, $B \in \mathcal{F}$ be arbitrary. Then*

1. $\Delta(\Omega) = \mathcal{P}^1_\infty$.

2. $\Delta(A \cap B) = \Delta(A) \cap \Delta(B)$.

3. $\Delta(A \cup B) = \Delta(A) \cup \Delta(B)$.

Proof. *Part 1* is an easy task. Let us show *Part 2*. In fact, let $p \in \Delta(A \cap B)$. Then $p(A \cap B) = 1$. Hence *Lemma 2.1* implies that $p(A) = 1$ and $p(B) = 1$, so that $p \in \Delta(A)$ and $p \in \Delta(B)$, i.e. $p \in \Delta(A) \cap \Delta(B)$. Consequently $\Delta(A \cap B) \subseteq \Delta(A) \cap \Delta(B)$. To show the reverse inclusion, pick an arbitrary $p \in \Delta(A) \cap \Delta(B)$. Then $p(A) = 1$ and $p(B) = 1$. Via *Lemma 2.1*, we have that $p(A \cap B) = 1$, i.e. $p \in \Delta(A \cap B)$. So $\Delta(A) \cap \Delta(B) \subseteq \Delta(A \cap B)$.

To end the proof, let us show the third part. In fact, let A and $B \in \mathcal{F}$ be arbitrary. Then making use of the second part of this proposition, it ensues that $\Delta(\overline{A} \cap \overline{B}) = \Delta(\overline{A}) \cap \Delta(\overline{B})$. By applying *Lemma 2.2* and De Morgan identities, we obtain that

$$\begin{aligned}\Delta(A \cup B) &= \overline{\overline{\Delta(A \cup B)}} = \overline{\Delta(\overline{A} \cap \overline{B})} = \overline{\Delta(\overline{A}) \cap \Delta(\overline{B})} \\ &= \overline{\Delta(\overline{A})} \cup \overline{\Delta(\overline{B})} = \overline{\overline{\Delta(A)}} \cup \overline{\overline{\Delta(B)}} = \Delta(A) \cup \Delta(B).\end{aligned}$$

This was to be proven. **q.e.d.**

Lemma 2.4. *Let A and $B \in \mathcal{F}$ be arbitrary nonempty sets. In order that $A \subset B$, it is necessary and sufficient that $\Delta(A) \subset \Delta(B)$.*

Proof. As the necessity is trivial, we need only show the sufficiency. In fact, assume that $A \backslash B$ is not an empty set. Then because of *Remark 2.2*, $\Delta(A \backslash B)$ is neither empty. Fix some $p \in \Delta(A \backslash B)$, i.e. $p(A \backslash B) = 1$. This implies that $p(B) < 1$. (Otherwise we would obtain via *Lemma 2.1* that $1 = p((A \backslash B) \cap B) = p(\emptyset) = 0$, which is absurd.) Then $p(A) = 1$ and $p(B) < 1$, i.e. $p \in \Delta(A) \backslash \Delta(B)$. So the set $\Delta(A) \backslash \Delta(B)$ is not empty. **q.e.d.**

Lemma 2.5. *Let A and $B \in \mathcal{F}$ be arbitrary nonempty sets. In order that $A \cap B = \emptyset$, it is necessary and sufficient that $\Delta(A) \cap \Delta(B) = \emptyset$.*

(The proof follows from *Proposition 2.3/2* and *Remark 2.2*.)

Lemma 2.6. *Let A and $B \in \mathcal{F}$ be arbitrary nonempty sets. In order that $A = B$ it is necessary and sufficient that $\Delta(A) = \Delta(B)$.*

Proof. As the necessity is trivial, we need only show the sufficiency. In fact, assume that A and $B \in \mathcal{F}$ are such that $\Delta(A) = \Delta(B)$, i.e. $\Delta(A) \subseteq \Delta(B)$ and $\Delta(B) \subseteq \Delta(A)$. By applying twice *Lemma 2.4* it ensues that $A \subseteq B$ and $B \subseteq A$. Therefore $A = B$. **q.e.d.**

Lemma 2.7. *Let A and $B \in \mathcal{F}$ be arbitrary nonempty sets. Then* $\Delta(A \backslash B) = \Delta(A) \backslash \Delta(B)$.

Proof. We simply note that *Proposition 2.3/2* and *Lemma 2.2* entail that

$$\begin{aligned} \Delta(A \backslash B) &= \Delta\left(A \cap \overline{B}\right) = \Delta(A) \cap \Delta\left(\overline{B}\right) \\ &= \Delta(A) \cap \left(\overline{\Delta(B)}\right) = \Delta(A) \backslash \Delta(B), \end{aligned}$$

which completes the proof. **q.e.d.**

Proposition 2.8. *Let $(A_n) \subset \mathcal{F}$ and $A \in \mathcal{F}$ be arbitrary. Then (A_n) converges increasingly to A if and only if $(\Delta(A_n))$ converges increasingly to $\Delta(A)$.*

Proof. Assume that (A_n) converges increasingly to A. Then by applying repeatedly *Lemma 2.4,* we have for every $n \in \mathbb{N}$ that

$$\Delta(A_n) \subset \Delta(A_{n+1}) \subset \Delta(A).$$

We need to prove that $\Delta(A) = \bigcup_{n=1}^{\infty} \Delta(A_n)$. To do this, it will be enough to show that $\Delta(A) \subseteq \bigcup_{n=1}^{\infty} \Delta(A_n)$ and $\bigcup_{n=1}^{\infty} \Delta(A_n) \subseteq \Delta(A)$. In fact, we note that the second inclusion is trivial. To prove the first one, let us pick an arbitrary class $p \in \Delta(A)$ and fix any representing member p^* of class p. We note that following the proof of *Lemma 0.1* (cf. [1], page 134), there can be found a positive integer n_0 such that $1 = p^*(A) = p^*\left(\bigcup_{k=1}^{\infty} A_k\right) = p^*(A_n)$, whenever $n \geq n_0$. Hence $p \in \bigcup_{n=n_0}^{\infty} \Delta(A_n) \subseteq \bigcup_{n=1}^{\infty} \Delta(A_n)$, i.e.

$$\Delta(A) \subseteq \bigcup_{n=n_0}^{\infty} \Delta(A_n) \subseteq \bigcup_{n=1}^{\infty} \Delta(A_n).$$

Conversely, assume that sequence $(\Delta(A_n))$ converges increasingly to $\Delta(A)$. Then for every $n \in \mathbb{N}$ we have that $\Delta(A_n) \subseteq \Delta(A_{n+1}) \subseteq \Delta(A)$, so that $A_n \subseteq A_{n+1} \subseteq A$ (because of *Lemma 2.4*). Hence $\bigcup_{n=1}^{\infty} A_n \subseteq A$. Now, suppose that set $A \backslash \bigcup_{n=1}^{\infty} A_n$ is not empty. Then via *Remark 2.2* and *Axiom 3* there can be found some $p \in \mathcal{P}_{\infty}^{1}$ and some representing member p^* of class p such that

$$1 = p^*\left(A \backslash \bigcup_{n=1}^{\infty} A_n\right) = p^*\left(\bigcap_{n=1}^{\infty} A \cap \overline{A_n}\right) = \bigwedge_{n=1}^{\infty} p^*\left(A \cap \overline{A_n}\right),.$$

since sequence $\left(\overline{A_n}\right)$ is a decreasing sequence. Consequently $1 = p^*\left(A \cap \overline{A_n}\right)$ for all $n \in \mathbb{N}$. But *Lemma 2.1* yields that $p^*(A) = 1$ and $p^*\left(\overline{A_n}\right) = 1$ for all $n \in \mathbb{N}$. Hence *Axiom 3* entails that

$$1 = \bigwedge_{n=1}^{\infty} p^*\left(\overline{A_n}\right) = p^*\left(\bigcap_{n=1}^{\infty} \overline{A_n}\right) = p^*\left(\overline{A}\right).$$

Nevertheless, this contradicts *Remark 2.1* **q.e.d.**

Proposition 2.9. *Let $(A_n) \subset \mathcal{F}$ and $A \in \mathcal{F}$ be arbitrary. Then (A_n) converges decreasingly to A if and only if $(\Delta(A_n))$ converges decreasingly to $\Delta(A)$.*

Proof. Assume that (A_n) converges decreasingly to A. Then by applying repeatedly *Lemma 2.4,* we have for every $n \in \mathbb{N}$ that

$$\Delta(A) \subset \Delta(A_{n+1}) \subset \Delta(A_n).$$

We need to prove that $\Delta(A) = \bigcap_{n=1}^{\infty} \Delta(A_n)$. To do this, it will be enough to show that $\Delta(A) \subseteq \bigcap_{n=1}^{\infty} \Delta(A_n)$ and $\bigcap_{n=1}^{\infty} \Delta(A_n) \subseteq \Delta(A)$. In fact, we note that the first inclusion is trivial. To prove the second inclusion let us pick some $p \in \bigcap_{n=1}^{\infty} \Delta(A_n)$. Then $p \in \Delta(A_n)$ for all $n \in \mathbb{N}$. Hence $p(A_n) = 1$ for all $n \in \mathbb{N}$. If we fix any representing member p^* in class p, we then obtain via *Axiom 3* that

$$p^*(A) = p^*\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigwedge_{n=1}^{\infty} p^*(A_n) = 1,$$

implying that $p(A) = 1$, i.e. $p \in \Delta(A)$. Consequently, $\bigcap_{n=1}^{\infty} \Delta(A_n) \subseteq \Delta(A)$.

Conversely, assume that sequence $(\Delta(A_n))$ converges decreasingly to $\Delta(A)$. Then for every $n \in \mathbb{N}$ we obtain that $\Delta(A) \subset \Delta(A_{n+1}) \subset \Delta(A_n)$ so that $A \subset A_{n+1} \subset A_n$, $n \in \mathbb{N}$ (by *Lemma 2.4*). Hence $A \subseteq \bigcap_{n=1}^{\infty} A_n$. To show the reverse inclusion let us assume that set $\left(\bigcap_{n=1}^{\infty} A_n\right) \setminus A$ is not empty. Then via *Remark 2.2* and *Axiom 3* there can be found some $p \in \mathcal{P}_{\infty}^1$ such that for every representing member p^* of class p

$$1 = p^*\left(\left(\bigcap_{n=1}^{\infty} A_n\right) \setminus A\right) = p^*\left(\bigcap_{n=1}^{\infty} A_n \cap \overline{A}\right) = \bigwedge_{n=1}^{\infty} p^*\left(A_n \cap \overline{A}\right),$$

since (A_n) is a decreasing sequence. Consequently, $1 = p^*\left(A_n \cap \overline{A}\right)$ for all $n \in \mathbb{N}$. Hence *Lemma 2.1* yields that $p\left(\overline{A}\right) = 1$ and $p(A_n) = 1$ for all $n \in \mathbb{N}$. But then $p \in \Delta(A_n)$ for all $n \in \mathbb{N}$ and hence $p \in \bigcap_{n=1}^{\infty} \Delta(A_n) = \Delta(A)$. Nevertheless, this is absurd since $p \in \Delta\left(\overline{A}\right) = \overline{\Delta(A)}$. We can thus conclude on the validity of the proposition. **q.e.d.**

Theorem 2.10. *Let* $(A_n) \subset \mathcal{F}$ *and* $A \in \mathcal{F}$ *be arbitrary. In order that* (A_n) *converge to* A, *it is necessary and sufficient that* $(\Delta(A_n))$ *converge to* $\Delta(A)$.

Proof. For every counting number $n \in \mathbb{N}$ write $E_n = \bigcap_{k=n}^{\infty} A_k$ and $B_n = \bigcup_{k=n}^{\infty} A_k$. It is clear that sequence (B_n) converges decreasingly to $\limsup_{n\to\infty} A_n$ and sequence (E_n) converges increasingly to $\liminf_{n\to\infty} A_n$. Consequently, by applying *Theorems 2.8* and *2.9* to these sequences, we can conclude on the validity of the theorem. **q.e.d.**

Definition 2.3. A mapping $\Delta : \mathcal{F} \to \mathbb{P}\left(\mathcal{P}^1_\infty\right)$ is said to be *powering* if it is defined by:

$$\Delta(A) = \begin{cases} \emptyset & \text{if } A = \emptyset \\ \left\{p \in \mathcal{P}^1_\infty : p(A) = 1\right\} & \text{if } A \neq \emptyset \end{cases}$$

Remark 2.3. If H is the unitary atom of a class $p \in \mathcal{P}^1_\infty$, then $\Delta(H) = \{p\}$.

The following result can easily be derived from *Lemma 2.6* and *Remark 2.2*.

Proposition 2.11. *If* $\Delta : \mathcal{F} \to \mathbb{P}\left(\mathcal{P}^1_\infty\right)$ *is a powering mapping, then it is an injection.*

Definition 2.4. If $\Gamma \subseteq \mathcal{P}^1_\infty$ is a nonempty set, then the collection $\mathcal{C}$ of all the unitary atoms of the classes $p \in \Gamma$ will be called unitary-atomic collection of Γ.

Postulate of powering. *If* $\Gamma \in \mathbb{P}\left(\mathcal{P}^1_\infty\right) \setminus \{\emptyset\}$ *and* $\mathcal{C}$ *denotes the governing-atomic collection of* Γ, *then* $\bigcup \mathcal{C}$ *is measurable and* $\Delta(\bigcup \mathcal{C}) \subseteq \Gamma$.

Theorem 2.12. *The powering mapping* $\Delta : \mathcal{F} \to \mathbb{P}\left(\mathcal{P}^1_\infty\right)$ *is surjective if and only if the postulate of powering is valid.*

Proof. Assume that *Postulate of powering* is valid. Let $\Gamma \in \mathbb{P}\left(\mathcal{P}^1_\infty\right)$ be arbitrarily fixed. We note that if $\Gamma = \emptyset$, then there is nothing to be proven. Suppose that Γ is a nonempty subset of $\mathcal{P}^1_\infty$, and denote by $\mathcal{C}$ its corresponding governing-atomic collection. Then $\bigcup \mathcal{C}$ is measurable and $\Delta(\bigcup \mathcal{C}) \subseteq \Gamma$ (by the postulate). Let us show that $\Gamma \subseteq \Delta(\bigcup \mathcal{C})$. In fact, pick any class $p \in \Gamma$ and p^* any representing member of p, with H the unitary atom of p. Since $H \subseteq \bigcup \mathcal{C}$, it ensues from *Lemma 2.2* that $\Delta(H) \subseteq \Delta(\bigcup \mathcal{C})$. But, via *Remark 2.3* we have that $\{p\} = \Delta(H)$ and $p \in \Delta(\bigcup \mathcal{C})$, i.e. $\Gamma \subseteq \Delta(\bigcup \mathcal{C})$. Therefore $\Gamma = \Delta(\bigcup \mathcal{C})$.

To prove the converse biconditional, let us assume that the powering mapping Δ is a surjection. We note that Δ is a bijection, since it is also an injection (by *Proposition 2.11*). Let $\Gamma \in \mathbb{P}\left(\mathcal{P}^1_\infty\right) \setminus \{\emptyset\}$ be arbitrary and write $\mathcal{C}$ for the corresponding unitary-atomic collection. Obviously we have that $\Gamma = \bigcup\{\Delta(H) : H \in \mathcal{C}\}$ is a subset of $\mathcal{P}^1_\infty$.

Then via the bijective property it ensues that $\Delta^{-1}(\Gamma) \in \mathcal{F}$. Clearly $\Delta(H) \subset \Gamma$ for every $H \in \mathcal{C}$. By *Lemma 2.2* together with the bijective property, we obtain that

$$H = \Delta^{-1}(\Delta(H)) \subset \Delta^{-1}(\Gamma)$$

whenever $H \in \mathcal{C}$. Consequently the inclusion $\bigcup \mathcal{C} \subseteq \Delta^{-1}(\Gamma)$ follows. Now let us show that if $\omega \in \Delta^{-1}(\Gamma)$, then there is some $H \in \mathcal{C}$ such that $\omega \in H$. Assume on the contrary that there can be found some $\omega_1 \in \Delta^{-1}(\Gamma)$ such that $\omega_1 \notin H$ for all $H \in \mathcal{C}$. We can thus define an optimal measure $q^* : \mathcal{F} \to [0, 1]$ so that

$$q^*(B) \begin{cases} = & 1 \quad \text{if } \omega_1 \in B \\ < & 1 \quad \text{if } \omega_1 \notin B. \end{cases}$$

(See *Example 1*) Then there is a unique indecomposable q^*-atom (to be denoted by $\widetilde{H}$) such that $q^*\left(\widetilde{H}\right) = 1$. It is clear that $\omega_1 \in \widetilde{H}$ and $q^*\left(\Delta^{-1}(\Gamma)\right) = 1$. We further note that

$$\bigcup \{\Delta(H) : H \in \mathcal{C}\} = \Gamma = \Delta\left(\Delta^{-1}(\Gamma)\right) = \left\{p \in \mathcal{P}^1_\infty : p\left(\Delta^{-1}(\Gamma)\right) = 1\right\}.$$

From this fact and the identity $q^*\left(\Delta^{-1}(\Gamma)\right) = 1$, there must exist some class $p_0 \in \mathcal{P}^1_\infty$ with $p_0\left(\Delta^{-1}(\Gamma)\right) = 1$, such that $q^*\left(\widetilde{H} \cap H \cap \Delta^{-1}(\Gamma)\right) = 1$, where H is the unitary atom of class p_0. Nevertheless, this is possible only if $\omega_1 \in H$, which is absurd, since we have supposed that $\omega_1 \notin H$ for all $H \in \mathcal{C}$. Therefore, if $\omega \in \Delta^{-1}(\Gamma)$, then there is some $H \in \mathcal{C}$ such that $\omega \in H$. It ensues that $\omega \in \bigcup \mathcal{C}$ for all $\omega \in \Delta^{-1}(\Gamma)$, as $H \subset \bigcup \mathcal{C}$ whenever $H \in \mathcal{C}$. Thus $\Delta^{-1}(\Gamma) \subseteq \bigcup \mathcal{C}$. Therefore, $\bigcup \mathcal{C} = \Delta^{-1}(\Gamma)$, which leads to the postulate. **q.e.d.**

Theorem 2.12 entails that an infinite σ-algebra is equinumerous with a power set if and only if *Postulate 1* is valid. This suggests that there are infinite σ-algebras that are not equinumerous with infinite power sets.

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