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A NOTE ON DERIVATIONS IN MV-ALGEBRAS

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Abstract. The aims of this paper are introduce the notions of symmetric bi-derivation and generalized derivation in MV-algebras and investigate some of their properties.

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1. Introduction

In [5], C. C. Chang invented the notion of MV-algebra in order to provide an algebraic proof of the completeness theorem of infinite valued Lukasiewicz propositional calculus. Recently, algebraic theory of MV-algebras is intensively studied, see [3, 6, 7, 9].

Let R be a ring. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. During the past few decades there has been on ongoing interest concerning the relation ship between the commutativity of a ring and the existance of certain specific types of derivations of R.

The concept of a symmetric bi-derivation has been introduced by Maksa [8]. Let R be a ring. A mapping $B : R \times R \to R$ is said to be symmetric if $B(x, y) = B(y, x)$ holds for all pairs $x, y \in R$. A mapping $f : R \to R$ defined by $f(x) = B(x, x)$, where $B : R \times R \to R$ is a symmetric mapping, is called the trace of B. A symmetric bi-additive (i. e. additive in both arguments) mapping $D : R \times R \to R$ is called a symmetric bi-derivation if $D(xy, z) = D(x, z)y + xD(y, z)$ is fullfilled for all $x, y, z \in R$. In recent years, many mathematicians studied the commutativity of prime and semi-prime rings admitting suitably-constrained symmetric bi-derivations. In [4], Y. Çeven applied the notion of symmetric bi-derivation in ring and near ring theory to lattices. In this paper, we introduce the notion of symmetric bi-derivation in MV-algebras and investigate some of its properties.

M. Bresar [2] defined the following notation. An additive mapping $f : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that

$f(xy) = f(x)y + xd(y)$ for all $x, y \in R$. One may observe that the concept of derivations, also of the left multipliers when $d = 0$.

In [1], N. O. Alshehri introduced the concept of derivation in MV-algebras and discussed some related properties. In this paper, we introduce the notions of symmetric bi-derivation and generalized derivation in MV-algebras and investigate some of their properties.

2. Preliminaries

Definition 1 ([9]). An MV-algebra is a structure $(M, +, *, 0)$ where $+$ is a binary operation, $*$ is a unary operation and 0 is a constant such that the following axioms are satisfied for any $a, b \in M$,

$(MV1)$ $(M, +, 0)$ is a commutative monoid,
$(MV2)$ $(a^*)^* = a$,
$(MV3)$ $0^* + a = 0^*$,
$(MV4)$ $(a^* + b)^* + b = (b^* + a)^* + a$.

If we define the constant $1 = 0^*$ and the auxiliary $\odot$, $\vee$, $\wedge$ by $a \odot b = (a^* + b^*)^*$, $a \vee b = a + (b \odot a^*)$, $a \wedge b = a \odot (b \oplus a^*)$ then $(M, \odot, 1)$ is a commutative monoid and the structure $(M, \vee, \wedge, 0, 1)$ is a bounded distributive lattice. Also, we define the binary operation $\ominus$ by $x \ominus y = x \odot y^*$. A subset of X an MV-algebra M is called subalgebra of M if and only if X is closed under the MV-operations defined in M. In any MV-algebras one can define a partial order $\leq$ by putting $x \leq y$ if and only if $x \wedge y = x$ for each $x, y \in M$. If the order relation $\leq$, defined over M, is total then we say that M is linearly ordered. For an MV-algebra M, if we define $B(M) = \{x \in M : x + x = x\} = \{x \in M : x \odot x = x\}$. Then $(B(M), +, *, 0)$ is both largest subalgebra of M and a Boolean algebra.

An MV-algebra M has the following properties for all $x, y, z \in M$,

(1) $x + 1 = 1$,
(2) $x + x^* = 1$,
(3) $x + x^* = 0$,
(4) If $x + y = 0$, then $x = y = 0$,
(5) If $x \odot y = 1$, then $x = y = 1$,
(6) If $x \leq y$, then $x \vee z \leq y \vee z$ and $x \wedge z \leq y \wedge z$,
(7) If $x \leq y$, then $x + z \leq y + z$ and $x \odot z \leq y \odot z$,
(8) $x \leq y$ if and only if $y^* \leq x^*$,
(9) $x + y = y$ if and only if $x \odot y = x$.

Theorem 1 ([5]). *The following conditions are equivalent for all $x, y \in M$,*
(i) $x \leq y$,
(ii) $y + x^* = 1$,
(iii) $x \odot y^* = 0$.

Definition 2 ([5]). Let M be an MV-algebra and I be a nonempty subset of M. Then we say that I is an ideal if the following conditions are satisfied,
(i) $0 \in I$,
(ii) $x, y \in I$ imply $x \oplus y \in I$,
(iii) $x \in I$ and $y \leq x$ imply $y \in I$.

Proposition 1 ([5]). *Let M be a linearly ordered MV-algebra, then $x + y = x + z$ and $x + z \neq 1$ imply that $y = z$.*

Definition 3 ([1]). Let M be an MV-algebra and $d : M \to M$ be a function. We called d a derivation of M, if it satisfies the following condition for all $x, y \in M$,

$$d(x \odot y) = (dx \odot y) + (x \odot dy)$$

Definition 4. Let M be an MV-algebra. A mapping $D : M \times M \to M$ is a called symmetric if $D(x, y) = D(y, x)$ holds for all $x, y \in M$.

Definition 5. Let M be an MV-algebra. A mapping $d : M \to M$ defined by $d(x) = D(x, x)$ is called trace of D, where $D : M \times M \to M$ is a symmetric mapping.

We often abbreviate $d(x)$ to dx.

3. Symmetric bi-derivation of MV-algebras

Definition 6. Let M be an MV-algebra and $D : M \times M \to M$ be a symmetric mapping. We call D a symmetric bi-derivation on M, if it satisfies the following condition,

$$D(x \odot y, z) = (D(x, z) \odot y) + (x \odot D(y, z))$$

for all $x, y, z \in M$.

Obviously, a symmetric bi-derivation D on M satisfies the relation $D(x, y \odot z) = (D(x, y) \odot z) + (y \odot D(x, z))$ for all $x, y, z \in M$.

Example 1. Let $M = \{0, a, b, 1\}$. Consider the following tables:

+	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

*	0	a	b	1
	1	b	a	0

Then $(M,+,*,0)$ is an MV-algebra. Define a map $D : M \times M \to M$ by

$$D(x,y) = \begin{cases} b, & (x,y) = (b,b),(b,1),(1,b) \\ 0, & \text{otherwise} \end{cases}$$

Then we can see that D is a symmetric bi-derivation of M.

Proposition 2. *Let M be an MV-algebra, D be a symmetric bi-derivation on M and d be a trace of D. Then, for all $x \in M$,*
(i) $d0 = 0$,
(ii) $dx \odot x^* = x \odot dx^* = 0$,
(iii) $dx = dx + (x \odot D(x,1))$,
(iv) $dx \leq x$,
(v) If I is an ideal of an MV-algebra, $d(I) \subseteq I$.

Proof. (i) For all $x \in M$,

$$\begin{aligned} D(x,0) &= D(x,0\odot 0) = (D(x,0)\odot 0) + (0 \odot D(x,0)) \\ &= 0+0=0. \end{aligned}$$

Since d is the trace of D,

$$\begin{aligned} d0 &= D(0,0) = D(0\odot 0,0) = (D(0,0)\odot 0) + (0\odot D(0,0)) \\ &= 0+0=0. \end{aligned}$$

(ii) For all $x \in M$,

$$\begin{aligned} 0 &= D(x,0) = D\left(x, x\odot x^*\right) \\ &= \left(D(x,x)\odot x^*\right) + \left(x \odot D\left(x,x^*\right)\right) \end{aligned}$$

and so, $dx \odot x^* = 0$ and $x \odot D(x,x^*) = 0$.
Similarly, $x \odot dx^* = 0$ for all $x \in M$.
(iii) For all $x \in M$,

$$\begin{aligned} dx &= D(x,x) = D(x,x\odot 1) = (D(x,x)\odot 1) + (x \odot D(x,1)) \\ &= dx + (x\odot D(x,1)) \end{aligned}$$

(iv) For all $x \in M$,

$$\begin{aligned} 1 &= 0^* = \left(dx\odot x^*\right)^* = \left[\left((dx)^* + \left(x^*\right)^*\right)^*\right]^* \\ &= (dx)^* + x \end{aligned}$$

From Theorem 1 (*ii*), $dx \leq x$ for all $x \in M$.
(v) Let $y \in d(I)$, then $d(x) = y$ for some $x \in I$. From (iv), $d(x) \leq x$ and so $y \in I$, since I is an ideal of M. Hence $d(I) \subseteq I$. □

Corollary 1. *For all $x \in M$, since $x \odot D(x, x^*) = 0$, we get $D(x, x^*) \le x^*$ and $x \le (D(x, x^*))^*$.*
For all $x, y \in M$, since

$$0 = D\left(x \odot x^*, y\right) = \left(D(x, y) \odot x^*\right) + \left(x \odot D\left(x^*, y\right)\right)$$

we get, $D(x, y) \le x$ and $D(x^*, y) \le x^*$.
Similarly, $D(x, y) \le y$ and $D(x, y^*) \le y^*$ for all $x, y \in M$.

Proposition 3. *Let M be an MV-algebra, D be a symmetric bi-derivation on M and d be a trace of D. If $x \le y$ for $x, y \in M$, then the followings hold:*
(i) $d(x \odot y^*) = 0$,
(ii) $dy^* \le x^*$,
(iii) $dx \odot dy^* = 0$.

Proof. (i) Let $x \le y$, for $x, y \in M$. From (7), since $x \odot y^* \le y \odot y^* = 0$, we get $x \odot y^* = 0$. Since $d0 = 0$, we have $d(x \odot y^*) = 0$.

(ii) Let $x \le y$, for $x, y \in M$. Since $x \odot dy^* \le y \odot dy^* \le y \odot y^* = 0$, we get $x \odot dy^* = 0$ and so $dy^* \le x^*$.

(iii) Since $x \le y$, we get $dx \le y$ and so $dx \odot dy^* \le y \odot dy^* \le y \odot y^* = 0$. Hence $dx \odot dy^* = 0$. □

Proposition 4. *Let M be an MV-algebra, D be a symmetric bi-derivation on M and d be a trace of D. The the followings hold:*
(i) $dx \odot dx^* = 0$,
(ii) $dx^* = (dx)^*$ if and only d is the identity on M.

Proof. (i) Since $dx \odot dy^* = 0$, replacing y by x, we get $dx \odot dx^* = 0$.

(ii) Since $x \odot dy^* = 0$ for $x, y \in M$, we get $x \odot dx^* = x \odot (dx)^* = 0$. Since $x \le dx$ and $dx \le x$, we have $x = dx$. Hence d is the identity on M.

If d is the identity on M, $dx^* = (dx)^*$ for all $x \in M$. □

Definition 7. Let M is an MV-algebra, D be a symmetric bi-derivation on M. If $x \le y$ implies $D(x, z) \le D(y, z)$ for all $x, y, z \in M$, D is called an isotone.

If d is the trace of D and D is an isotone, $x \le y$ implies $d(x) \le d(y)$ for all $x, y \in M$.

Example 2. Let M be an MV-algebra as in Example 1. Define a map $D : M \times M \to M$ by

$$D(x, y) = \begin{cases} 0, & (x, y) \in \{(0,0), (a,0), (0,a), (b,0), (0,b), (1,0), (0,1), (a,b), (b,a)\} \\ b, & (x, y) \in \{(b,b), (b,1), (1,b)\} \\ a, & (x, y) \in \{(a,a), (1,a), (a,1)\} \\ 1, & (x, y) \in \{(1,1)\} \end{cases}$$

Then we can see that D is an isotone symmetric bi-derivation on M. Since $d0 = 0$, $d1 = 1$, $da = a$ and $db = b$, d is the identity on M and so $x \le y$ implies $d(x) \le$

$d(y)$ for all $x, y \in M$.

In Example 1, $b \leq 1$, $D(b,1) = b$, $D(1,1) = 0$, but $0 \leq b$. That is, D is not isotone.

Proposition 5. *Let M be an MV-algebra, D be a symmetric bi-derivation on M and d be a trace of D. If $dx^* = dx$ for all $x \in M$, then the followings hold:*
(i) $d1 = 0$,
(ii) $dx \odot dx = 0$,
(iii) If D is an isotone on M, then $d = 0$.

Proof. (i) Replacing x by 0 in $dx^* = dx$, we get $d1 = 0$.
(ii) For all $x \in M$, $dx \odot dx = dx \odot dx^* = 0$.
(iii) Let D is an isotone on M. For $x \in M$, since $dx \leq d1 = 0$, we get $dx = 0$. Thus $d = 0$. □

Definition 8. Let M be an MV-algebra and D be a symmetric mapping on M. If $D(x+y,z) = D(x,z) + D(y,z)$ for all $x, y, z \in M$, D is called bi-additive mapping.

Theorem 2. *Let M be an MV-algebra, D be a bi-additive symmetric bi-derivation on M and d be a trace of D. Then $d(B(M)) \subseteq B(M)$.*

Proof. Let $y \in d(B(M))$. Thus $y = d(x)$ for some $x \in B(M)$. Then

$$\begin{aligned} y + y = dx + dx &= D(x,x) + D(x,x) = D(x+x,x) \\ &= D(x,x) = y. \end{aligned}$$

Hence $y \in B(M)$. That is, $d(B(M)) \subseteq B(M)$. □

Theorem 3. *Let M be a linearly ordered MV-algebra, D be a bi-additive symmetric bi-derivation on M and d be a trace of D. Then $d = 0$ or $d1 = 1$.*

Proof. Since $x + x^* = 1$ and $x + 1 = 1$ for all $x \in M$,

$$d1 = D(1,1) = D\left(x + x^*, 1\right) = D(x,1) + D\left(x^*, 1\right)$$

and

$$\begin{aligned} d1 = D(1,1) &= D(x+1,1) \\ &= D(x,1) + d1 \end{aligned}$$

If $d1 \neq 1$, Proposition 1, we get $D(x^*,1) = d1$. Replacing x by 1, we get $d1 = 0$.
For all $x \in M$,

$$0 = d1 = D(x,1) + d1 = D(x,1)$$

and

$$D(x,1) = D(x,x+1) = dx = D(x,1) = dx.$$

Thus $dx = 0$ for all $x \in M$. That is, $d = 0$. □

Theorem 4. *Let M be a linearly ordered MV-algebra, D_1 and D_2 bi-additive symmetric bi-derivations on M and d_1, d_2 be traces of D_1, D_2, respectively. If $d_1 d_2 = 0$ where $(d_1 d_2)(x) = d_1(d_2 x)$ for all $x \in M$, then $d_1 = 0$ or $d_2 = 0$.*

Proof. Let $d_1 d_2 = 0$ and $d_2 \neq 0$. Thus $d_2 1 = 1$. For all $x \in M$,

$$0 = (d_1 d_2)(x) = d_1(d_2 x) = d_1(d_2 x + (x \odot D_2(x,1))).$$

Also, since $d_2 1 = 1$, we have

$$\begin{aligned} D_2(x,1) &= D_2(x \odot 1, 1) = (D_2(x,1) \odot 1) + (x \odot D_2(1,1)) \\ &= D_2(x,1) + x \end{aligned}$$

From (9), we get $x \odot D_2(x,1) = x$.

Thus,

$$\begin{aligned} 0 &= d_1(d_2 x + x) = D_1(d_2 x + x, d_2 x + x) \\ &= D_1(d_2 x, d_2 x) + D_1(d_2 x, x) + D_1(x, d_2 x) + D_1(x,x) \\ &= D_1(d_2 x, x) + D_1(x, d_2 x) + d_1 x. \end{aligned}$$

From (4), we get $D_1(d_2 x, x) = 0$ or $d_1 x = 0$ for all $x \in M$.

Let $D_1(d_2 x, x) = 0$ for all $x \in M$. Replacing x by 1, we get $D_1(1,1) = 0$, that is, $d_1 1 = 0$. For all $x \in M$,

$$0 = d_1 1 = D_1(x+1, 1) = D_1(x,1) + d_1 1$$

and so, $D_1(x,1) = 0$. Therefore,

$$0 = D_1(x,1) = D_1(x, x+1) = d_1 x + d_1 1 = d_1 x.$$

Thus $d_1 = 0$. □

4. Generalized derivations on MV-algebras

Definition 9. Let M be an MV-algebra. A mapping $f : M \to M$ is called generalized derivation on M if there exists a derivation $d : M \to M$ such that

$$f(x \odot y) = (f(x) \odot y) + (x \odot d(y)),$$

for all $x, y \in M$.

Example 3. Let M be an MV-algebra in Example 1. Define a function $d : M \to M$ as the following,

$$d(x) = \begin{cases} 0, & x = 0, a, 1 \\ b, & x = b \end{cases}$$

Example 4. It is obvious that d is derivation on M. If we define f by

$$f(x) = \begin{cases} 0, & x = 0, a \\ b, & x = b, 1 \end{cases}$$

Then f is generalized derivation determined by d on M. Also, f is derivation on M.

Example 5. Let $M = \{0, a, b, c, d, 1\}$. Consider the following tables:

+	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	c	d	c	1	1
b	b	d	b	1	d	1
c	c	c	1	c	1	1
d	d	1	d	1	1	1
1	1	1	1	1	1	1

*	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(M, +, *, 0)$ is an MV-algebra. Define a function $d : M \to M$ as the following

$$d(x) = \begin{cases} 0, & x = 0, a, c \\ b, & x = b, d, 1 \end{cases}$$

It is obvious that d is derivation on M. If we define a function f by $f(x) = x$, for all $x \in M$.

Then f is generalized derivation determined by d on M. But, since

$$\begin{aligned} f(ac) &= f(a)c + af(c) \\ &= ac + ac = a + a = c \end{aligned}$$

and $f(ac) = f(c) = a$, f is not derivation on M.

Proposition 6. *Let M be an MV-algebra, f be a generalized derivation determined by d on M. Then the followings hold for all $x \in M$,*
(i) $f(0) = 0$,
(ii) $f(x) \odot x^* = 0$,
(iii) $f(x) = f(x) + (x \odot d(1))$,
(iv) $f(x) \leq x$,
(v) If I is an ideal of an MV-algebra, then $f(I) \subseteq I$.

Proof. (i) $f(0) = f(0 \odot 0) = (f(0) \odot 0) + (0 \odot d(0)) = 0$.
(ii) For all $x \in M$,

$$0 = f(0) = f\left(x \odot x^*\right) = \left(f(x) \odot x^*\right) + \left(x \odot d\left(x^*\right)\right)$$

and so, $f(x) \odot x^* = 0$.

(iii) For all $x \in M$,

$$f(x) = f(x \odot 1) = (f(x) \odot 1) + (x \odot d(1))$$
$$= f(x) + (x \odot d(1)).$$

(iv) For all $x \in M$,

$$1 = 0^* = \left(f(x) \odot x^*\right)^* = (f(x))^* + x$$

From Theorem 1 (ii), $f(x) \leq x$ for all $x \in M$.

(v) Let $y \in f(I)$, then $d(x) = y$ for some $x \in I$. From (iv), $f(x) \leq x$ and so $y \in I$, since I is an ideal of M. Hence $f(I) \subseteq I$. □

Corollary 2. *Let M be an MV-algebra, f be a generalized derivation determined by d on M. If $x \leq y$ for some $x, y \in M$, then the followings hold,*
(i) $f(x \odot y^*) = 0$,
(ii) $f(x) \leq y$,
(iii) $f(x) \odot f(y^*) = 0$,
(iv) $f(x^*) = (f(x))^*$ *if and only if f is the identity on M.*

Definition 10. Let M is an MV-algebra, f be a generalized derivation determined by d on M. If $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in M$, f is called an isotone.

Example 6. In Example 5, since f is an identity function, f is isotone.

Proposition 7. *Let M be an MV-algebra, f be a generalized derivation determined by d on M. If $f(x^*) = f(x)$ for all $x \in M$, then the followings hold,*
(i) $f(1) = 0$,
(ii) $f(x) \odot f(x) = 0$,
(iii) *If f is an isotone on M, then $f = 0$.*

Proof. It is clear. □

Definition 11. Let M be an MV-algebra and f be a generalized derivation determined by d on M. If $f(x + y) = f(x) + f(y)$ for all $x, y \in M$, f is called additive generalized derivation on M.

Example 7. In Example 4, f is additive generalized derivation on M.

Theorem 5. *Let M be an MV-algebra and f be a nonzero additive derivation on M. Then $f(B(M)) \subseteq B(M)$.*

Proof. Let $y \in f(B(M))$. Thus $y = f(x)$ for some $x \in B(M)$. Then

$$y + y = f(x) + f(x) = f(x + x) = f(x) = y$$

Hence $y \in B(M)$. That is, $f(B(M)) \subseteq B(M)$. □

Theorem 6. *Let f be an additive generalized derivation on a linearly ordered MV-algebra M. Then either $f = 0$ or $f(1) = 1$.*

Proof. Let f be an additive generalized derivation on a linearly ordered MV-algebra M. Hence

$$f(1) = f\left(x + x^*\right) = f(x) + f\left(x^*\right)$$

and

$$f(1) = f(x+1) = f(x) + f(1)$$

for all $x \in M$. If $f(1) \neq 1$, from Proposition 1, we get $f(1) = 0$. Therefore

$$0 = f(1) = f(1) + f(x) = f(x)$$

for all $x \in M$. That is, $f = 0$. □

Corollary 3. *Let M be a linearly ordered MV-algebra and f additive generalized derivation determined by d on M. If $f^2 = 0$ where $f^2(x) = f(f(x))$ for all $x \in M$, then $f = 0$.*

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