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ON THE DIOPHANTINE EQUATION $x^2 - kxy + y^2 + 2^n = 0$

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Abstract. In this paper, we determine when the equation in the title has an infinite number of positive integer solutions x and y when $0 \leq n \leq 10$. Moreover, we give all the positive integer solutions of the same equation for $0 \leq n \leq 10$.

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1. INTRODUCTION

In [14], Yuan and Hu determined when the two equations

$$x^2 - kxy + y^2 + 2x = 0 \tag{1.1}$$

and

$$x^2 - kxy + y^2 + 4x = 0 \tag{1.2}$$

have an infinite number of positive integer solutions x and y. They showed that Eq.(1.1) has an infinite number of positive integer solutions x and y if and only if $k = 3,4$ and Eq.(1.2) has an infinite number of positive integer solutions x and y if and only if $k = 3,4,6$. In the present paper, we consider the equation

$$x^2 - kxy + y^2 + 2^r x = 0, \tag{1.3}$$

where k is a positive integer and r is a nonnegative integer. Eq.(1.3) is a generalization of Eq.(1.1) and Eq.(1.2). In order to decide when Eq.(1.3) has an infinite number of positive integer solutions x and y, it is sufficient to determine when the equation

$$x^2 - kxy + y^2 + 2^n = 0 \tag{1.4}$$

has an infinite number of positive integer solutions x and y for nonnegative integer n. Let us assume that Eq.(1.3) has positive integer solutions x and y. Then it follows that $x | y^2$ and thus $y^2 = xz$ for some positive integer z. A simple computation shows that $\gcd(x,z) = 2^j$ for some nonnegative integer j. Thus $x = 2^j a^2$ and $z = 2^j b^2$

for some positive integers a and b with $(a,b)=1$. Then it follows that $y=2^j ab$. Substituting these values of x and y into Eq.(1.3), we obtain

$$a^2-kab+b^2+2^{r-j}=0.$$

Therefore it is sufficient to know when $x^2-kxy+y^2+2^{r-j}=0$ has an infinite number of positive integer solutions for $0\le j\le r$.

Now we begin with some well known elementary properties about Pell equations. Let d be a positive integer which is not a perfect square and N be any nonzero fixed integer. Then the equation $x^2-dy^2=N$ is known as the Pell equation. For $N=\pm1$, the equation $x^2-dy^2=\pm1$ is known as the classical Pell equation. We use the notations (x,y), and $x+y\sqrt{d}$ interchangeably to denote solutions of the equation $x^2-dy^2=N$. Also, if x and y are both positive, we say that $x+y\sqrt{d}$ is positive solution to the equation $x^2-dy^2=N$. It is well known that the equation $x^2-dy^2=1$ always has a positive solution when $d\ge2$. The least positive integer solution $x_1+y_1\sqrt{d}$ of the equation $x^2-dy^2=N$ is called the fundamental solution. If $x_1+y_1\sqrt{d}$ is the fundamental solution of the equation $x^2-dy^2=-1$, it is well known that $(x_1+y_1\sqrt{d})^2$ is the fundamental solution to the equation $x^2-dy^2=1$. Moreover, if $x_1+y_1\sqrt{d}$ is the fundamental solution to the equation $x^2-dy^2=1$, then all positive integer solutions to the equation $x^2-dy^2=1$ are given by

$$(x_n+y_n\sqrt{d})=(x_1+y_1\sqrt{d})^n \tag{1.5}$$

with $n\ge1$. It can be seen that $x_n=(\alpha^n+\beta^n)/2$ and $y_n=(\alpha^n-\beta^n)/2\sqrt{d}$, where $\alpha=x_1+y_1\sqrt{d}$ and $\beta=x_1-y_1\sqrt{d}$. If $x+y\sqrt{d}$ is a solution of the equation $x^2-dy^2=N$ and $a+b\sqrt{d}$ is a solution of the equation $x^2-dy^2=1$, then $(a+b\sqrt{d})(x+y\sqrt{d})=(ax+dby)+(ay+bx)\sqrt{d}$ is also a solution of the equation $x^2-dy^2=N$. This means that if the equation $x^2-dy^2=N$ has a solution, then it has infinitely many solutions. For more information, see [10], [13], and [2].

In section 2, we determine when Eq.(1.4) has an infinite number of positive integer solutions x and y for $0\le n\le10$. Then in section 3, we give all positive integer solutions to Eq.(1.4) for $0\le n\le10$.

2. Main Theorems

In this section, we determine when Eq.(1.4) has an infinite number of positive integer solutions x and y for $0\le n\le10$. Before discussing this, we give the following lemma and theorem, which will be needed in the proof of the main theorems.

Lemma 1. *Let $d>2$. If $u_1+v_1\sqrt{d}$ is the fundamental solution of the equation $u^2-dv^2=\pm2$, then $(u_1^2+dv_1^2)/2+u_1v_1\sqrt{d}$ is the fundamental solution of the equation $x^2-dy^2=1$.*

Proof. Assume that $d>2$ and $\omega=u_1+v_1\sqrt{d}$ is the fundamental solution of the equation $u^2-dv^2=\pm2$. Our purpose is to show that $\alpha=\omega^2/2=(u_1^2+dv_1^2)/2+$

$u_1 v_1 \sqrt{d}$ is the fundamental solution of the equation $x^2 - dy^2 = 1$. On the contrary, assume that α is not the fundamental solution of the equation $x^2 - dy^2 = 1$. Then there exists a fundamental solution $\beta = x_1 + y_1\sqrt{d}$ of the equation $x^2 - dy^2 = 1$ such that $\alpha = \beta^n$ with $n > 1$. Assume that n is an even integer. Then $n = 2k$ for some positive integer k. By using $\alpha = \omega^2/2$, we obtain $\omega^2 = 2\beta^{2k}$, i.e., $(\omega/\beta^k)^2 = 2$. If we write $\omega/\beta^k = a + b\sqrt{d}$, then it follows that $(\omega/\beta^k)^2 = a^2 + b^2 d + 2ab\sqrt{d} = 2$. Thus $ab = 0$. This shows that $a = 0$ or $b = 0$. If $a = 0$, then $b^2 d = 2$, which implies that $d = 2$. This contradicts with the fact that $d > 2$. If $b = 0$, then $a^2 = 2$, which is impossible. Now assume that n is an odd integer. Then $n = 2t + 1$ for some positive integer t. Thus it follows that $\omega^2 = 2\alpha = 2\beta^{2t+1}$, i.e., $(\omega/\beta^t)^2 = 2\beta$. It is obvious that $\beta^t > 1$. Writing $\omega/\beta^t = a + b\sqrt{d}$ gives $(\omega/\beta^t)^2 = a^2 + b^2 d + 2ab\sqrt{d} = 2\beta = 2x_1 + 2y_1\sqrt{d}$. Since β is the fundamental solution of the equation $x^2 - dy^2 = 1$, it follows that $ab > 0$. Assume that $a > 0$ and $b > 0$. Since ω is the fundamental solution of the equation $u^2 - dv^2 = \pm 2$ and ω/β^t is a positive solution of the same equation, we get $\omega \leq \omega/\beta^t$, which implies that $\beta^t \leq 1$. This is impossible since $\beta^t > 1$. Assume that $a < 0$ and $b < 0$. Then $\omega/\beta^t = -(e + f\sqrt{d})$ for some positive integers e and f. This shows that $-\omega/\beta^t = e + f\sqrt{d}$ is a positive solution of the equation $u^2 - dv^2 = \pm 2$. Since ω is the fundamental solution of the same equation, we get $\omega \leq -\omega/\beta^t$. From here, we find that $\beta^t \leq -1$, a contradiction. □

The following theorem is given in [10].

Theorem 1. *Let d be a positive integer which is not a perfect square. If x_1 and y_1 are natural numbers satisfying the inequality*

$$x_1 > \frac{y_1^2}{2} - 1$$

and if $\alpha = x_1 + y_1\sqrt{d}$ is a solution of the equation $x^2 - dy^2 = 1$, then α is the fundamental solution of this equation.

The proof of the following theorem is given in [6], and [7].

Theorem 2. *Let $k > 3$. Then the equation $x^2 - kxy + y^2 + 1 = 0$ has no positive integer solutions.*

Corollary 1. *The equation $x^2 - kxy + y^2 + 1 = 0$ has an infinite number of positive integer solutions x and y if and only if $k = 3$.*

Proof. By the above theorem, $x^2 - kxy + y^2 + 1 = 0$ has no positive integer solutions when $k > 3$. It is clear that the equation $x^2 - kxy + y^2 + 1 = 0$ has no positive integer solutions x and y for $k = 1, 2$. For $k = 3$, $x^2 - 3xy + y^2 + 1 = 0$ has an infinite number of positive integer solutions $(x, y) = (F_{2n+1}, F_{2n-1})$ with $n \geq 0$, where F_n is the n-th Fibonacci number defined in section 3 (see [6], Theorem 1.6). □

Theorem 3. *The equation* $x^2-kxy+y^2+2=0$ *has an infinite number of positive integer solutions* x *and* y *if and only if* $k=4$.

Proof. Assume that $x^2-kxy+y^2+2=0$ for some positive integers x and y. It is clear that x and y must be odd integers. Then it follows that k is even. Let $k=2t$ for some positive integer t. Then $x^2-kxy+y^2+2=0$ implies that $(x-ty)^2-(t^2-1)y^2=-2$. Let $u_1+v_1\sqrt{t^2-1}$ be the fundamental solution of the equation $u^2-\sqrt{t^2-1}v^2=-2$. Then from Lemma 1, it follows that $(u_1^2+(t^2-1)v_1^2)/2+u_1v_1\sqrt{t^2-1}$ is the fundamental solution of the equation $x^2-(t^2-1)y^2=1$. For $t>1$, since $(t,1)$ is the fundamental solution of the equation $x^2-(t^2-1)y^2=1$ by Theorem 1, we get $(u_1^2+(t^2-1)v_1^2)/2=t$ and $u_1v_1=1$. From this, it follows that $t=2$ and thus $k=4$. □

Theorem 4. *The equation* $x^2-kxy+y^2+4=0$ *has an infinite number of positive integer solutions* x *and* y *if and only if* $k=3,6$.

Proof. Assume that $x^2-kxy+y^2+4=0$ for some positive integers x and y. Assume that x is even. Then y is even and thus $x=2a$ and $y=2b$ for some positive integers a and b. Then it follows that $a^2-kab+b^2+1=0$, which implies that $k=3$ by Corollary 1. Now assume that x and y are odd integers. Then k is even and $4\nmid k$. Therefore $k=2t$ for some odd positive integer t. Completing the square gives $(x-ty)^2-(t^2-1)y^2=-4$. Since $8|t^2-1$, it follows that $x-ty=2m$ and thus $m^2-((t^2-1)/4)y^2=-1$. Let $d=(t^2-1)/4$ and assume that $u_1+v_1\sqrt{d}$ is the fundamental solution of the equation $u^2-dv^2=-1$. Then $(u_1+v_1\sqrt{d})^2=u_1^2+dv_1^2+2u_1v_1\sqrt{d}$ is the fundamental solution of the equation $x^2-dy^2=1$. For $t>1$, since $(t,2)$ is the fundamental solution of the equation $x^2-dy^2=1$ by Theorem 1, we get $u_1^2+dv_1^2+2u_1v_1\sqrt{d}=t+2\sqrt{d}$. Then it follows that $u_1v_1=1$ and $u_1^2+((t^2-1)/4)v_1^2=t$. From this, we see that $t=3$ and thus $k=6$. □

Theorem 5. *The equation* $x^2-kxy+y^2+8=0$ *has an infinite number of positive integer solutions* x *and* y *if and only if* $k=4,6,10$.

Proof. Assume that x is even. Then y is even and thus $x=2a$ and $y=2b$ for some positive integers a and b. Thus we get $a^2-kab+b^2+2=0$. By Theorem 3, it follows that $k=4$. Now assume that x and y are odd positive integers. Then k is even and $4\nmid k$. Thus $k=2t$ for some odd positive integer t. Completing the square gives $(x-ty)^2-(t^2-1)y^2=-8$, which implies that $x-ty=2m$ for some positive integer m. Thus we get $m^2-((t^2-1)/4)y^2=-2$. If $t=3$, then we get $m^2-2y^2=-2$. Since $4+3\sqrt{2}$ is a solution of the equation $m^2-2y^2=-2$, this equation has infinitely many solutions. Thus we get $k=6$. Let $d=(t^2-1)/4$ and assume that $t>3$. If $u_1+v_1\sqrt{d}$ is the fundamental solution of the equation $u^2-dv^2=-2$, then by Lemma 1, $(u_1^2+dv_1^2)/2+u_1v_1\sqrt{d}$ is the fundamental solution of the equation $x^2-dy^2=1$. For $t>1$, since $(t,2)$ is the fundamental solution of the equation

$x^2-dy^2=1$ by Theorem 1, we get $(u_1^2+dv_1^2)/2+u_1v_1\sqrt{d}=t+2\sqrt{d}$. From this, it follows that $u_1v_1=2$ and $u_1^2+((t^2-1)/4)v_1^2=2t$. Solving these equations, we see that $t=5$ and thus we get $k=10$. □

The proofs of the following theorems are similar to that of the above theorems and therefore we omit their proofs.

Theorem 6. *The equation $x^2-kxy+y^2+16=0$ has an infinite number of positive integer solutions x and y if and only if $k=3,6,18$.*

Theorem 7. *The equation $x^2-kxy+y^2+32=0$ has an infinite number of positive integer solutions x and y if and only if $k=4,6,10,14,34$.*

Now, we consider the equation

$$x^2-dy^2=N, \tag{2.1}$$

where $N\neq 0$ and d is a positive integer which is not a perfect square. If $u^2-dv^2=N$, then we say that $\alpha=u+v\sqrt{d}$ is a solution to Eq.(2.1). Let α_1 and α_2 be any two solutions to Eq.(2.1). Then α_1 and α_2 are called associated solutions if there exists a solution α to $x^2-dy^2=1$ such that

$$\alpha_1=\alpha\alpha_2.$$

The set of all solutions associated with each other forms a class of solutions to Eq.(2.1). If K is a class, then $\overline{K}=\left\{u-v\sqrt{d}\mid u+v\sqrt{d}\in K\right\}$ is also a class. We say that the class is ambiguous if $K=\overline{K}$.

Now we give the following definitions from [1].

Definition 1. Assume that $N<0$ or $N=1$. Let $u_0+v_0\sqrt{d}$ be a solution to Eq.(2.1) given in a class K such that v_0 is the least positive value of v which occurs in K. If K is not ambiguous then the number u_0 is uniquely determined. If K is ambiguous we get a uniquely determined u_0 by prescribing that $u_0\geq 0$.

Now we can give the following theorem from [10].

Theorem 8. *Let $N<0$ and $x_1+y_1\sqrt{d}$ be the fundamental solution to $x^2-dy^2=1$. If $u_0+v_0\sqrt{d}$ is the fundamental solution to the equation $u^2-dv^2=N$ in its class, then*

$$0<v_0\leq\frac{y_1\sqrt{-N}}{\sqrt{2(x_1-1)}}\ \text{and}\ 0\leq|u_0|\leq\sqrt{\frac{-1}{2}(x_1-1)N}$$

Now we can give the following theorems.

Theorem 9. *The equation $x^2-kxy+y^2+64=0$ has an infinite number of positive integer solutions x and y if and only if $k=3,6,18,66$.*

Proof. Assume that $x^2-kxy+y^2+64=0$ for some positive integers x and y. If x is even, then y is even and thus $x=2a$ and $y=2b$ for some positive integers a and b. Substituting these values of x and y into the equation $x^2-kxy+y^2+64=0$, we get $a^2-kab+b^2+16=0$, which implies that $k=3,6,18$ by Theorem 6. Now assume that x and y are odd integers. Then k is even and $4\nmid k$. Thus $k=2t$ for some positive odd integer t. Completing the square gives $(x-ty)^2-(t^2-1)y^2=-64$. Since $8|t^2-1$, it follows that $x-ty=2n$ and $t^2-1=8s$ for some positive integers n and s. So we get $n^2-2sy^2=-16$. It is seen that n is even. Then $n=2m$ and thus $2m^2-sy^2=-8$. Since y is odd, it is seen that s is even and thus we get $m^2-((t^2-1)/16)y^2=-4$. Now we consider the equation

$$u^2-\left(\frac{t^2-1}{16}\right)v^2=-4. \tag{2.2}$$

Let $u_0+v_0\sqrt{d}$ be the fundamental solution to Eq.(2.2) in a given class K. If (m,y) is a solution in the class K, then it is seen that v_0 is odd. Since $(t,4)$ is the fundamental solution to the equation $x^2-((t^2-1)/16)y^2=1$ for $t>7$ by Theorem 1, we get

$$0<v_0\le\frac{4\sqrt{4}}{\sqrt{2(t-1)}}\le\frac{4\sqrt{4}}{\sqrt{2(9-1)}}=2$$

by Theorem 8. Since v_0 is odd, $v_0=1$. If we substitute the value of v_0 into Eq.(2.2), we get $(t-4u)(t+4u)=65$. First assume that $t-4u=1$ and $t+4u=65$. Then we get $t=33$ and thus $k=66$. In a similar way, if $t-4u=5$ and $t+4u=13$, then we get $t=9$ and thus $k=18$. Now assume that $1<t\le 7$. Since $(t^2-1)/16$ is not an integer for $1<t<7$, t must be 7. But if we substitute the value of t into Eq.(2.2), we get $u^2-3v^2=-4$, which has no positive integer solutions u and v. This completes the proof. □

Theorem 10. *The equation $x^2-kxy+y^2+128=0$ has an infinite number of positive integer solutions x and y if and only if $k=4,6,10,14,34,46,130$.*

Proof. Assume that $x^2-kxy+y^2+128=0$ for some positive integers x and y. If x is even, then y is even and thus $x=2a$ and $y=2b$ for some positive integers a and b. Substituting these values of x and y into the equation $x^2-kxy+y^2+128=0$, we get $a^2-kab+b^2+32=0$, which implies that $k=4,6,10,14,34$ by Theorem 7. Now assume that x and y are odd integers. Then k is even and $4\nmid k$. Thus $k=2t$ for some positive odd integer t. Completing the square gives $(x-ty)^2-(t^2-1)y^2=-128$. Since $8|t^2-1$, it is seen that $x-ty=4m$ and thus we get $m^2-((t^2-1)/16)y^2=-8$. Now we consider the equation

$$u^2-\left(\frac{t^2-1}{16}\right)v^2=-8. \tag{2.3}$$

Let $u_0+v_0\sqrt{d}$ be the fundamental solution to Eq.(2.3) in a given class K. If (m,y) is a solution in the class K, then it is seen that v_0 is odd. Since $(t,4)$ is the fundamental solution to the equation $x^2 - ((t^2-1)/16)y^2 = 1$ for $t > 7$ by Theorem 1, we get

$$0 < v_0 \le \frac{4\sqrt{8}}{\sqrt{2(t-1)}} \le \frac{4\sqrt{8}}{\sqrt{2(9-1)}} < 3$$

by Theorem 8. Since v_0 is odd, $v_0 = 1$. Substituting this value of v_0 into Eq.(2.3), we get $(t-4u)(t+4u) = 129$. A simple computation shows that $t = 23$ and $t = 65$. Thus we get $k = 46$ and $k = 130$. Now assume that $1 < t \le 7$. Since $(t^2-1)/16$ is not an integer for $1 < t < 7$, t must be 7. If we substitute the value of t into Eq.(2.3), we get $u^2 - 3v^2 = -8$, which has no positive integer solutions u and v. This completes the proof. □

Since the proofs of the following theorems are similar to that of above theorems, we omit them.

Theorem 11. *The equation $x^2 - kxy + y^2 + 256 = 0$ has an infinite number of positive integer solutions x and y if and only if $k = 3, 6, 18, 66, 258$.*

Theorem 12. *The equation $x^2 - kxy + y^2 + 512 = 0$ has an infinite number of positive integer solutions x and y if and only if $k = 4, 6, 10, 14, 34, 46, 66, 130, 174, 514$.*

Theorem 13. *The equation $x^2 - kxy + y^2 + 1024 = 0$ has an infinite number of positive integer solutions x and y if and only if $k = 3, 6, 18, 66, 210, 258, 1026$.*

Assume that x and y are solutions of Eq.(1.4), where $1 \le n \le 10$. Then it can be shown that x and y have same parity. The equation $x^2 - 66xy + y^2 + 1024 = 0$ has positive integer solutions $(4,4)$ and $(41,1)$. The equation $x^2 - 46xy + y^2 + 512 = 0$ has positive integer solutions $(6,2)$ and $(19,1)$. Morever, the equation $x^2 - 18xy + y^2 + 64 = 0$ has positive integer solutions $(2,2)$ and $(5,1)$. It is seen from the proofs of the above theorems that all x and y solutions of Eq.(1.4) are either odd or even for $(k,n) \notin \{(66,10), (46,9), (18,6)\}$.

3. Solutions of some of the equations $x^2 - kxy + y^2 + 2^n = 0$

In this section, we will give solutions of the equation $x^2 - kxy + y^2 + 2^n = 0$ for $0 \le n \le 10$. Solutions of the equation $x^2 - kxy + y^2 + 2^n = 0$ are related to the generalized Fibonacci and Lucas numbers. Now we briefly mention the generalized Fibonacci and Lucas sequences $(U_n(k,s))$ and $(V_n(k,s))$. Let k and s be two integers with $k^2 + 4s > 0$. Generalized Fibonacci sequence is defined by $U_0(k,s) = 0$, $U_1(k,s) = 1$ and $U_{n+1}(k,s) = kU_n(k,s) + sU_{n-1}(k,s)$ for $n \geqslant 1$ and generalized Lucas sequence is defined by $V_0(k,s) = 2$, $V_1(k,s) = k$ and $V_{n+1}(k,s) = kV_n(k,s) + sV_{n-1}(k,s)$ for $n \geqslant 1$, respectively. For negative subscript, U_{-n} and V_{-n} are defined by

$$U_{-n}(k,s) = \frac{-U_n(k,s)}{(-s)^n} \text{ and } V_{-n}(k,s) = \frac{V_n(k,s)}{(-s)^n} \tag{3.1}$$

for $n \geqslant 1$. We will use U_n and V_n instead of $U_n(k,1)$ and $V_n(k,1)$, respectively. For $s=-1$, we represent (U_n) and (V_n) by $(u_n)=(U_n(k,-1))$ and $(v_n)=(V_n(k,-1))$ or briefly by (u_n) and (v_n) respectively. Also, it is seen from Eq.(3.1) that

$$u_{-n}=-U_n(k,-1) \text{ and } v_{-n}=V_n(k,-1)$$

for all $n \in \mathbb{Z}$. For $k=s=1$, the sequences (U_n) and (V_n) are called Fibonacci and Lucas sequences and they are denoted as (F_n) and (L_n), respectively. For $k=2$ and $s=1$, the sequences (U_n) and (V_n) are called Pell and Pell Lucas sequences and they are denoted as (P_n) and (Q_n), respectively. Let α and β are the roots of the equation $x^2-kx-s=0$. Then it is well known that

$$U_n=\frac{\alpha^n-\beta^n}{\alpha-\beta} \text{ and } V_n=\alpha^n+\beta^n \tag{3.2}$$

where $\alpha=(k+\sqrt{k^2+4s})/2$ and $\beta=(k-\sqrt{k^2+4s})/2$. The above identities are known as Binet's formulae. Clearly $\alpha+\beta=k$, $\alpha-\beta=\sqrt{k^2+4s}$, and $\alpha\beta=-s$ for every $n \in \mathbb{Z}$. Moreover, it is well known that

$$U_n^2-kU_nU_{n-1}-U_{n-1}^2=(-1)^{n-1}, \tag{3.3}$$

$$v_n=u_{n+1}-u_{n-1}$$

and

$$u_n^2-ku_nu_{n-1}+u_{n-1}^2=1, \tag{3.4}$$

where $U_n=U_n(k,1)$ and $u_n=U_n(k,-1)$. For more information about generalized Fibonacci and Lucas sequences, one can consult [11], [4], [12], [5], [8], and [9].

Now we give the following two theorems that help us to find solutions of some of the equations $x^2-kxy+y^2+2^n=0$. Since the proofs of these theorems can be found in [6], [5], [8], [9], and [3], we omit their proofs.

Theorem 14. *Let $k>3$. Then all nonnegative integer solutions of the equation $x^2-kxy+y^2-1=0$ are given by $(x,y)=(u_n,u_{n-1})$ with $n \geq 0$, where $u_n=U_n(k,-1)$.*

Theorem 15. *All nonnegative integer solutions of the equation $x^2-kxy-y^2+1=0$ are given by $(x,y)=(U_{2n},U_{2n-1})$ with $n \geq 0$, where $U_n=U_n(k,1)$.*

Theorem 16. *Let $r \geq 0$ be an integer. Then all positive integer solutions of the equation $x^2-(2^{2r}+2)xy+y^2+2^{2r}=0$ are given by $(x,y)=(U_{2n+1}(2^r,1),U_{2n-1}(2^r,1))$ with $n \geq 0$.*

Proof. Assume that $x^2-(2^{2r}+2)xy+y^2+2^{2r}=0$ for some positive integers x and y. It is easily seen that $2^r|x-y$. Without loss of generality, we may suppose $x \geq y$. Let $u=(x-y)/2^r$ and $v=y$. Then we get $x=2^ru+v$ and $y=v$. Substituting these values of x and y into the equation $x^2-(2^{2r}+2)xy+y^2+2^{2r}=0$, we obtain

$$(2^ru+v)^2-(2^{2r}+2)(2^ru+v)v+v^2+2^{2r}=0$$

and this implies that $u^2-2^r uv-v^2+1=0$. Therefore by Theorem 15, we get $u=U_{2n}(2^r,1)$ and $v=U_{2n-1}(2^r,1)$ with $n\geq 0$. Thus $x=2^r U_{2n}+U_{2n-1}=U_{2n+1}$ and $y=U_{2n-1}$ with $n\geq 0$. Conversely, if $(x,y)=(U_{2n+1},U_{2n-1})$, then from identity (3.3), it follows that $x^2-(2^{2r}+2)xy+y^2+2^{2r}=0$. □

Theorem 17. *Let $r\geq 1$ be an odd integer. Then all positive integer solutions of the equation $x^2-(2^r+2)xy+y^2+2^r=0$ are given by $(x,y)=(u_{n+1}-u_n,u_n-u_{n-1})$ with $n\geq 0$, where $u_n=U_n(2^r+2,-1)$.*

Proof. Assume that $x^2-(2^r+2)xy+y^2+2^r=0$ for some positive integers x and y. It is seen that x and y have the same parity. Without loss of generality, we may suppose $x\geq y$. It can be easily seen that $2^{(r+1)/2}|x-y$. Moreover, it can be shown that

$$\frac{2^r}{4}(x+y)^2-(\frac{2^r}{4}+1)(x-y)^2=2^r.$$

This implies that

$$\left(\frac{x+y}{2}\right)^2-(2^{r-1}+2)\left(\frac{x-y}{2^{(r+1)/2}}\right)^2=1.$$

Since $\alpha=\left(2^{r-1}+1+2^{(r-1)/2}\sqrt{2^{r-1}+2}\right)$ is the fundamental solution to the equation $x^2-(2^{r-1}+2)y^2=1$ by Theorem 1, it follows from (1.5) that

$$(x+y)/2=x_n \text{ and } (x-y)/2^{(r+1)/2}=y_n$$

for some $n\geq 0$, where $x_n+y_n\sqrt{2^{r-1}+2}=\alpha^n$. It is easily seen that $x_n=v_n(2^r+2,-1)/2$ and $y_n=2^{(r-1)/2}u_n(2^r+2,-1)$. Then we get $x=(v_n+2^r u_n)/2$ and $y=(v_n-2^r u_n)/2$. Since $v_n=u_{n+1}-u_{n-1}$, it follows that $x=(u_{n+1}-u_{n-1}+2^r u_n)/2=(u_{n+1}+u_{n+1}-2u_n)/2=u_{n+1}-u_n$. In a similar way, it is seen that $y=u_n-u_{n-1}$. This shows that $(x,y)=(u_{n+1}-u_n,u_n-u_{n-1})$ with $n\geq 0$. Conversely, if $(x,y)=(u_{n+1}-u_n,u_n-u_{n-1})$, then from identity (3.4), it follows that $x^2-(2^r+2)xy+y^2+2^r=0$. □

As an alternative to Theorem 16, we can give the following theorem without proof, since its proof is similar to that of Theorem 17.

Theorem 18. *Let $r\geq 1$ be an integer. Then all positive integer solutions of the equation $x^2-(2^{2r}+2)xy+y^2+2^{2r}=0$ are given by $(x,y)=(u_{n+1}-u_n,u_n-u_{n-1})$ with $n\geq 0$, where $u_n=U_n(2^{2r}+2,-1)$.*

Now we can give the following corollaries from above theorems.

Corollary 2. *All positive integer solutions of the equation $x^2-3xy+y^2+1=0$ are given by $(x,y)=(F_{2n+1},F_{2n-1})$ with $n\geq 0$.*

Corollary 3. *All positive integer solutions of the equation $x^2-4xy+y^2+2=0$ are given by $(x,y)=(u_{n+1}-u_n,u_n-u_{n-1})$ with $n\geq 0$, where $u_n=U_n(4,-1)$.*

Corollary 4. *All positive integer solutions of the equation* $x^2-3xy+y^2+4=0$ *are given by* $(x,y)=(2F_{2n+1},2F_{2n-1})$ *with* $n\geq 0$.

Corollary 5. *All positive integer solutions of the equation* $x^2-6xy+y^2+4=0$ *are given by* $(x,y)=(P_{2n+1},P_{2n-1})$ *with* $n\geq 0$.

Corollary 6. *All positive integer solutions of the equation* $x^2-4xy+y^2+8=0$ *are given by* $(x,y)=(2u_{n+1}-2u_n,2u_n-2u_{n-1})$ *with* $n\geq 0$, *where* $u_n=U_n(4,-1)$.

Theorem 19. *All positive integer solutions of the equation* $x^2-6xy+y^2+8=0$ *are given by* $(x,y)=(3u_{n+1}-u_n,3u_n-u_{n-1})$ *with* $n\geq 0$, *where* $u_n=U_n(6,-1)$.

Proof. Assume that $x^2-6xy+y^2+8=0$ for some positive integers x and y. Then by Theorem 3, it is seen that x and y are both odd integers. Also it is easily seen that $8|x-3y$. Without loss of generality, we may suppose $x\geq 3y$. Let $u=(x-3y)/8$ and $v=(3x-17y)/8$. Then we get $x=17u-3v$ and $y=3u-v$. Substituting these values of x and y into the equation $x^2-6xy+y^2+8=0$, we obtain

$$(17u-3v)^2-6(17u-3v)(3u-v)+(3u-v)^2+8=0$$

and this shows that $u^2-6uv+v^2-1=0$. From Theorem 14, we get $(u,v)=(u_n,u_{n-1})$ with $n\geq 0$. If we substitute these values of u and v into the equations $x=17u-3v$ and $y=3u-v$, then it follows that $x=17u_n-3u_{n-1}=3u_{n+1}-u_n$ and $y=3u_n-u_{n-1}$ with $n\geq 0$. Conversely, if $(x,y)=(3u_{n+1}-u_n,3u_n-u_{n-1})$, then from identity (3.4), it follows that $x^2-6xy+y^2+8=0$. This completes the proof. □

Corollary 7. *All positive integer solutions of the equation* $x^2-10xy+y^2+8=0$ *are given by* $(x,y)=(u_{n+1}-u_n,u_n-u_{n-1})$ *with* $n\geq 0$, *where* $u_n=U_n(10,-1)$.

Now we can give the following corollaries.

Corollary 8. *All positive integer solutions of the equation* $x^2-3xy+y^2+16=0$ *are given by* $(x,y)=(4F_{2n+1},4F_{2n-1})$ *with* $n\geq 0$.

Corollary 9. *All positive integer solutions of the equation* $x^2-6xy+y^2+16=0$ *are given by* $(x,y)=(2P_{2n+1},2P_{2n-1})$ *with* $n\geq 0$.

Corollary 10. *All positive integer solutions of the equation* $x^2-18xy+y^2+16=0$ *are given by* $(x,y)=(U_{2n+1},U_{2n-1})$ *with* $n\geq 0$, *where* $U_n=U_n(4,1)$.

Corollary 11. *All positive integer solutions of the equation* $x^2-4xy+y^2+32=0$ *are given by* $(x,y)=(4u_{n+1}-4u_n,4u_n-4u_{n-1})$ *with* $n\geq 0$, *where* $u_n=U_n(4,-1)$.

Corollary 12. *All positive integer solutions of the equation* $x^2-6xy+y^2+32=0$ *are given by* $(x,y)=(6u_{n+1}-2u_n,6u_n-2u_{n-1})$ *with* $n\geq 0$, *where* $u_n=U_n(6,-1)$.

Corollary 13. *All positive integer solutions of the equation* $x^2 - 10xy + y^2 + 32 = 0$ *are given by* $(x, y) = (2u_{n+1} - 2u_n, 2u_n - 2u_{n-1})$ *with* $n \geq 0$, *where* $u_n = U_n(10, -1)$.

Corollary 14. *All positive integer solutions of the equation* $x^2 - 34xy + y^2 + 32 = 0$ *are given by* $(x, y) = (u_{n+1} - u_n, u_n - u_{n-1})$ *with* $n \geq 0$, *where* $u_n = U_n(34, -1)$.

Theorem 20. *All positive integer solutions of the equation* $x^2 - 14xy + y^2 + 32 = 0$ *are given by* $(x, y) = (3u_{n+1} - u_n, u_n - u_{n-1})$ *with* $n \geq 0$, *where* $u_n = U_n(4, -1)$.

Proof. Assume that $x^2 - 14xy + y^2 + 32 = 0$ for some positive integers x and y. Then by Theorem 5, it is seen that x and y must be odd integers. Also, it is easily seen that $8|x - 3y$. Without loss of generality, we may suppose $x \geq 3y$. Let $u = (x - 3y)/8$ and $v = (x - 11y)/8$. Then we get $x = 11u - 3v$ and $y = u - v$. Substituting these values of x and y into the equation $x^2 - 14xy + y^2 + 32 = 0$, we obtain

$$(11u - 3v)^2 - 14(11u - 3v)(u - v) + (u - v)^2 + 32 = 0$$

and this shows that $u^2 - 4uv + v^2 - 1 = 0$. Therefore by Theorem 14, we get $u = U_n(4, -1)$ and $v = U_{n-1}(4, -1)$ with $n \geq 0$. Thus it follows that $x = 11u_n - 3u_{n-1} = 3u_{n+1} - u_n$ and $y = u_n - u_{n-1}$ with $n \geq 0$. Conversely, if $(x, y) = (3u_{n+1} - u_n, u_n - u_{n-1})$, then from identity (3.4), it follows that $x^2 - 14xy + y^2 + 32 = 0$. □

In order to find all positive integer solutions of the equation $x^2 - 46xy + y^2 + 128 = 0$, we need the following theorem given in [1].

Theorem 21. *If* $u + v\sqrt{d}$ *is a solution in nonnegative integers to the Diophantine equation* $u^2 - dv^2 = N$, *where* $N < 0$, *then there exists a nonnegative integer* m *such that*

$$u + v\sqrt{d} = (u_1 + v_1\sqrt{d})(x_1 + y_1\sqrt{d})^m$$

where $u_1 + v_1\sqrt{d}$ *is the fundamental solution to the class of solutions of the equation* $u^2 - dv^2 = N$ *to which* $u + v\sqrt{d}$ *belongs and* $x_1 + y_1\sqrt{d}$ *is the fundamental solution to the equation* $x^2 - dy^2 = 1$.

Lemma 2. *All positive integer solutions of the equation* $x^2 - 33y^2 = -8$ *are given by* $(x, y) = (|17u_n + 5u_{n-1}|, 3u_n - u_{n-1})$ *with* $n \in \mathbb{Z}$, *where* $u_n = U_n(46, -1)$.

Proof. Assume that $x^2 - 33y^2 = -8$ for some positive integers x and y. It can be seen from Theorem 8 that the equation $x^2 - 33y^2 = -8$ has two solution classes. And the fundamental solutions of these classes are $5 + \sqrt{33}$ and $-5 + \sqrt{33}$. By Theorem 21, all positive integer solutions of the equation $x^2 - 33y^2 = -8$ are given by

$$a_n + b_n\sqrt{33} = (5 + \sqrt{33})(x_n + y_n\sqrt{33})$$

with $n \geq 0$ or

$$c_n + d_n\sqrt{33} = (-5 + \sqrt{33})(x_n + y_n\sqrt{33})$$

with $n \geq 1$, where $x_n + y_n\sqrt{33}$ is the solution of the equation $x^2 - 33y^2 = 1$. Since the fundamental solution of this equation is $\alpha = 23 + 4\sqrt{33}$, we get $x_n + y_n\sqrt{33} = \alpha^n$ and therefore $x_n = (\alpha^n + \beta^n)/2$ and $y_n = (\alpha^n - \beta^n)/2\sqrt{33}$, where $\beta = 23 - 4\sqrt{33}$. Thus we get $b_n = 5y_n + x_n$ and $d_n = -5y_n + x_n$. It is seen that $x_n = V_n(46,-1)/2$ and $y_n = 4U_n(46,-1)$. This shows that $b_n = 20u_n + v_n/2 = u_{n+1} - 3u_n$ with $n \geq 0$ and $d_n = -20u_n + v_n/2 = 3u_n - u_{n-1}$ with $n \geq 1$. Since $3u_{-n} - u_{-n-1} = u_{n+1} - 3u_n$ for $n > 0$, we can take y as $y = u_{n+1} - 3u_n$ with $n \in \mathbb{Z}$. Substituting the value of y into the equation $x^2 - 33y^2 = -8$, we get $x = |17u_n + 5u_{n-1}|$ with $n \in \mathbb{Z}$. Conversely, if $(x,y) = (|17u_n + 5u_{n-1}|, 3u_n - u_{n-1})$, then from identity (3.4), it follows that $x^2 - 33y^2 = -8$. □

Theorem 22. *All positive integer solutions of the equation $x^2 - 46xy + y^2 + 128 = 0$ are given by $(x,y) = (3u_{n+1} - u_n, 3u_n - u_{n-1})$ with $n \in \mathbb{Z}$, where $u_n = U_n(46,-1)$.*

Proof. Assume that $x^2 - 46xy + y^2 + 128 = 0$ for some positive integers x and y. Completing the square gives $(x - 23y)^2 - 528y^2 = -128$, which implies that $x - 23y = \pm 4m$ for some positive integer m. Rearranging the equation gives $m^2 - 33y^2 = -8$. By Lemma 2, we get $(m,y) = (|17u_n + 5u_{n-1}|, 3u_n - u_{n-1})$ with $n \in \mathbb{Z}$. Thus $x = 23y \pm 4m = 23(3u_n - u_{n-1}) \pm 4(17u_n + 5u_{n-1})$ and therefore we get $x = 3u_{n+1} - u_n$ or $x = 3u_{n-1} - u_{n-2}$ with $n \in \mathbb{Z}$. Since $3u_{(n+2)-1} - u_{(n+2)-2} = 3u_{n+1} - u_n$, we can take $(x,y) = (3u_{n+1} - u_n, 3u_n - u_{n-1})$ with $n \in \mathbb{Z}$. Conversely, if $(x,y) = (3u_{n+1} - u_n, 3u_n - u_{n-1})$, then from identity (3.4), it follows that $x^2 - 46xy + y^2 + 128 = 0$. □

The proofs of the following theorems are similar to that of Theorem 22 and therefore we omit them.

Theorem 23. *All positive integer solutions of the equation $x^2 - 174xy + y^2 + 512 = 0$ are given by $(x,y) = (3u_{n+1} - u_n, 3u_n - u_{n-1})$ with $n \in \mathbb{Z}$, where $u_n = U_n(174,-1)$.*

Theorem 24. *All positive integer solutions of the equation $x^2 - 66xy + y^2 + 512 = 0$ are given by $(x,y) = (9u_{n+1} - u_n, 9u_n - u_{n-1})$ with $n \in \mathbb{Z}$, where $u_n = U_n(66,-1)$.*

Theorem 25. *All positive integer solutions of the equation $x^2 - 210xy + y^2 + 1024 = 0$ are given by $(x,y) = (5u_{n+1} - u_n, 5u_n - u_{n-1})$ with $n \in \mathbb{Z}$, where $u_n = U_n(210,-1)$.*

Theorem 26. *All positive integer solutions of the equation $x^2 - 66xy + y^2 + 1024 = 0$ are given by $(x,y) = (41u_{n+1} - u_n, 41u_n - u_{n-1})$ with $n \in \mathbb{Z}$, where $u_n = U_n(66,-1)$ or $(x,y) = (4U_{2n+1}, 4U_{2n-1})$ with $n \geq 0$, where $U_n = U_n(8,1)$.*

Since all positive integer solutions of the following equations

$$x^2 - kxy + y^2 + 64 = 0,\ k \in \{3,6,18,66\},$$
$$x^2 - kxy + y^2 + 128 = 0,\ k \in \{4,6,10,14,34,130\},$$
$$x^2 - kxy + y^2 + 256 = 0,\ k \in \{3,6,18,66,258\},$$
$$x^2 - kxy + y^2 + 512 = 0,\ k \in \{4,6,10,14,34,46,130,514\},$$

and

$$x^2 - kxy + y^2 + 1024 = 0,\ k \in \{3,6,18,258,1026\}$$

can be given easily by using the previous theorems and corollaries, we do not give their solutions.

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